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The palindromic generalized eigenvalue problem $A^*x = \lambda Ax$: Numerical solution and applications

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A B S T R A C T

In this paper, we propose the palindromic doubling algorithm (PDA) for the palindromic generalized eigenvalue problem (PGEP) $A^*x = \lambda Ax$. We establish a complete convergence theory of the PDA for PGEPs without unimodular eigenvalues, or with unimodular eigenvalues of partial multiplicities two (one or two for eigenvalue 1). Some important applications from the vibration analysis and the optimal control for singular descriptor linear systems will be presented to illustrate the feasibility and efficiency of the PDA.

1. Introduction

In this paper, we develop the palindromic doubling algorithm (PDA) for the numerical solution of the palindromic generalized eigenvalue problem (PGEP)

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$$A^*x = \lambda Ax, \tag{1.1}$$

where A is a real or complex $N \times N$ matrix, $\lambda \in \mathbb{C}$ and $x \in \mathbb{C}^N\backslash\{0\}$ are eigenvalue and the corresponding eigenvector of (1.1), respectively. Here, the symbol "$*$"$= \top$ (Transpose) or H (Hermitian). The pencil $A^* - \lambda A$ and the pair (A^*, A) are usually called a palindromic linear pencil and a palindromic matrix pair, respectively. It is easily seen that the eigenvalues of (1.1) satisfy the reciprocal property, i.e., they appear in pairs as in $\{\lambda, \ 1/\lambda^*\}$.

The PGEPs with complex coefficient matrices were firstly suggested as "good" linearizations [5,6] of palindromic polynomial/quadratic matrix pencils, arising from the study of vibration analysis [2, 4,12]. A PGEP with real coefficient matrices can also be shown to be equivalent to the generalized continuous/discrete-time algebraic Riccati equations, associated with the continuous/discrete-time, linear-quadratic optimal control problems (see [11] for details).

The standard approach for solving the PGEP is to compute its generalized Schur form (e.g., by `qz` in MATLAB), ignoring its symmetric or palindromic structure in (A^*, A). However, the reciprocal property of eigenvalues of (1.1) is not preserved by computation generally, producing large numerical errors [7]. Recently, a QR-like algorithm [8] and a hybrid method [7] (which combines Jacobi-type method with the Laub's trick) were proposed for the PGEP. The QR-like algorithm generally requires $O(N^4)$ flops and the hybrid method requires $O(N^3 \log(N))$ flops. Alternatively, for methods of cubic complexity, a URV-decomposition based structured method [9] and a structure-preserving algorithm [3] for PGEPs were proposed, producing eigenvalues which are paired to working precision. Unfortunately for PGEPs, these more efficient (and equivalent) methods require the transformation of the PGEP to the quadratic form $(\mu^2 A^* + \mu \cdot 0 + A)x = 0$, leading to operations in larger $2N \times 2N$ matrices. The PDA is a unique and more direct, thus more efficient, algorithm for the PGEP.

The purpose of this paper is to develop the PDA for solving the PGEP structurally. We establish quadratic convergence and linear convergence with rate 1/2 of the PDA, respectively, when (A^*, A) has no unimodular eigenvalues and has unimodular eigenvalues with partial multiplicities two. In application to discrete-time optimal control problems, we especially develop a new algorithm combined with the PDA (as in Algorithm 4.1) for solving the optimal control of *singular* descriptor linear systems. To our knowledge, the associated generalized discrete-time algebraic Riccati equation (GDARE) has not been solved successfully in a structure-preserving manner.

This paper is organized as follows. In Section 2 we develop the palindromic doubling algorithm (PDA) for solving PGEPs. In Section 3 we establish the convergence theory for the PDA. In Section 4 we use the PDA to compute numerical solutions structurally in different applications in PGEPs, GCAREs and GDAREs. Concluding remarks are given in Section 5.

Throughout this paper, $\mathbb{C}^{m\times n}$ and $\mathbb{R}^{m\times n}$ denote the sets of $m \times n$ complex and real matrices, respectively. For convenience, we denote $\mathbb{C}^n = \mathbb{C}^{n\times 1}$, $\mathbb{C} = \mathbb{C}^1$, $\mathbb{R}^n = \mathbb{R}^{n\times 1}$ and $\mathbb{R} = \mathbb{R}^1$. The open left-half plane and the open unit disk are denoted by $\mathbb{C}_-$ and $\mathbb{D}_1$, respectively; $0_{m\times n}(0_m)$ and I_m are the $m \times n(m \times m)$ zero matrix and the $m \times m$ identity matrix, respectively. We use $\sigma(A, B)$ to denote the spectrum of the matrix pair (A, B), and $\|\cdot\|$ denotes the 2-norm of a matrix.

2. Palindromic doubling algorithm

For a given palindromic matrix pair (A^*, A), we shall develop a doubling algorithm for solving the associated PGEP which preserves the palindromic structure at each iterative step.

Suppose $-1 \notin \sigma(A^*, A)$ (the assumption can be removed later in Remark 3.1). We then have

$$\begin{aligned} A^*(A^* + A)^{-1}A &= ((A^* + A) - A)(A^* + A)^{-1}(A + A^* - A^*) \\ &= (I - A(A^* + A)^{-1})((A^* + A) - A^*) \\ &= A(A^* + A)^{-1}A^*. \end{aligned} \tag{2.1}$$

From (2.1), it is easily seen that

$$\left[A(A^* + A)^{-1}, A^*(A^* + A)^{-1}\right] \begin{bmatrix} A^* \\ -A \end{bmatrix} = 0. \tag{2.2}$$

We now define the doubling transform $A \to \widehat{A}$ by

$$\widehat{A} = A(A^* + A)^{-1}A. \tag{2.3}$$

Theorem 2.1. *The matrix pair* $(\widehat{A}^*, \widehat{A})$ *has the doubling property*; *i.e., if*

$$A^*U = AUS, \tag{2.4}$$

where $U \in \mathbb{C}^{N\times\ell}$ *and* $S \in \mathbb{C}^{\ell\times\ell}$, *then*

$$\widehat{A}^*U = \widehat{A}US^2. \tag{2.5}$$

Proof. Multiplying the both sides of (2.4) by $A^*(A^* + A)^{-1}$, and (2.1) and (2.4) imply (2.5). □

From Theorem 2.1, we see that the doubling transform (2.3) preserves the palindromic structure. So, for a palindromic matrix pair (A_0^*, A_0) with $A_0 \in \mathbb{C}^{N\times N}$, we can develop the PDA to generate the sequence $\{(A_k^*, A_k)\}$ if no breakdown occurs in the iterative process.

