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Applied Mathematical Modelling 32 (2008) 958–970

APPLIED MATHEMATICAL MODELLING

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A recursive method for the *F*-policy G/M/1/K queueing system with an exponential startup time

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Received 1 May 2006; received in revised form 1 February 2007; accepted 28 February 2007
Available online 13 March 2007

Abstract

This paper deals with the optimal control of a finite capacity G/M/1 queueing system combined the *F*-policy and an exponential startup time before start allowing customers in the system. The *F*-policy queueing problem investigates the most common issue of controlling arrival to a queueing system. We provide a recursive method, using the supplementary variable technique and treating the supplementary variable as the remaining interarrival time, to develop the steady-state probability distribution of the number of customers in the system. We illustrate a recursive method by presenting three simple examples for exponential, 3-stage Erlang, and deterministic interarrival time distributions, respectively. A cost model is developed to determine the optimal management *F*-policy at minimum cost. We use an efficient Maple computer program to determine the optimal operating *F*-policy and some system performance measures. Sensitivity analysis is also studied.

Keywords: *F*-Policy; G/M/1/K queue; Recursive method; Server startup; Supplementary variable

1. Introduction

We use a supplementary variable technique to analyze the optimal control of the *F*-policy G/M/1/K queueing system where the server needs a startup time before start allowing customers in the system and $K < \infty$ denotes the maximum capacity of the system. The method of controlling arrivals focuses on reducing the number of customers in the system. The model proposed in this paper is very useful in real-life situations since the controlling of arriving customers is considered.

Steady-state analytical solutions of the *F*-policy M/M/1/K queueing system with an exponential startup time were first developed by Gupta [1]. However, steady-state analytical solutions of the *F*-policy queueing systems with interarrival times or service times distribution of the general type have not been found. It is extremely difficult, if not possible, to obtain the explicit expressions for the steady-state probability distribution of

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0307-904X/$ - see front matter
doi:10.1016/j.apm.2007.02.023

the number of customers in the system. This becomes particularly helpful when the supplementary variable technique to the non-Markovian queueing system having general interarrival times or general service times is used. Cox [2] first introduced the supplementary variable technique. Basis on this technique, Gupta and Rao [3,4] presented a recursive method to develop the steady-state probability distributions of the number of failed machines for the no-spare M/G/1 machine repair problem and the cold-standby M/G/1 machine repair problem, respectively.

Past work regarding queues may be divided into two parts according to whether the system is considered to control the service or the arrival. In the first category of controlling the service, the *N*-policy M/M/1 queueing system without startup was first introduced by Yadin and Naor [5]. The extension of this model can be referred to Bell [6,7], Heyman [8], Kimura [9], Teghem [10], Wang and Ke [11], and others. Wang and Ke [11] provided a recursive method and used the supplementary variable technique to develop the steady-state probability distributions of the number of customers for the *N*-policy M/G/1/L queueing system. Ke and Wang [12] presented a recursive method and applied the supplementary variable technique to obtain the steady-state probability distributions of the number of customers for the *N*-policy G/M/1/L queueing system. The server startup corresponds to the preparatory work of the server before starting the service. In some real-life situations, the server often needs a startup time before beginning to provide the service. Several authors research on queueing systems with startup time focus mainly on the *N*-policy M/G/1 queues. Baker [13] first studied the *N*-policy M/M/1 queueing system with an exponential startup time. Borthakur et al. [14] extended Baker's model to the general startup time. The *N*-policy M/G/1 queueing system with startup time was investigated by several authors such as Medhi and Templeton [15], Takagi [16], Lee and Park [17], Hur and Paik [18], Krishna et al. [19], and so on. Ke [20] presented a recursive method and used the supplementary variable technique to compute the operating characteristics for the *N*-policy G/M/1/L queueing system with an exponential startup time. In the second category of controlling the arrivals, the analytical developments for controlling the arrivals in queueing problems are rarely found in the literature, which are particularly for service time and interarrival time following general type. The work of related problems in the past mainly concentrates on Markovian system. The pioneering work in steady-state analytical solutions of the *F*-policy M/M/1/K queueing system with an exponential startup time was first derived by Gupta [1]. Through a series of propositions, the relationship between the operating *N*-policy and the operating *F*-policy are established by Gupta [1].

Practically, the memoryless property of the arrival (input) process does not always meets the needs of applications because, for interarrival time, general distribution, rather than exponential distribution, appears to be more appropriate and reasonable. General distribution can include the special cases of exponential, Erlang, hyper-exponential, and deterministic, etc. However, aside from theoretical arguments, many real-life situations satisfy the assumptions of Markovian conditions for service time. Hence, we may consider inevitably to analyze the *F*-policy G/M/1/K queueing system.

In Section 2, the queueing model is briefly described. Practical justification of the model is also included. Section 3 provides a recursive method using the supplementary variable technique and treating the supplementary variable as the remaining interarrival time, to obtain the steady-state probability distributions of the number of customers in the *F*-policy G/M/1/K queueing system. We illustrate the solution algorithm by presenting three simple examples for three different interarrival time distributions: exponential (denoted M), 3-stage Erlang (denoted E_3), and deterministic (denoted D). In Section 4, various system performance measures are presented. The total expected cost function per unit time for the *F*-policy G/M/1/K queueing system with startup times is developed in Section 4. Numerical and comparative results are shown in Section 5. Finally, Section 6 consists of some concluding remarks.

2. Description of the system

We consider the category of controlling the arrivals to the *F*-policy G/M/1/K queueing system with exponential startup time. It is assumed that the times elapsing between successive arrivals are independent and identically distributed (i.i.d) random variables having general distribution $A(v)$ ($v \geqslant 0$), a probability density function $a(v)$ ($v \geqslant 0$) and mean interarrival time b_1. The service times of the customers are independent random variables having a common exponential distribution with mean $1/\mu$. Let us assume that customers arriving at the server form a single waiting line and are served in the order of their arrivals; that is, according to the first-come, first-served (FCFS) discipline. Suppose that the server can serve only one customer at a time, and that the service is independent of the arrival of the customers. Customers entering into the service facility and finding that the server is busy have to wait in the queue until the server is available. Gupta [1] first introduced the concept of a F-policy. The definition of a F-policy is described as follows: When the number of customers in the system reaches its capacity K (i.e. the system becomes full), no further arriving customers are allowed to enter the system until there are enough customers in the system have been served so that the number of customers in the system decreases to a threshold value $F\,(0 \leqslant F < K-1)$. At that time, the server needs to take an exponential startup time with parameter β to start allowing customers in the system. Thus, the system operates normally until the number of customers in the system reaches its capacity at which time the above process is repeated all over again.