PDA Algorithm Given $A_0 \in \mathbb{C}^{N\times N}$, τ (a small tolerance),

for $k = 0, 1, 2, \ldots$, **compute**

$$A_{k+1} = A_k\left(A_k^* + A_k\right)^{-1}A_k, \tag{2.6}$$

if dist(Null(A_{k+1}), Null(A_k)) $< \tau$, then stop,

end for

Here, "Null(·)" denotes the null space of the given matrix and "dist(·, ·)" denotes the distance between two subspaces.

To develop the PDA further, denote

$$A_k = H_k + K_k, \tag{2.7a}$$

where

$$H_k = \frac{1}{2}\left(A_k^* + A_k\right) = H_k^*, \quad K_k = \frac{1}{2}\left(A_k - A_k^*\right) = -K_k^* \tag{2.7b}$$

are the ∗-symmetric and ∗-anti-symmetric parts of A_k, respectively. Then the iteration (2.6) can be rewritten as

$$\begin{aligned} A_{k+1} = H_{k+1} + K_{k+1} &= \frac{1}{2}(H_k + K_k)H_k^{-1}(H_k + K_k) \\ &= \frac{1}{2}\left(H_k + K_kH_k^{-1}K_k\right) + K_k. \end{aligned}$$

The iteration (2.6) in the PDA can be simplified to

$$\begin{aligned} H_{k+1} &= \frac{1}{2}\left(H_k + K_0H_k^{-1}K_0\right), \\ K_{k+1} &= K_k = \cdots = K_0. \end{aligned}$$

3. Convergence of PDA

Let $A_0 \in \mathbb{C}^{N\times N}$. Suppose the eigenvalue "1" of (A_0^*, A_0) (if exists) has partial multiplicity one or two, and the other unimodular eigenvalues of (A_0^*, A_0) (if exist) have exactly partial multiplicities two. By the theorem of Kronecker canonical form there are nonsingular matrices Q and Z such that

$$QA_0^*Z = \begin{bmatrix} J_0 \oplus \Omega_0 & 0_{\ell,\tilde{\ell}} \oplus I_r \\ 0_{\tilde{n},n} & I_{\tilde{\ell}} \oplus \Omega_0 \end{bmatrix} \equiv C_0, \tag{3.1a}$$

$$QA_0Z = \begin{bmatrix} I_n & 0_{n,\tilde{n}} \\ 0_{\tilde{n},n} & \widetilde{J}_0^* \oplus I_r \end{bmatrix} \equiv D_0, \tag{3.1b}$$

where $\Omega_0 = \operatorname{diag}\left(e^{i\omega_1}, \ldots, e^{i\omega_r}\right)$, $J_0 \in \mathbb{C}^{\ell\times\ell}$ consists of stable Jordan blocks (i.e., $\rho(J_0) < 1$, where $\rho(\cdot)$ is the radius of the spectrum) and $\widetilde{J}_0^* = J_0 \oplus I_m$ with $n = \ell + r$, $\tilde{\ell} = \ell + m$, $\tilde{n} = n + m = \ell + r + m$ and $N = 2n + m$. Here "$\oplus$" denotes the direct sum of matrices.

Since $C_0D_0 = D_0C_0$, from (3.1b) we have that $A_0^*ZD_0 = A_0ZC_0$. From Theorem 2.1 and steps in the PDA, it follows that

$$A_k^*ZD_0^{2^k} = A_kZC_0^{2^k}. \tag{3.2}$$

Substituting (3.1b) into (3.2), we get

$$A_k^*Z\begin{bmatrix} I_n & 0_{n,\tilde{n}} \\ 0_{\tilde{n},n} & (\widetilde{J}_0^*)^{2^k} \oplus I_r \end{bmatrix} = A_kZ\begin{bmatrix} J_0^{2^k} \oplus \Omega_0^{2^k} & 0_{\ell,\tilde{\ell}} \oplus \Gamma_k \\ 0_{\tilde{n},n} & I_{\ell} \oplus \Omega_0^{2^k} \end{bmatrix}, \tag{3.3}$$

where $\Gamma_k = 2^k\Omega_0^{2^k-1}$. On the other hand, we can interchange the role of (A_0^*, A_0) by considering the pair (A_0, A_0^*) which has the same Kronecker structure as (A_0^*, A_0). Therefore, there are nonsingular P and Y such that

$$PA_0Y = \begin{bmatrix} J_0^* \oplus \Omega_0^* & 0_{\ell,\tilde{\ell}} \oplus I_r \\ 0_{\tilde{n},n} & I_{\tilde{\ell}} \oplus \Omega_0^* \end{bmatrix} \equiv E_0, \tag{3.4a}$$

$$PA_0^*Y = \begin{bmatrix} I_n & 0_{n,\tilde{n}} \\ 0_{\tilde{n},n} & \widetilde{J}_0 \oplus I_r \end{bmatrix} \equiv F_0, \tag{3.4b}$$

Since $E_0F_0 = F_0E_0$, we deduce that $A_0YF_0 = A_0^*YE_0$. Using the similar arguments as in (3.2) and (3.3), we obtain

$$A_kY\begin{bmatrix} I_n & 0_n \\ 0_n & \widetilde{J}_0^{2^k} \oplus I_r \end{bmatrix} = A_k^*Y\begin{bmatrix} (J_0^*)^{2^k} \oplus (\Omega_0^*)^{2^k} & 0_{\ell,\tilde{\ell}} \oplus \Gamma_k^* \\ 0_{\tilde{n},n} & I_{\tilde{\ell}} \oplus (\Omega_0^*)^{2^k} \end{bmatrix}. \tag{3.5}$$

We partition A_k, H_k and K_0 in (2.7a) into four sub-blocks as in

$$A_k = \begin{bmatrix} A_{k1} & A_{k3} \\ A_{k2} & A_{k4} \end{bmatrix}, \quad H_k = \begin{bmatrix} H_{k1} & H_{k2}^* \\ H_{k2} & H_{k4} \end{bmatrix}, \quad K_0 = \begin{bmatrix} K_{01} & -K_{02}^* \\ K_{02} & K_{04} \end{bmatrix}, \tag{3.6a}$$

where $A_{k1}, H_{k1}, K_{01} \in \mathbb{C}^{n\times n}$, $A_{k2}^*, A_{k3}, H_{k2}^*, K_{02}^* \in \mathbb{C}^{n\times\tilde{n}}$ and $A_{k4}, H_{k4}, K_{04} \in \mathbb{C}^{\tilde{n}\times\tilde{n}}$. From (2.7a) and (3.6a), we also have

$$A_{k1} = H_{k1} + K_{01}, \quad A_{k2} = H_{k2} + K_{02}, \tag{3.6b}$$

$$A_{k3} = H_{k2}^* - K_{02}^*, \quad A_{k4} = H_{k4} + K_{04}. \tag{3.6c}$$

Furthermore, we partition Z in (3.3) and Y in (3.5) as in

$$Z = \begin{bmatrix} Z_1 & Z_3 \\ Z_2 & Z_4 \end{bmatrix}, \quad Y = \begin{bmatrix} Y_1 & Y_3 \\ Y_2 & Y_4 \end{bmatrix}, \tag{3.7}$$

where $Z_1, Y_1 \in \mathbb{C}^{n\times n}$; $Z_2^*, Z_3, Y_2^*, Y_3 \in \mathbb{C}^{n\times\tilde{n}}$ and $Z_4,\ Y_4 \in \mathbb{C}^{\tilde{n}\times\tilde{n}}$. For convenience, we denote

$$Z_{i,a} \equiv Z_i(:, 1:\ell), \quad Y_{i,a} \equiv Y_i(:, 1:\ell), \quad i = 3, 4, \tag{3.8}$$

$$Z_{i,b} \equiv Z_i(:, \ell+1:n), \quad Y_{i,b} \equiv Y_i(:, \ell+1:n), \quad i = 1, 2. \tag{3.9}$$

Theorem 3.1. *Let $A_0 \in \mathbb{C}^{N\times N}$. Suppose that the eigenvalue "1" of (A_0^*, A_0) (if exists) has partial multiplicity one or two, the other unimodular eigenvalues of $(A_0^*,\ A_0)$ (if exist) have exactly partial multiplicities two, and (3.1b) and (3.4b) hold with $\tilde{n} \leqslant 2\ell$ (i.e., $r+m \leqslant \ell$). Suppose that Z_1 and Y_1 in (3.7) are invertible, and $W \equiv [\Phi Z_{3,a} - Z_{4,a} | \Psi Y_{3,a} - Y_{4,a}] \in \mathbb{C}^{\tilde{n}\times 2\ell}$ is of full row rank, where $\Phi \equiv Z_2 Z_1^{-1}$, $\Psi \equiv Y_2 Y_1^{-1}$. If $-1 \notin \bigcup_{j=1}^{r} \left\{e^{2^k i\omega_j}, k \geqslant 0\right\}$, then the sequence $\{(A_k^*, A_k)\}$ generated by the PDA is well defined and satisfies*

$$A_k^* \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} \to 0, \text{ linearly as } k \to \infty, \tag{3.10a}$$

$$A_k \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \to 0, \text{ linearly as } k \to \infty, \tag{3.10b}$$

with convergence rate $1/2$, where $\operatorname{span}\begin{Bmatrix} Z_1 \\ Z_2 \end{Bmatrix}$ and $\operatorname{span}\begin{Bmatrix} Y_3 \\ Y_4 \end{Bmatrix}$ form the weakly stable and the unstable invariant subspaces of (A_0^*, A_0) corresponding to $(J_0 \oplus \Omega_0, I_n)$ and $(I_n, J_0^* \oplus \Omega_0^*)$, respectively.

Proof. Since $-1 \notin \bigcup_{j=1}^{r} \left\{e^{2^k i\omega_j}, k \geqslant 0\right\}$, from (2.6) we see that $-1 \notin \sigma(A_k^*, A_k)$, thus, $A_k^* + A_k$ is invertible for all k.