2.1. Practical justification of the model

A number of practical problems arise which may be formulated as one in which the server requires take a startup time to start allowing customers in the system. Such models have potentially useful in practical real-life. For example, in computer process and service systems, messages are transmitted among the computers (processors). If the processor is free the message is accepted; otherwise the message is temporarily stored in a buffer to be served some time later. When the buffer is full, the arriving messages will be restricted entrance until the number of messages drops to a specified threshold level. When system buffer reduces to the threshold level, the messages are immediately admitted to enter the system. This will help to prevent the system from becoming over-loaded. Another application of our model is transportation. In order to avoid traffic jams caused by motorists returning home for Thanksgiving day, the entrance ramps along the highway will be controlled by a metering system. When traffic flow is congested, entrance ramps are closed to keep expressway traffic smooth. Vehicles are allowed to re-enter once the traffic is improved. The entrance ramps may need to maintain and the service may be temporarily shut down.

2.2. Notation

The following notation and definitions are used throughout the paper:

- F threshold level
- K system capacity $(K > F+1)$
- A interarrival time random variable
- V remaining interarrival time random variable
- $A(v)$ distribution function (d.f.) of A
- $a(v)$ probability density function (p.d.f.) of A
- $a^*(\theta)$ Laplace–Stieltjes transform (LST) of A
- $a^{*(l)}(\theta)$ lth order derivative of $a^*(\theta)$ with respect to θ
- $P_{0,0}(t)$ probability of no customers in the system at time t when the arrivals are not allowed to enter the system
- $P_{0,n}(t)$ probability of n customers in the system at time t when the arrivals are not allowed to enter the system, where $n = 1, 2, \ldots, K$
- $P_{1,0}(t)$ probability of no customers in the system at time t when the arrivals are allowed to enter the system
- $P_{1,n}(t)$ probability of n customers in the system at time t when the arrivals are allowed to enter the system, where $n = 1, 2, \ldots, K-1$

3. Steady-state results

The state of the system at time t is given by

$N(t) \equiv$ number of customers in the system, and
$V(t) \equiv$ remaining interarrival time for the customer who is arriving.

Let us define

$$P_{0,n}(v,t)\mathrm{d}v = \Pr\{N(t)=n,\ v<V(t)\leqslant v+\mathrm{d}v\},\quad v\geqslant 0,\ n=0,1,\ldots,F,$$
$$P_{1,n}(v,t)\mathrm{d}v = \Pr\{N(t)=n,\ v<V(t)\leqslant v+\mathrm{d}v\},\quad v\geqslant 0,\ n=0,1,\ldots,K-1,$$
$$P_{0,n}(t)=\int_0^\infty P_{0,n}(v,t)\mathrm{d}v,\quad n=0,1,\ldots,F,$$
$$P_{1,n}(t)=\int_0^\infty P_{1,n}(v,t)\mathrm{d}v,\quad n=0,1,\ldots,K-1.$$

In steady state, we define

$$P_{0,n}=\lim_{t\to\infty}P_{0,n}(t),\quad n=0,1,\ldots,K,$$
$$P_{1,n}=\lim_{t\to\infty}P_{0,n}(t),\quad n=0,1,\ldots,K-1,$$
$$P_{0,n}(v)=\lim_{t\to\infty}P_{0,n}(v,t),\quad n=0,1,\ldots,F,$$
$$P_{1,n}(v)=\lim_{t\to\infty}P_{1,n}(v,t),\quad n=0,1,\ldots,K-1$$

and further define

$$P_{0,n}(v)=P_{0,n}a(v),\quad n=0,1,\ldots,F. \tag{1}$$

For the F-policy G/M/1/K queueing system with server startup, we can easily obtain the steady-state equations as follows:

$$0=-\beta P_{0,0}+\mu P_{0,1}, \tag{2}$$
$$0=-(\beta+\mu)P_{0,n}+\mu P_{0,n+1},\quad 1\leqslant n\leqslant F, \tag{3}$$
$$0=-\mu P_{0,n}+\mu P_{0,n+1},\quad F+1\leqslant n\leqslant K-1, \tag{4}$$
$$0=-\mu P_{0,K}+P_{1,K-1}(0), \tag{5}$$
$$-\frac{\mathrm{d}}{\mathrm{d}v}P_{1,0}(v)=\beta P_{0,0}a(v)+\mu P_{1,1}(v), \tag{6}$$
$$-\frac{\mathrm{d}}{\mathrm{d}v}P_{1,n}(v)=-\mu P_{1,n}(v)+\beta P_{0,n}a(v)+P_{1,n-1}(0)a(v)+\mu P_{1,n+1}(v),\quad 1\leqslant n\leqslant F, \tag{7}$$
$$-\frac{\mathrm{d}}{\mathrm{d}v}P_{1,n}(v)=-\mu P_{1,n}(v)+P_{1,n-1}(0)a(v)+\mu P_{1,n+1}(v),\quad F+1\leqslant n\leqslant K-2, \tag{8}$$
$$-\frac{\mathrm{d}}{\mathrm{d}v}P_{1,K-1}(v)=-\mu P_{1,K-1}(v)+P_{1,K-2}(0)a(v). \tag{9}$$