From (3.6a), (3.3) and (3.7), we have

$$A_{k1}^* Z_1 + A_{k2}^* Z_2 = A_{k1} Z_1 \left(J_0^{2^k} \oplus \Omega_0^{2^k}\right) + A_{k3} Z_2 \left(J_0^{2^k} \oplus \Omega_0^{2^k}\right), \tag{3.11}$$

$$A_{k3}^* Z_1 + A_{k4}^* Z_2 = A_{k2} Z_1 \left(J_0^{2^k} \oplus \Omega_0^{2^k}\right) + A_{k4} Z_2 \left(J_0^{2^k} \oplus \Omega_0^{2^k}\right), \tag{3.12}$$

$$\begin{aligned} A_{k1}^* Z_3 \left(\left(\tilde{J}_0^*\right)^{2^k} \oplus I_r\right) + A_{k2}^* Z_4 \left(\left(\tilde{J}_0^*\right)^{2^k} \oplus I_r\right) = {} & A_{k1} \left[Z_1 (0_{\ell,\tilde{\ell}} \oplus \Gamma_k) + Z_3 \left(I_{\tilde{\ell}} \oplus \Omega_0^{2^k}\right)\right] \\ & + A_{k3} \left[Z_2 (0_{\ell,\tilde{\ell}} \oplus \Gamma_k) + Z_4 \left(I_{\tilde{\ell}} \oplus \Omega_0^{2^k}\right)\right], \end{aligned} \tag{3.13}$$

$$\begin{aligned} A_{k3}^* Z_3 \left(\left(\tilde{J}_0^*\right)^{2^k} \oplus I_r\right) + A_{k4}^* Z_4 \left(\left(\tilde{J}_0^*\right)^{2^k} \oplus I_r\right) = {} & A_{k2} \left[Z_1 (0_{\ell,\tilde{\ell}} \oplus \Gamma_k) + Z_3 \left(I_{\tilde{\ell}} \oplus \Omega_0^{2^k}\right)\right] \\ & + A_{k4} \left[Z_2 (0_{\ell,\tilde{\ell}} \oplus \Gamma_k) + Z_4 \left(I_{\tilde{\ell}} \oplus \Omega_0^{2^k}\right)\right]. \end{aligned} \tag{3.14}$$

Post-multiplying (3.13) by $0_{\tilde{\ell},\ell} \oplus \Gamma_k^{-1} \Omega_0^{2^k}$, we get

$$\begin{aligned} & A_{k1}^* Z_3 \left(0_{\tilde{\ell},\ell} \oplus \Gamma_k^{-1} \Omega_0^{2^k}\right) + A_{k2}^* Z_4 \left(0_{\tilde{\ell},\ell} \oplus \Gamma_k^{-1} \Omega_0^{2^k}\right) \\ & \quad = (A_{k1} Z_1 + A_{k3} Z_2) \left(0_\ell \oplus \Omega_0^{2^k}\right) + (A_{k1} Z_3 + A_{k3} Z_4) \left(0_{\tilde{\ell},\ell} \oplus \Omega_0^{2^k} \Gamma_k^{-1} \Omega_0^{2^k}\right). \end{aligned} \tag{3.15}$$

Substituting (3.15) into (3.11) and using $\Omega_0^{2^k} \Gamma_k^{-1} \Omega_0^{2^k} = 2^{-k} \Omega_0^{2^k+1}$, we have

$$\begin{aligned} A_{k1}^* Z_1 + A_{k2}^* Z_2 = {} & (A_{k1} Z_1 + A_{k3} Z_2) \left(J_0^{2^k} \oplus 0_r\right) + (A_{k1} Z_1 + A_{k3} Z_2) \left(0_\ell \oplus \Omega_0^{2^k}\right) \\ = {} & (A_{k1} Z_1 + A_{k3} Z_2) \left(J_0^{2^k} \oplus 0_r\right) + (A_{k1}^* Z_3 + A_{k2}^* Z_4) \left(0_{\tilde{\ell},\ell} \oplus \Gamma_k^{-1} \Omega_0^{2^k}\right) \\ & - (A_{k1} Z_3 + A_{k3} Z_4) \left(0_{\tilde{\ell},\ell} \oplus 2^{-k} \Omega_0^{2^k+1}\right). \end{aligned} \tag{3.16}$$

Using (3.6a) and re-arranging (3.16), we get

$$H_{k1} Z \left\{Z_1 \left[I_n - \left(J_0^{2^k} \oplus 0_r\right)\right] - Z_3 \left[0_{\tilde{\ell},\ell} \oplus 2^{-k} \Omega_0 \left(I_r - \Omega_0^{2^k}\right)\right]\right\}$$

$$+ H_{k2}^* \left\{ Z_2 \left[I_n - \left(J_0^{2^k} \oplus 0_r \right) \right] - Z_4 \left[0_{\tilde{\ell},\ell} \oplus 2^{-k} \Omega_0 \left(I_r - \Omega_0^{2^k} \right) \right] \right\}$$
$$= K_{01} \left\{ Z_1 \left[I_n + \left(J_0^{2^k} \oplus 0_r \right) \right] - Z_3 \left[0_{\tilde{\ell},\ell} \oplus 2^{-k} \Omega_0 \left(I_r + \Omega_0^{2^k} \right) \right] \right\}$$
$$- K_{02}^* \left\{ Z_2 \left[I_n + \left(J_0^{2^k} \oplus 0_r \right) \right] - Z_4 \left[0_{\tilde{\ell},\ell} \oplus 2^{-k} \Omega_0 \left(I_r + \Omega_0^{2^k} \right) \right] \right\}. \tag{3.17}$$

Denote

$$\epsilon_k \equiv \max \left\{ \rho(J_0)^{2^k}, 2^{-k} \right\} \to 0, \quad \text{as } k \to \infty. \tag{3.18}$$

Since $\left\| \Omega_0^{2^k} \right\|$ is bounded and Z_1 is invertible, by letting $\Phi \equiv Z_2 Z_1^{-1}$, (3.17) can be simplified to

$$H_{k1} = -H_{k2}^*(\Phi + O(\epsilon_k)) + K_{01} - K_{02}^* \Phi + O(\epsilon_k). \tag{3.19}$$