We introduce the following Laplace–Stieltjes transforms (LST):

$$a^*(\theta)=\int_0^\infty \mathrm{e}^{-\theta v}\mathrm{d}A(v)=\int_0^\infty \mathrm{e}^{-\theta v}a(v)\mathrm{d}v,$$
$$P^*_{i,n}(\theta)=\int_0^\infty \mathrm{e}^{-\theta v}P_{i,n}(v)\mathrm{d}v,\quad i=0,1,$$
$$P_{i,n}=P^*_{i,n}(0)=\int_0^\infty P_{i,n}(v)\mathrm{d}v,\quad i=0,1,$$
$$\int_0^\infty \mathrm{e}^{-\theta v}\frac{\partial}{\partial u}P_{i,n}(v)\mathrm{d}v=\theta P^*_{i,n}(\theta)-P_{i,n}(0),\quad i=0,1.$$

Taking the LST on both sides of (6)–(9), it implies that

$$-\theta P^*_{1,0}(\theta)=\beta P_{0,0}a^*(\theta)+\mu P^*_{1,1}(\theta)-P_{1,0}(0), \tag{10}$$
$$(\mu-\theta)P^*_{1,n}(\theta)=\beta P_{0,n}a^*(\theta)+\mu P^*_{1,n+1}(\theta)+P_{1,n-1}(0)a^*(\theta)-P_{1,n}(0),\quad 1\leqslant n\leqslant F, \tag{11}$$

$$(\mu-\theta)P^*_{1,n}(\theta)=\mu P^*_{1,n+1}(\theta)+P_{1,n-1}(0)a^*(\theta)-P_{1,n}(0),\quad F+1\leqslant n\leqslant K-2, \tag{12}$$
$$(\mu-\theta)P^*_{1,K-1}(\theta)=P_{1,K-2}(0)a^*(\theta)-P_{1,K-1}(0). \tag{13}$$

3.1. Recursive method

Our main work is to develop the steady-state probabilities $P^*_{0,n}(0)$ and $P^*_{1,n}(0)$, where $1\leqslant n\leqslant K$. Our solution algorithm will first obtain $P^*_{0,n}(0)$ $(1\leqslant n\leqslant K)$.

Using (2)–(5) yields

$$P^*_{0,n}(0)=\phi_n P_{0,0},\quad 1\leqslant n\leqslant K, \tag{14}$$
$$P_{1,K-1}(0)=\mu\phi_{F+1}P_{0,0}, \tag{15}$$

where

$$\phi_n=\begin{cases}1, & n=0,\\ \dfrac{\beta}{\mu}\left(1+\dfrac{\beta}{\mu}\right)^{\zeta_{n-1}}, & 1\leqslant n\leqslant K,\end{cases} \tag{16}$$

and

$$\zeta_n=\begin{cases}n, & 0\leqslant n\leqslant F-1,\\ F, & F\leqslant n\leqslant K.\end{cases}$$

Thus, $P^*_{0,1}(0),P^*_{0,2}(0),\ldots,P^*_{0,K}(0)$ can be obtained by using (14).

Next, we derive the expressions of $P_{1,n}(0)$ $(0\leqslant n\leqslant K-2)$ in terms of $P_{0,0}$. Substituting (14), (15) into (11)–(13) and then setting $\theta=\mu$ in (11)–(13) we finally have

$$P_{1,n}(0)=\frac{P_{1,n+1}(0)-\mu P^*_{1,n+2}(\mu)-\beta\varphi_{n+1,F}\phi_{n+1}P_{0,0}a^*(\mu)}{a^*(\mu)},\quad 0\leqslant n\leqslant K-3, \tag{17}$$

where

$$\varphi_{n,F}=\begin{cases}1, & 1\leqslant n\leqslant F,\\ 0, & \text{otherwise},\end{cases}$$
$$P_{1,K-2}(0)=\frac{\mu\phi_{F+1}}{a^*(\mu)}P_{0,0}. \tag{18}$$

To obtain $P^*_{1,n+2}(\mu)$ $(0\leqslant n\leqslant K-3)$ in (17), substituting (14) and (18) into (11)–(13) then differentiating (11)–(13) $(l-1)$ times with respect to θ, and finally setting $\theta=\mu$, we get

$$P^{*(l-1)}_{1,n}(\mu)=-\frac{1}{l}[P_{1,n-1}(0)a^{*(l)}(\mu)+\beta\varphi_{n,F}\phi_nP_{0,0}a^{*(l)}(\mu)+\mu P^{*(l)}_{1,n+1}(\mu)], \tag{19}$$

where $2\leqslant n\leqslant K-2$, $l=1,\ldots,K-n-1$,

$$P^*_{1,K-1}(\mu)=-\frac{\mu\phi_{F+1}a^{*(1)}(\mu)}{a^*(\mu)}P_{0,0}, \tag{20}$$

where $P^{*(0)}_{1,n}(\mu)=P^*_{1,n}(\mu)$.

Solving (19) and (20) recursively, we finally obtain

$$P^*_{1,n}(\mu)=-\frac{\beta\varphi_{n+2,F}a^*(\mu)}{\mu}\sum_{i=\zeta_{n+2}}^{F}\ell_{n-i-1}\phi_iP_{0,0}-\frac{a^*(\mu)}{\mu}\sum_{i=n+2}^{K-1}\ell_{n-i-1}P_{1,i-1}(0),\quad 2\leqslant n\leqslant K-2, \tag{21}$$

where

$$\ell_n = \begin{cases} -\dfrac{(-\mu)^n a^{*(n)}(\mu)}{n! a^*(\mu)}, & 1 \leqslant n \leqslant K-1, \\ 0, & \text{otherwise.} \end{cases} \quad (22)$$

Using (20) and (21) in (17), we obtain

$$P_{1,n}(0) = \frac{P_{1,n+1}(0)}{a^*(\mu)} + \sum_{i=n+2}^{K-1} \ell_{i-n-1} P_{1,i-1}(0) + \beta \left[\varphi_{n+2,F} \sum_{i=\zeta_{n+2}}^{F} \ell_{i-n-1}\phi_i - \varphi_{n+1,F}\phi_{n+1} \right] P_{0,0}, \quad 0 \leqslant n \leqslant K-3. \quad (23)$$

Further, let us define

$$\Psi_n = \begin{cases} 1, & n = 0, \\ \sum_{1 \leqslant k \leqslant n} \sum_{\substack{\tau_1+\tau_2+\cdots+\tau_k=n \\ \tau_1,\tau_2,\cdots,\tau_k \in \{1,2,\cdots,n\}}} \kappa_{\tau_1}\kappa_{\tau_2}\cdots\kappa_{\tau_k}, & n = 1,2,\ldots,K-2, \\ 0, & \text{otherwise,} \end{cases} \quad (24)$$

where

$$\kappa_n = \begin{cases} \dfrac{1}{a^*(\mu)} + \ell_1, & n = 1, \\ \ell_n, & n = 2,3,\ldots,K-2, \\ 0, & \text{otherwise.} \end{cases} \quad (25)$$

Remark. The representative meaning of the above formulation (24) is to sum up all possible products of k κs in which the total of subscript values of κ equals n. We may present an easily understood example for $n = 4$:

$$\Psi_4 = \kappa_4 + \kappa_3\kappa_1 + \kappa_2\kappa_2 + \kappa_1\kappa_3 + \kappa_1\kappa_1\kappa_2 + \kappa_1\kappa_2\kappa_1 + \kappa_2\kappa_1\kappa_1 + \kappa_1\kappa_1\kappa_1\kappa_1 = \kappa_4 + 2\kappa_3\kappa_1 + \kappa_2^2 + 3\kappa_1^2\kappa_2 + \kappa_1^4. \quad (26)$$

Using (24) and (25) to solve (23) recursively, and including (18), we finally have

$$P_{1,n}(0) = \sum_{i=1}^{K-n-1} \Psi_{K-n-i-1} \Lambda(K-i-1) P_{0,0}, \quad 0 \leqslant n \leqslant K-2, \quad (27)$$

where

$$\Lambda(n) = \begin{cases} \beta\varphi_{n+2,F} \sum_{i=\zeta_{n+2}}^{F} \ell_{i-n-1}\phi_i - \beta\varphi_{n+1,F}\phi_{n+1}, & 0 \leqslant n \leqslant K-3, \\ \dfrac{\mu\phi_{F+1}}{a^*(\mu)}, & n = K-2. \end{cases} \quad (28)$$

Finally, we develop the steady-state probabilities $P^*_{1,n}(0)$ in terms of $P_{0,0}$. Setting $\theta = 0$ in (10)–(13) yields

$$P^*_{1,n}(0) = \frac{1}{\mu} \left[P_{1,n-1}(0) - \beta \sum_{i=0}^{\zeta_{n-1}} \phi_i P_{0,0} \right], \quad 1 \leqslant n \leqslant K-1. \quad (29)$$

As $P_{1,1}(0), P_{1,2}(0), \ldots, P_{1,K-1}(0)$ are known, $P^*_{1,1}(0), P^*_{1,2}(0), \ldots, P^*_{1,K-1}(0)$ can be solved recursively using (29) in terms of $P_{0,0}$.

Now the only unknown quantity is $P^*_{1,0}(0)$ which can be determined from (10)–(13). To find it, differentiating (10)–(13) with respect to θ and then setting $\theta = 0$, we get

$$P^*_{1,0}(0) = -\beta P_{0,0} a^{*(1)}(0) - \mu P^{*(1)}_{1,1}(0), \tag{30}$$

$$P^{*(1)}_{1,n}(0) = \frac{P_{1,n} + \beta\varphi_{n,F}\phi_n P_{0,0} a^{*(1)}(0) + \mu P^{*(1)}_{1,n+1}(0) + P_{1,n-1}(0) a^{*(1)}(0)}{\mu}, \quad 1 \leqslant n \leqslant K-1. \tag{31}$$

The values $P^{*(1)}_{1,n}(0)$ for $n = 1, 2, \ldots, K-1$ can be found recursively from (31). Therefore, we obtain

$$P^*_{1,0}(0) = -\left[\beta a^{*(1)}(0) \sum_{i=0}^{F} \phi_n P_{0,0} + \sum_{i=1}^{K-1} P_{1,i} + a^{*(1)}(0) \sum_{i=0}^{K-2} P_{1,i}(0)\right]. \tag{32}$$

So $P^*_{1,0}(0), P^*_{1,1}(0), \ldots, P^*_{1,K-1}(0)$ are known in terms of $P_{0,0}$, which can be determined using the normalizing condition

$$\sum_{i=0}^{K} P_{0,i} + \sum_{i=0}^{K-1} P_{1,i} = 1. \tag{33}$$

3.2. The solution algorithm

The steps of the solution algorithm are stated as follows:

Step 1. For $n = 0, 1, \ldots, K$, compute ϕ_n using (16).
Step 2. For $n = 1, 2, \ldots, K$, compute $P^*_{0,n}(0)$ using (14) in terms of $P_{0,0}$.
Step 3. Compute ℓ_n $(1 \leqslant n \leqslant K-2)$ and κ_n $(1 \leqslant n \leqslant K-2)$ using (22) and (25), respectively.
Step 4. For $n = 0, 1, \ldots, K-2$, compute Ψ_n using (24).
Step 5. For $n = 0, 1, \ldots, K-2$, compute $\Lambda(n)$ using (28).
Step 6. For $n = 0, 1, \ldots, K-2$, compute $P_{1,n}(0)$ using (27) in terms of $P_{0,0}$.
Step 7. For $n = 1, 2, \ldots, K-1$, compute $P^*_{1,n}(0)$ using (29) in terms of $P_{0,0}$.
Step 8. Compute $P^*_{1,0}(0)$ using (32) in terms of $P_{0,0}$.
Step 9. Determine $P_{0,0}$ using (33). Thus $P^*_{0,n}(0)$ $(n = 1, 2, \ldots, K)$ are achieved from *Step* 2, and $P^*_{1,n}(0)$ $(n = 0, 1, \ldots, K-1)$ are achieved from *Steps* 7 and 8.

3.3. Simple examples

We use the solution algorithm to illustrate a recursive method. We provide three simple examples for three different interarrival time distributions such as exponential, 3-stage Erlang, and deterministic, respectively.