Post-multiplying (3.13) by $I_{\tilde{\ell}} \oplus \Gamma_k^{-1}$, we have

$$A_{k1}^* Z_3 \left((\tilde{J}_0^*)^{2^k} \oplus \Gamma_k^{-1} \right) + A_{k2}^* Z_4 \left((\tilde{J}_0^*)^{2^k} \oplus \Gamma_k^{-1} \right)$$
$$= A_{k1} \left[Z_1 (0_{\ell,\tilde{\ell}} \oplus I_r) + Z_3 \left(I_{\tilde{\ell}} \oplus \Omega_0^{2^k} \Gamma_k^{-1} \right) \right]$$
$$+ A_{k3} \left[Z_2 (0_{\ell,\tilde{\ell}} \oplus I_r) + Z_4 \left(I_{\tilde{\ell}} \oplus \Omega_0^{2^k} \Gamma_k^{-1} \right) \right]. \tag{3.20}$$

From (3.6a) and (3.18), (3.20) becomes

$$H_{k1} [Z_3 (I_\ell \oplus 0_{m+r}) + Z_1 (0_{\ell,\tilde{\ell}} \oplus I_r) + O(\epsilon_k)] + H_{k2}^* [Z_4 (I_\ell \oplus 0_{m+r}) + Z_2 (0_{\ell,\tilde{\ell}} \oplus I_r) + O(\epsilon_k)]$$
$$= -K_{01} [Z_3 (I_\ell \oplus 2I_m \oplus 0_r) + Z_1 (0_{\ell,\tilde{\ell}} \oplus I_r) + O(\epsilon_k)]$$
$$+ K_{02}^* [Z_4 (I_\ell \oplus 2I_m \oplus 0_r) + Z_2 (0_{\ell,\tilde{\ell}} \oplus I_r) + O(\epsilon_k)]. \tag{3.21}$$

Substituting (3.19) into (3.21) we get

$$H_{k2}^* \Big\{ (\Phi + O(\epsilon_k)) [Z_3 (I_\ell \oplus 0_{m+r}) + Z_1 (0_{\ell,\tilde{\ell}} \oplus I_r) + O(\epsilon_k)] - Z_4 (I_\ell \oplus 0_{m+r})$$
$$- Z_2 (0_{\ell,\tilde{\ell}} \oplus I_r) + O(\epsilon_k) \Big\} = O(1).$$

Since $\Phi Z_{1,b} = Z_{2,b}$, it holds that

$$H_{k2}^* ([\Phi Z_{3,a} - Z_{4,a}] + O(\epsilon_k)) = O(1) \in \mathbb{C}^{\tilde{n} \times \ell}. \tag{3.22}$$

On the other hand, from (3.6a), (3.5) and (3.7), we have

$$A_{k1} Y_1 + A_{k3} Y_2 = A_{k1}^* Y_1 \left((J_0^*)^{2^k} \oplus (\Omega_0^*)^{2^k} \right) + A_{k2}^* Y_2 \left((J_0^*)^{2^k} \oplus (\Omega_0^*)^{2^k} \right), \tag{3.23}$$

$$A_{k2} Y_1 + A_{k4} Y_2 = A_{k3}^* Y_1 \left((J_0^*)^{2^k} \oplus (\Omega_0^*)^{2^k} \right) + A_{k4}^* Y_2 \left((J_0^*)^{2^k} \oplus (\Omega_0^*)^{2^k} \right), \tag{3.24}$$

$$A_{k1} Y_3 \left(\tilde{J}_0^{2^k} \oplus I_r \right) + A_{k3} Y_4 \left(\tilde{J}_0^{2^k} \oplus I_r \right) = (A_{k1}^* Y_3 + A_{k2}^* Y_4) \left(I_{\tilde{\ell}} \oplus (\Omega_0^*)^{2^k} \right)$$
$$+ (A_{k1}^* Y_1 + A_{k2}^* Y_2) \left(0_{\ell,\tilde{\ell}} \oplus \Gamma_k^* \right), \tag{3.25}$$

$$A_{k2} Y_3 \left(\tilde{J}_0^{2^k} \oplus I_r \right) + A_{k4} Y_4 \left(\tilde{J}_0^{2^k} \oplus I_r \right) = (A_{k3}^* Y_3 + A_{k4}^* Y_4) \left(I_{\tilde{\ell}} \oplus (\Omega_0^*)^{2^k} \right)$$
$$+ (A_{k3}^* Y_1 + A_{k4}^* Y_2) \left(0_{\ell,\tilde{\ell}} \oplus \Gamma_k^* \right). \tag{3.26}$$

As in (3.15) and (3.17), post-multiplying (3.25) by $0_{\tilde{\ell},\ell} \oplus (\Gamma_k^*)^{-1} (\Omega_0^*)^{2^k}$ and substituting it into (3.23), we have

$$A_{k1}Y_1 + A_{k3}Y_2 = (A_{k1}^*Y_1 + A_{k2}^*Y_2)\left((J_0^*)^{2^k} \oplus 0_r\right) + (A_{k1}^*Y_3 + A_{k3}^*Y_4)\left(0_{\tilde{\ell},\ell} \oplus 2^{-k}\Omega_0^*\right) - (A_{k1}^*Y_3 + A_{k2}^*Y_4)\left(0_{\tilde{\ell},\ell} \oplus 2^{-k}(\Omega_0^*)^{2^k+1}\right). \tag{3.27}$$

From (3.6a) and (3.18), (3.27) becomes

$$\begin{aligned} &H_{k1}\left\{Y_1\left[I_n - \left((J_0^*)^{2^k} \oplus 0_r\right)\right] - Y_3\left[0_{\tilde{\ell},\ell} \oplus 2^{-k}\Omega_0^*\left(I_r - (\Omega_0^*)^{2^k}\right)\right]\right\} \\ &+ H_{k2}^*\left\{Y_2\left[I_n - \left((J_0^*)^{2^k} \oplus 0_r\right)\right] - Y_4\left[0_{\tilde{\ell},\ell} \oplus 2^{-k}\Omega_0^*\left(I_r - (\Omega_0^*)^{2^k}\right)\right]\right\} \\ &= -K_{01}\left\{Y_1\left[I_n + \left((J_0^*)^{2^k} \oplus 0_r\right)\right] - Y_3\left[0_{\tilde{\ell},\ell} \oplus 2^{-k}\Omega_0^*(I_r + (\Omega_0^*)^{2^k})\right]\right\} \\ &+ K_{02}^*\left\{Y_2\left[I_n + \left((J_0^*)^{2^k} \oplus 0_r\right)\right] - Y_4\left[0_{\tilde{\ell},\ell} \oplus 2^{-k}\Omega_0^*\left(I_r + (\Omega_0^*)^{2^k}\right)\right]\right\}, \end{aligned} \tag{3.28}$$

and then

$$H_{k1} = -H_{k2}^*(\Psi + O(\epsilon_k)) - K_{01} + K_{02}^*\Psi + O(\epsilon_k),$$

where $\Psi = Y_2Y_1^{-1}$.

Post-multiplying (3.25) by $I_{\tilde{\ell}} \oplus (\Gamma_k^*)^{-1}$ and substituting (3.28) into it, we have

$$\begin{aligned} &H_{k2}^*\left\{Y_4(I_\ell \oplus 0_{m+r}) + Y_2(0_{\ell,\tilde{\ell}} \oplus I_r) + O(\epsilon_k) - (\Psi + O(\epsilon_k))[Y_3(I_\ell \oplus 0_{m+r})\right. \\ &\left.+ Y_1(0_{\ell,\tilde{\ell}} \oplus I_r) + O(\epsilon_k)]\right\} = O(1). \end{aligned}$$

Since $\Psi Y_{1,b} = Y_{2,b}$, it holds that

$$H_{k2}^*([\Psi Y_{3,a} - Y_{4,a}] + O(\epsilon_k)) = O(1) \in \mathbb{C}^{\tilde{n}\times\ell}. \tag{3.29}$$

Combining (3.22) and (3.29) we get

$$H_{k2}^*([\Phi Z_{3,a} - Z_{4,a}|\Psi Y_{3,a} - Y_{4,a}] + O(\epsilon_k)) = O(1) \in \mathbb{C}^{\tilde{n}\times 2\ell}. \tag{3.30}$$

By the assumption that $W \equiv [\Phi Z_{3,a} - Z_{4,a}|\Psi Y_{3,a} - Y_{4,a}] \in \mathbb{C}^{\tilde{n}\times 2\ell}$ is of full row rank, it follows that $\|H_{k2}^*\|$ is uniformly bounded on k. Consequently, (3.19) implies that $\|H_{k1}\|$, and in turn $\|A_{k1}\|$ and $\|A_{k2}^*\|$, are uniformly bounded on k. From (3.16), it follows that

$$A_{k1}^*Z_1 + A_{k2}^*Z_2 = O(\epsilon_k) \to 0, \quad \text{as } k \to \infty. \tag{3.31}$$

Applying the similar argument as in (3.15) and (3.17) to (3.24) and (3.26), we deduce that

$$H_{k4} = -H_{k2}\left(Y_2Y_1^{-1} + O(\epsilon_k)\right) + K_{04} - K_{02}Y_2Y_1^{-1} + O(\epsilon_k).$$

Thus, (3.30) implies that $\|H_{k4}\|$, and in turn $\|A_{k4}\|$, are uniformly bounded on k.

To show $A_{k3}^*Z_1 + A_{k4}^*Z_2 = O(\epsilon_k)$, we post-multiply (3.14) by $0_{\tilde{\ell},\ell} \oplus \Gamma_k^{-1}\Omega_0^{2^k}$ and obtain

$$\begin{aligned} &A_{k3}^*Z_3\left(0_{\tilde{\ell},\ell} \oplus \Gamma_k^{-1}\Omega_0^{2^k}\right) + A_{k4}^*Z_4\left(0_{\tilde{\ell},\ell} \oplus \Gamma_k^{-1}\Omega_0^{2^k}\right) \\ &= (A_{k2}Z_1 + A_{k4}Z_2)\left(0_\ell \oplus \Omega_0^{2^k}\right) + (A_{k2}Z_3 + A_{k4}Z_4)\left(0_{\tilde{\ell},\ell} \oplus 2^{-k}\Omega_0^{2^k+1}\right). \end{aligned} \tag{3.32}$$

Substituting (3.32) into (3.12), as in (3.16) we have

$$\begin{aligned} A_{k3}^*Z_1 + A_{k4}^*Z_2 = &\ (A_{k2}Z_1 + A_{k4}Z_2)\left(J_0^{2^k} \oplus 0_r\right) + (A_{k3}^*Z_3 + A_{k4}^*Z_4)\left(0_{\tilde{\ell},\ell} \oplus 2^{-k}\Omega_0\right) \\ &- (A_{k2}Z_3 + A_{k4}Z_4)\left(0_{\tilde{\ell},\ell} \oplus 2^{-k}\Omega_0^{2^k+1}\right) \\ = &\ O(\epsilon_k) \to 0, \quad \text{as} \to \infty. \end{aligned} \tag{3.33}$$

Combining (3.31) and (3.33), we have shown that

$$\begin{bmatrix} A_{k1}^* & A_{k2}^* \\ A_{k3}^* & A_{k4}^* \end{bmatrix}\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = O(\epsilon_k) \to 0, \quad \text{as } k \to \infty.$$