Example 1 (*For M/M/1/K queueing system*). We set the mean interarrival time $b_1 = 1/\lambda$, where λ is the interarrival rate. Assume that $F = 2$ and $K = 5$. In this case, we have

$$a^*(\theta) = \frac{\lambda}{\lambda + \theta}.$$

Step 1. For $n = 0, 1, \ldots, 5$, compute ϕ_n using (16).
Using (16), we have

$$\phi_0 = 1,\ \phi_1 = \frac{1-\alpha}{\alpha},\ \phi_2 = \frac{1-\alpha}{\alpha^2},\ \text{and}\ \phi_3 = \phi_4 = \phi_5 = \frac{1-\alpha}{\alpha^3}, \quad \text{where}\ \alpha = \mu/(\mu+\beta).$$

Step 2. For $n = 1, 2, \ldots, 5$, compute $P^*_{0,n}(0)$ using (14) in terms of $P_{0,0}$.
Using (14), we finally obtain

$$P^*_{0,1}(0) = \phi_1 P_{0,0} = \frac{1-\alpha}{\alpha} P_{0,0}, \quad P^*_{0,2}(0) = \phi_2 P_{0,0} = \frac{1-\alpha}{\alpha^2} P_{0,0},$$

$$P^*_{0,3}(0) = P^*_{0,4}(0) = P^*_{0,5}(0) = \phi_3 P_{0,0} = \frac{1-\alpha}{\alpha^3} P_{0,0}.$$

Step 3. For $n = 1,2,3$, compute ℓ_n and κ_n using (22) and (24), respectively.
We first compute ℓ_n. For $n = 1,2,3$, using (22) yields $\ell_1 = -\sigma/(1+\sigma)$, $\ell_2 = -\sigma^2/(1+\sigma)^2$ and $\ell_3 = -\sigma^3/(1+\sigma)^3$, where $\sigma = \mu/\lambda$.
Next, we compute κ_n. For $n = 1,2,3$, using (24) yields

$$\kappa_1 = -(1+\sigma+\sigma^2)/(1+\sigma), \quad \kappa_2 = -\sigma^2/(1+\sigma)^2 \quad \text{and} \quad \kappa_3 = -\sigma^3/(1+\sigma)^3.$$

Step 4. For $n = 0,1,2,3$, compute Ψ_n using (23).
It follows from (23) that

$$\Psi_0 = 1, \quad \Psi_1 = (1+\sigma+\sigma^2)/(1+\sigma), \quad \Psi_2 = 1+\sigma^2, \quad \text{and} \quad \Psi_3 = (1+\sigma+\sigma^2+\sigma^3+\sigma^4)/(1+\sigma).$$

Step 5. For $n = 0,1,2,3$, compute $\Lambda(n)$ using (28).
From (28) we have

$$\Lambda(0) = -\frac{\mu(1-\alpha)^2(\alpha+\alpha\sigma+\sigma^2)}{\alpha^3(1+\sigma)}, \quad \Lambda(1) = -\frac{\mu(1-\alpha)^2}{\alpha^3}, \quad \Lambda(2) = 0, \quad \text{and} \quad \Lambda(3) = \frac{\mu(1-\alpha)(1+\sigma)}{\alpha^3}.$$

Step 6. For $n = 0,1,2,3$, compute $P_{1,n}(0)$ using (27) in terms of $P_{0,0}$.
Using (27) yields

$$P_{1,0}(0) = [\Psi_3\Lambda(3) + \Psi_2\Lambda(2) + \Psi_1\Lambda(1) + \Psi_0\Lambda(0)]P_{0,0} = \frac{\mu(1-\alpha)(\sigma^2+\sigma^3+\sigma^4+\alpha\sigma+\alpha^2)}{\alpha^3}P_{0,0},$$

$$P_{1,1}(0) = [\Psi_2\Lambda(3) + \Psi_1\Lambda(2) + \Psi_0\Lambda(1)]P_{0,0} = \frac{\mu(1-\alpha)(\sigma+\sigma^2+\sigma^3+\alpha)}{\alpha^3}P_{0,0},$$

$$P_{1,2}(0) = [\Psi_1\Lambda(3) + \Psi_0\Lambda(2)]P_{0,0} = \frac{\mu(1-\alpha)(1+\sigma+\sigma^2)}{\alpha^3}P_{0,0},$$

$$P_{1,3}(0) = \Psi_3\Lambda(3)P_{0,0} = \frac{\mu(1-\alpha)(1+\sigma)}{\alpha^3}P_{0,0}.$$

Step 7. For $n = 1,2,3,4$, compute $P^*_{1,n}(0)$ using (29) in terms of $P_{0,0}$.
It implies from (29) that

$$P^*_{1,1}(0) = \frac{\sigma(1-\alpha)(\sigma+\sigma^2+\sigma^3+\alpha)}{\alpha^3}P_{0,0}, \quad P^*_{1,2}(0) = \frac{\sigma(1-\alpha)(1+\sigma+\sigma^2)}{\alpha^3}P_{0,0},$$

$$P^*_{1,3}(0) = \frac{\sigma(1-\alpha)(1+\sigma)}{\alpha^3}P_{0,0}, \quad \text{and} \quad P^*_{1,4}(0) = \frac{\sigma(1-\alpha)}{\alpha^3}P_{0,0}.$$

Step 8. Compute $P^*_{1,0}(0)$ using (32) in terms of $P_{0,0}$.
Using (32) yields $P^*_{1,0}(0) = \frac{\sigma(1-\alpha)(\sigma^2+\sigma^3+\sigma^4+\alpha\sigma+\alpha^2)}{\alpha^3}P_{0,0}$.

Step 9. Determine $P_{0,0}$ using (33). Thus $P^*_{0,n}(0)$ $(n = 1,2,\ldots,5)$ are achieved from *Step* 2, and $P^*_{1,n}(0)$ $(n = 0,1,\ldots,4)$ are achieved from *Step* 7 and *Step* 8.

$$P_{0,0} = \frac{\alpha^3}{\alpha^3 + (1-\alpha)(3+\alpha+\alpha^2) + \sigma(1-\alpha)(3+\alpha+\alpha^2+\alpha\sigma+3+3\sigma+3\sigma^2+2\sigma^3+\sigma^4)}.$$

It is to be noted that these results are in accordance with the expressions given by Gupta [1, p. 1006].