Similarly, as in (3.15) and (3.16), from (3.24) and (3.26) we have

$$\begin{aligned} A_{k2}Y_1 + A_{k4}Y_2 = &\ (A_{k3}^*Y_1 + A_{k4}^*Y_2)\left((J_0^*)^{2^k} \oplus 0_r\right) + (A_{k2}Y_3 + A_{k4}Y_4)\left(0_{\tilde{\ell},\ell} \oplus 2^{-k}\Omega_0^*\right) \\ &- (A_{k3}^*Y_3 + A_{k4}^*Y_4)\left(0_{\tilde{\ell},\ell} \oplus 2^{-k}\left(\Omega_0^*\right)^{2^k+1}\right) = O(\epsilon_k). \end{aligned} \tag{3.34}$$

Using the boundedness of $\|A_{ki}\|$, $i = 1, \ldots, 4$, and combining (3.27) and (3.34), we have shown that

$$\begin{bmatrix} A_{k1} & A_{k3} \\ A_{k2} & A_{k4} \end{bmatrix}\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = O(\epsilon_k) \to 0, \quad \text{as } k \to \infty,$$

because $\frac{1}{2^k}$ dominates $\rho(J_0)^{2^k}$ in (3.18) for sufficiently large values of k. □

Remark 3.1. Consider the assumption $-1 \notin U \equiv \bigcup_{j=1}^r \left\{e^{2^k i\omega_j}, k \geqslant 0\right\}$ in Theorem 3.1. Since U is a countable set (possibly dense on the unit circle only when $r \to \infty$), there exist an $-e^{i\theta_0} \notin U$. With $A_{\text{new}}^* \equiv e^{-i\theta_0/2}A_0^*$, we have $A_{\text{new}}^* + A_{\text{new}} = e^{i\theta_0/2}A_0 + e^{-i\theta_0/2}A_0 = e^{-i\theta_0/2}\left(A_0^* + e^{i\theta_0}A_0\right)$ being invertible. It is unclear how the "optimal" θ_0 can be found.

Theorem 3.2. *Suppose (A_0^*, A_0) has no unimodular eigenvalues. The sequence $\{(A_k^*, A_k)\}$ generated by the PDA satisfies*

$$A_k^*\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} \to 0, \quad A_k\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \to 0,$$

quadratically, as $k \to \infty$, with convergence rate $\rho(J_0)$.

Proof. Since (A_0^*, A_0) has no unimodular eigenvalues, Theorem 3.1 implies (A_k^*, A_k) has no unimodular eigenvalues and $(A_k^* + A_k)$ is invertible. So, the PDA is well-defined.

From (3.6a), (3.3) and (3.7), we have

$$A_{k1}^*Z_1 + A_{k2}^*Z_2 = A_{k1}Z_1J_0^{2^k} + A_{k3}Z_2J_0^{2^k}, \tag{3.35}$$

$$A_{k3}^*Z_1 + A_{k4}^*Z_2 = A_{k2}Z_1J_0^{2^k} + A_{k4}Z_2J_0^{2^k}, \tag{3.36}$$

From (3.6a), it holds that

$$H_{k1}Z_1 + H_{k2}^*Z_2 = (K_{01}Z_1 - K_{01}^*Z_2)\left(I + J_0^{2^k}\right)\left(I - J_0^{2^k}\right)^{-1}.$$

Therefore,

$$\|H_{k1}Z_1 + H_{k2}^*Z_2\| \leqslant O(1).$$

This implies

$$\|(H_{k1} + K_{01})Z_1 + (H_{k2}^* - K_{02}^*)\, Z_2\| = \|A_{k1}Z_1 + A_{k3}Z_2\| \leqslant O(1). \tag{3.37}$$

From (3.35) and (3.37), we have

$$A_{k1}^* Z_1 + A_{k2}^* Z_2 = O\left(\rho(J_0)^{2^k}\right) \to 0, \quad \text{as } k \to \infty.$$

Similarly, from (3.36), we obtain

$$H_{k2}Z_1 + H_{k4}^* Z_2 = (K_{02}Z_1 + K_{04}^* Z_2)\left(I + J_0^{2^k}\right)\left(I - J_0^{2^k}\right)^{-1},$$

which is uniformly bounded on k. This implies

$$A_{k3}^* Z_1 + A_{k4}^* Z_2 = O\left(\rho(J_0)^{2^k}\right) \to 0, \quad \text{as } k \to \infty.$$

This shows that $A_k^*\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} \to 0$, quadratically, with convergence rate $\rho(J_0)$. Similarly, from (3.6a), (3.5) and (3.7), we can also show that $A_k\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \to 0$ quadratically, with rate $\rho(J_0)$. $\square$

4. Numerical solution and applications

In this section, we want to apply the PDA to find all the eigenpairs of a general PGEP, and solve the c-/d-stabilizing solutions of generalized continuous/discrete-time algebraic Riccati equations (GCARE/GDARE). We especially develop Algorithm 4.1 in subsection 4.3 for the computation of the d-semi-stabilizing solution of GDAREs arising in the optimal control of singular descriptor linear systems. To our knowledge, Algorithm 4.1 is the first structure-preserving algorithm for solving GDAREs associated with singular descriptor systems.

For operation counts or complexity, it depends on the details in the individual applications and whether efficiency can be squeezed from these fine structures. From the PDA, it is suffice to say that the algorithm is of $O(N^3)$ complexity per iteration. In addition, for problems without unimodular eigenvalues, the convergence is quadratic and typically less than ten iterations are required for convergence to machine accuracy.