Example 2. (For $E_3/M/1/K$ queueing system). Let λ denote the interarrival rate. The 3-stage Erlang distribution is made up of three independent and identical exponential stages, each with mean $1/3\lambda$. We set the mean interarrival time $b_1 = 1/\lambda$, $F = 1$, and $K = 3$. In this case, we have

$$a^*(\theta) = \left(\frac{3\lambda}{3\lambda+\theta}\right)^3.$$

Step 1. For $n = 0, 1, \ldots, 3$, compute ϕ_n using (16).
Using (16), we obtain

$$\phi_0 = 1,\ \phi_1 = 3(1-\gamma)/\gamma,\ \text{and}\ \phi_2 = \phi_3 = 3(1-\gamma)(3-2\gamma)/\gamma^2, \quad \text{where}\ \gamma = 3\mu/(3\mu+\beta).$$

Step 2. For $n = 1, 2, 3$, compute $P^*_{0,n}(0)$ using (14) in terms of $P_{0,0}$.
Using (14) yields

$$P^*_{0,1}(0) = \phi_1 P_{0,0} = 3\frac{1-\gamma}{\gamma}P_{0,0}, \quad P^*_{0,2}(0) = P^*_{0,3}(0) = \phi_2 P_{0,0} = 3\frac{(1-\gamma)(3-2\gamma)}{\gamma^2}P_{0,0}.$$

Step 3. Compute ℓ_1 and κ_1 using (22) and (25), respectively.
Using (22) yields $\ell_1 = -3\tau/(1+\tau)$, where $\tau = \mu/3\lambda$.
Form (25), we obtain $\kappa_1 = (1+\tau+6\tau^2+4\tau^3+\tau^4)/(1+\tau)$.

Step 4. For $n = 0, 1$, compute Ψ_n using (23).
It follows from (23) that $\Psi_0 = 1$ and $\Psi_1 = (1+\tau+6\tau^2+4\tau^3+\tau^4)/(1+\tau)$.

Step 5. For $n = 0, 1$, compute $\Lambda(n)$ using (28).
It finds from (28) that

$$\Lambda(0) = -9\frac{\mu(1-\gamma)^2}{\gamma^2} \quad \text{and} \quad \Lambda(1) = 3\frac{\mu(1-\gamma)(3-2\gamma)(1+\tau)^2}{\gamma^2}.$$

Step 6. For $n = 0, 1$, compute $P_{1,n}(0)$ using (27) in terms of $P_{0,0}$.
Using (27), we finally get

$$P_{1,0}(0) = [\Psi_1\Lambda(1) + \Psi_0\Lambda(0)]P_{0,0} = \frac{3\mu(1-\gamma)(3-2\gamma)(1+\tau)^2(1+\tau+6\tau^2+4\tau^3+\tau^4) - 9\mu(1-\gamma)^2}{\gamma^2}P_{0,0},$$

$$P_{1,1}(0) = \Psi_0\Lambda(1)P_{0,0} = \frac{3\mu(1-\gamma)(3-2\gamma)(1+\tau)^3}{\gamma^2}P_{0,0}.$$

Step 7. For $n = 1, 2$, compute $P^*_{1,n}(0)$ using (29) in terms of $P_{0,0}$.
It implies from (29) that

$$P^*_{1,1}(0) = \frac{3(1-\gamma)(3-2\gamma)\tau(3+9\tau+17\tau^2+15\tau^3+6\tau^4+\tau^5)}{\gamma^2}P_{0,0},$$

$$P^*_{1,2}(0) = \frac{3(1-\gamma)(3-2\gamma)\tau(3+3\tau+\tau^2)}{\gamma^2}P_{0,0}.$$

Step 8. Compute $P^*_{1,0}(0)$ using (32) in terms of $P_{0,0}$.
Using (32) yields

$$P^*_{1,0}(0) = \frac{1}{\gamma^2}\{3\tau(1-\gamma)[\gamma(3-12\tau-36\tau^2-78\tau^3-78\tau^4-34\tau^5-6\tau^6) + \tau(18+54\tau+117\tau^2+117\tau^3+51\tau^4+9\tau^5)]\}P_{0,0}.$$

Step 9. Determine $P_{0,0}$ using (33). Thus $P^*_{0,n}(0)$ $(n = 1, 2, 3)$ are achieved from *Step* 2, and $P^*_{1,n}(0)$ $(n = 0, 1, 2)$ are achieved from *Step* 7 and *Step* 8.

$$P_{0,0} = \gamma^2[18 - \gamma(27-10\gamma) + 27\tau(2+6\tau+12\tau^2+18\tau^3+15\tau^4+6\tau^5+\tau^6) - 9\tau\gamma(27+30\tau+60\tau^2+270\tau^3 + 75\tau^4+30\tau^5+5\tau^6) + 9\tau\gamma^2(3+12\tau+24\tau^2+36\tau^3+30\tau^4+12\tau^5+2\tau^6)]^{-1}.$$

Example 3 (*For D/M/1/K queueing system*). We set the mean interarrival time $b_1 = 1/\lambda$, $F = 1$, and $K = 3$, where λ is the interarrival rate. In this case, we have

$$a^*(\theta) = e^{-\theta/\lambda}.$$

Step 1. For each $n = 0, 1, 2, 3$, compute ϕ_n using (16).
Using (16), we obtain

$$\phi_0 = 1,\ \phi_1 = (1-\alpha)/\alpha,\ \text{and}\ \phi_2 = \phi_3 = (1-\alpha)/\alpha^2, \quad \text{where}\ \alpha = \mu/(\mu+\beta).$$

Step 2. For each $n = 1, 2, 3$, compute $P^*_{0,n}(0)$ using (14) in terms of $P_{0,0}$.
Using (14), we finally get

$$P^*_{0,1}(0) = \phi_1 P_{0,0} = \frac{1-\alpha}{\alpha} P_{0,0}, \quad P^*_{0,2}(0) = P^*_{0,3}(0) = \phi_2 P_{0,0} = \frac{1-\alpha}{\alpha^2} P_{0,0}.$$

Step 3. Compute ℓ_1 and κ_1 using (22) and (25), respectively.
Using (22) yields $\ell_1 = -\sigma$, where $\sigma = \mu/\lambda$. Form (25), we obtain $\kappa_1 = e^{\sigma} - \sigma$.

Step 4. For each $n = 0, 1$, compute Ψ_n using (23).
It implies from (23) that $\Psi_0 = 1$ and $\Psi_1 = e^{\sigma} - \sigma$.

Step 5. For each $n = 0, 1$, compute $\Lambda(n)$ using (28).
It follows from (28) that $\Lambda(0) = -\frac{\mu(1-\alpha)^2}{\alpha^2}$ and $\Lambda(1) = \frac{\mu(1-\alpha)e^{\sigma}}{\alpha^2}$.

Step 6. For each $n = 0, 1$, compute $P_{1,n}(0)$ using (27) in terms of $P_{0,0}$.
Using (27) yields

$$P_{1,0}(0) = [\Psi_1 \Lambda(1) + \Psi_0 \Lambda(0)] P_{0,0} = \frac{\mu(1-\alpha)(e^{2\sigma} - \sigma e^{\sigma} + \alpha - 1)}{\alpha^2} P_{0,0},$$

$$P_{1,1}(0) = \Psi_0 \Lambda(1) P_{0,0} = \frac{\mu(1-\alpha)e^{\sigma}}{\alpha^2} P_{0,0}.$$

Step 7. For each $n = 1, 2$, compute $P^*_{1,n}(0)$ using (29) in terms of $P_{0,0}$.
It implies from (29) that

$$P^*_{1,1}(0) = \frac{(1-\alpha)(e^{2\sigma} - \sigma e^{\sigma} - 1)}{\alpha^2} P_{0,0} \quad \text{and} \quad P^*_{1,2}(0) = \frac{(1-\alpha)(e^{\sigma} - 1)}{\alpha^2} P_{0,0}.$$

Step 8. Compute $P^*_{1,0}(0)$ using (32) in terms of $P_{0,0}$.
Using (32) yields $P^*_{1,0}(0) = \frac{(1-\alpha)[2+\alpha\sigma-(1-\sigma)e^{\sigma}(e^{\sigma}+1-\sigma)]}{\alpha^2} P_{0,0}$.

Step 9. Determine $P_{0,0}$ using (33). Thus $P^*_{0,n}(0)$ $(n = 1, 2, 3)$ are achieved from *Step* 2, and $P^*_{1,n}(0) (n = 0, 1, 2)$ are achieved from *Steps* 7 to 8.

$$P_{0,0} = \frac{\alpha^2}{\sigma e^{2\sigma}(1-\alpha) + \sigma e^{\sigma}(1-\alpha)(1-\sigma) + 2 - \alpha + \alpha\sigma - \alpha^2\sigma}.$$

4. Optimal *F*-policy

Some important system performance measures of the *F*-policy G/M/1/K queueing system with exponential startup time are first defined as follows:

$L_s \equiv$ the expected number of customers in the system;
$P_b \equiv$ the probability that the server is busy;
$P_s \equiv$ the probability that the server requires a startup time before starting the service;
$P_{bl} \equiv$ the probability that the server is blocked.

The expressions for L_s, P_b, P_s, and P_{bl} are given by

$$L_s = \sum_{n=1}^{K} n P_{0,n} + \sum_{n=1}^{K-1} n P_{1,n},$$

$$P_b = \sum_{n=0}^{K} P_{0,n} + \sum_{n=0}^{K-1} P_{1,n},$$

$$P_s = \sum_{n=0}^{F} P_{0,n},$$

$$P_{bl} = \sum_{n=0}^{K} P_{0,n}.$$

Next, we develop the total expected cost function per unit time for the F-policy G/M/1/K queueing system with startup times, in which F is a decision variable. The main purpose of this paper is to determine the optimum operating F-policy so as to minimize this total expected cost function. Let

$C_h \equiv$ holding cost per unit time for each customer present in the system;
$C_b \equiv$ cost per unit time for a busy server;
$C_s \equiv$ startup cost per unit time for the preparatory work of the server before starting the service;
$C_{bl} \equiv$ fixed cost for every lost customer when the system is blocked.

Utilizing the definitions of each cost element listed above, the total expected cost function per unit time is given by

$$\mathrm{TC}(F) = C_h L_s + C_b P_b + C_s P_s + C_{bl} \lambda P_{bl}. \tag{34}$$

The optimal value of F, F^* is determined by the following inequalities:

$$\mathrm{TC}(F^* - 1) \geqslant \mathrm{TC}(F^*) \quad \text{and} \quad \mathrm{TC}(F^* + 1) \geqslant \mathrm{TC}(F^*). \tag{35}$$

5. Numerical comparisons

We set the system capacity $K = 15$. We perform a sensitivity analysis for changes in the optimum value F^* along with changes in specific values of the system parameters. We consider three simple examples for three different interarrival time distributions such as exponential, 3-stage Erlang, and deterministic. The following cost elements are employed:

Case 1: $C_h = 10,\ C_b = 200,\ C_s = 250,\ C_{bl} = 350.$
Case 2: $C_h = 10,\ C_b = 200,\ C_s = 250,\ C_{bl} = 400.$
Case 3: $C_h = 10,\ C_b = 200,\ C_s = 300,\ C_{bl} = 400.$
Case 4: $C_h = 10,\ C_b = 225,\ C_s = 300,\ C_{bl} = 400.$
Case 5: $C_h = 15,\ C_b = 225,\ C_s = 300,\ C_{bl} = 400.$

Table 1
The optimal value of F and its minimum expected cost for exponential interarrival time

		$(\mu, \beta) = (1.0, 3.0)$			$(\lambda, \beta) = (0.7, 3.0)$			$(\lambda, \mu) = (0.7, 1.0)$		
		λ			μ			β		
		0.55	0.65	0.75	1.0	1.1	1.2	2.0	4.0	5.0
Case 1	F^*	6	4	3	4	6	8	4	4	4
	$\mathrm{TC}(F^*)$	122.209	148.361	177.914	162.635	144.660	130.652	162.654	162.626	162.620
Case 2	F^*	8	6	5	5	8	11	5	5	5
	$\mathrm{TC}(F^*)$	122.215	148.425	178.303	162.803	144.705	130.663	162.823	162.793	162.787
Case 3	F^*	8	6	5	5	8	11	5	5	5
	$\mathrm{TC}(F^*)$	122.216	148.428	178.318	162.810	144.708	130.664	162.834	162.798	162.791
Case 4	F^*	7	5	4	4	7	10	5	4	4
	$\mathrm{TC}(F^*)$	135.963	164.647	196.879	180.230	160.598	145.243	180.253	180.218	180.211
Case 5	F^*	4	3	2	2	4	6	2	2	2
	$\mathrm{TC}(F^*)$	142.056	173.713	210.174	191.254	169.196	152.207	191.276	191.242	191.235

Table 2
The optimal value of F and its minimum expected cost for 3-stage Erlang interarrival time

		$(\mu, \beta) = (1.0, 3.0)$			$(\lambda, \beta) = (0.7, 3.0)$			$(\lambda, \mu) = (0.7, 1.0)$		
		λ			μ			β		
		0.55	0.65	0.75	1.0	1.1	1.2	2.0	4.0	5.0
Case 1	F^*	7	5	4	4	7	9	4	4	4
	$TC(F^*)$	118.991	143.288	170.687	156.448	139.842	126.868	156.450	156.447	156.447
Case 2	F^*	9	7	5	6	9	12	6	6	6
	$TC(F^*)$	118.991	143.291	170.747	156.463	139.844	126.868	156.465	156.462	156.462
Case 3	F^*	9	7	5	6	9	11	6	6	6
	$TC(F^*)$	118.991	143.291	170.750	156.464	139.844	126.868	156.466	156.463	156.462
Case 4	F^*	8	6	4	5	8	11	5	5	5
	$TC(F^*)$	132.741	159.539	189.471	173.957	155.752	141.451	173.959	173.956	173.955
Case 5	F^*	5	4	3	3	5	7	3	3	3
	$TC(F^*)$	137.236	166.178	199.711	182.155	162.033	146.552	182.157	182.153	182.153

Table 3
The optimal value of F and its minimum expected cost for deterministic interarrival time

		$(\mu, \beta) = (1.0, 3.0)$			$(\lambda, \beta) = (0.7, 3.0)$			$(\lambda, \mu) = (0.7, 1.0)$		
		λ			μ			β		
		0.55	0.65	0.75	1.0	1.1	1.2	2.0	4.0	5.0
Case 1	F^*	10	6	4	5	7	10	5	5	5
	$TC(F^*)$	117.440	140.710	166.469	153.130	137.435	125.030	153.130	153.129	153.129
Case 2	F^*	10	7	5	6	9	12	6	6	6
	$TC(F^*)$	117.440	140.710	166.477	153.131	137.435	125.030	153.131	153.130	153.130
Case 3	F^*	12	7	5	6	9	12	6	6	6
	$TC(F^*)$	117.440	140.710	166.478	153.131	137.435	125.030	153.131	153.131	153.130
Case 4	F^*	9	6	5	5	8	11	6	5	5
	$TC(F^*)$	131.190	156.960	185.224	170.630	153.344	139.613	170.630	170.630	170.630
Case 5	F^*	6	4	3	4	6	8	4	4	4
	$TC(F^*)$	134.911	162.315	193.445	177.193	158.425	143.794	177.193	177.193	177.193

In this section we provide the numerical results of the optimal value F^* and the minimum expected cost for three interarrival time distributions and specific values of λ, μ, β. We first fix $(\mu, \beta) = (1.0, 3.0)$ and choose different values of $\lambda = 0.55, 0.65, 0.75$. Next, we fix $(\lambda, \beta) = (0.7, 3.0)$ and consider various values of $\mu = 1.0, 1.1, 1.2$. Finally, we fix $(\lambda, \mu) = (0.7, 1.0)$ and select different values of $\beta = 2.0, 4.0, 5.0$.

The optimal value of F, F^*, and its minimum expected cost $TC(F^*)$ for the above five cases are shown in Tables 1–3. For fixed values of (μ, β) and various values of λ in Tables 1–3, we observe that (i) $TC(F^*)$ increases as λ increases for any case; and (ii) F^* decreases as λ increases for any case. For fixed values of (λ, β) and various values of μ in Tables 1–3, we find that (i) $TC(F^*)$ decreases as μ increases for any case; and (ii) F^* increases as μ increases for any case. Again, for fixed (λ, μ) and various values of β in Tables 1–3, we observe that (i) $TC(F^*)$ slightly decreases as β increases for any case; and (ii) F^* does not change at all when β changes from 2.0 to 5.0 for any case. Intuitively, F^* is insensitive to changes in β.

It can be easily seen from Tables 1–3 that (i) F^* increases as C_h decreases (see cases 4–5); and (ii) C_h has a larger effect on F^* than C_b, C_s and C_{bl} (see cases 3–4, cases 2–3 and cases 1–2).

6. Conclusions

The analytical steady-state results developed in this paper would be useful, which is significant to practitioners and system designers. The main objective of this paper is threefold. We have first provided a recursive method for obtaining the steady-state probability distributions of the number of customers in the system. Next, we have illustrated our recursive method by a study of three different interarrival time distributions: exponential, 3-stage Erlang, and deterministic. In addition, we provide a very efficient solution algorithm to calculate the optimal threshold F^* at minimum cost. Finally, we have performed a sensitivity analysis among the optimal value of F, specific values of system parameters, and the cost elements. Further, the developed controlling arrival systems in this paper can be modeled many quality and service (Q&S) system in real-life.

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