4.1. PGEP

In this subsection, we apply the PDA to solve the PGEP $A_0^* x = \lambda A_0 x$, where $A_0 \in \mathbb{C}^{2n\times 2n}$. First, we apply the PDA to A_0 until convergence to A_k. Then we compute the bases $Z_s, Y_s \in \mathbb{C}^{2n\times n}$ for the right and left null spaces of A_k^*, respectively, satisfying

$$A_k^* Z_s = 0, \quad Y_s^* A_k^* = 0.$$

This implies that there are S and $T \in \mathbb{C}^{n\times n}$ with $\rho(S) \leqslant 1$ and $\rho(T) \leqslant 1$ such that

$$A_0^* Z_s = A_0 Z_s S, \quad A_0 Y_s = A_0^* Y_s T. \tag{4.1}$$

From (4.1), S and T can be computed by

$$S = (Y_s^* A_0 Z_s)^{-1}\,(Y_s^* A_0^* Z_s) \equiv S_1^{-1} S_2,$$
$$T = (Z_s^* A_0^* Y_s)^{-1}\,(Z_s^* A_0 Y_s) \equiv S_1^{-*} S_2^*.$$

Rewrite the second equation of (4.1) as

$$A_0\left(Y_s S_1^{-*}\right) = A_0^*\left(Y_s S_1^{-*}\right) S_2^* S_1^{-*} = A_0^*\left(Y_s S_1^{-*}\right) S^*.$$

Compute $Sg_j = \lambda_j g_j$ and $S^* h_j = \lambda_j^* h_j$, as well as $z_j = Z_s g_j$ and $y_j = \left(Y_s S_1^{-*}\right) h_j$, for $j = 1, \ldots, n$. It holds that

Table 4.1
Results for Example 1.

ITs	20	21	22	23	24
Err_k	1.26e-6	6.29e-7	3.15e-7	1.58e-7	8.01e-8

$$A_0^* z_j = \lambda_j A_0 z_j, \quad \lambda_j^* A_0^* y_j = A_0 y_j, \quad j = 1, \ldots, n.$$

In the following example, we report the numerical results of the PDA to illustrate the linear convergence in the critical case. Recall that Theorem 3.1 shows the PDA converges linearly with rate $1/2$ when all unimodular eigenvalues of (A_0^*, A_0) have partial multiplicities two.

Example 4.1. Given $\alpha = \cos(\theta)$ and $\beta = \sin(\theta)$ with $\theta = 0.62$. Let

$$J_0 = \begin{bmatrix} 0_2 & \Gamma \\ I_2 & I_2 \end{bmatrix}, \quad J_s = \begin{bmatrix} 0_3 & \text{diag}(\lambda_1, \lambda_2, \lambda_3) \\ I_3 & 0_3 \end{bmatrix},$$

where $\Gamma = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$, and $|\lambda_i| < 1$ for $i = 1, 2, 3$. We construct

$$A_0 = Q^*(J_0 \oplus J_s)Q,$$

where Q is an unitary matrix. It is easily seen that (A_0^*, A_0) has eigenvalues $\{\alpha + \imath\beta, \alpha - \imath\beta, \lambda_1, \lambda_2, \lambda_3, 1/\lambda_1^*, 1/\lambda_2^*, 1/\lambda_3^*\}$ with partial multiplicities $\{2, 2, 1, 1, 1, 1, 1, 1\}$ which satisfy the assumptions in Theorem 3.1. The kth absolute error as in (3.10a) computed by the PDA is defined by

$$\text{Err}_k \equiv \|A_k^* Z_{s,k}\|,$$

where $Z_{s,k}$ is an orthogonal basis corresponding to the five smallest singular values of A_k.

In Table 4.1, we list the absolute errors from the 20th to 24th iterations computed by the PDA which is observed to be linearly convergent with rate $1/2$. Here, the tolerance τ in the PDA is chosen to be the optimal $\sqrt{1e-16} = 1e-8$, because the unimodular eigenvalues of (A_0^*, A_0) have partial multiplies two. Furthermore, the residual $\|A_0^* Z_s - A_0 Z_s \Lambda_s\|$ is given by $8.07e-8$, where $Z_s \equiv Z_{s,24}$ and Λ_s is the corresponding approximate stable eigenvalue matrix.

4.2. GCARE

In this subsection, we are interested in finding the c-stabilizing solution of the generalized continuous-time algebraic Riccati equation (GCARE)

$$A_c^\top X_c E_c + E_c^\top X_c A_c - \left(N_c + E_c^\top X_c B_c\right) R_c^{-1} \left(N_c + E_c^\top X_c B_c\right)^\top + M_c = 0, \tag{4.2}$$

which solves the continuous-time linear-quadratic control problem

$$\min_u \frac{1}{2} \int_0^\infty \begin{bmatrix} x \\ u \end{bmatrix}^\top \begin{bmatrix} M_c & N_c \\ N_c^\top & R_c \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} dt \tag{4.3a}$$

subject to the descriptor linear system

$$E_c \dot{x} = A_c x + B_c u, \quad x(0) = x^0, \tag{4.3b}$$

where $E_c, A_c, M_c = M_c^\top, X_c = X_c^\top \in \mathbb{R}^{n\times n}, B_c, N_c \in \mathbb{R}^{n\times m}$ and $R_c = R_c^\top \in \mathbb{R}^{m\times m}$ with E_c and R_c being nonsingular. Furthermore, the c-stabilizing closed-loop matrix pencil of (4.3b) is given by $A_c + B_c K_c - \lambda E_c$ with the $\sigma(A_c + B_c K_c, E_c) \subseteq \mathbb{C}_-$, where

$$K_c \equiv -R_c^{-1}\left(B_c^\top X_c E_c + N_c^\top\right).$$

Let

$$\mathcal{M}_c - \lambda \mathcal{L}_c \equiv \begin{bmatrix} 0 & A_c & B_c \\ A_c^\top & M_c & N_c \\ B_c^\top & N_c^\top & R_c \end{bmatrix} - \lambda \begin{bmatrix} 0 & E_c & 0 \\ -E_c^\top & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{4.4}$$

One common approach to solve (4.2) is to compute the n-dimensional, c-stable invariant subspace $\mathcal{U}_c$ of the symmetric/skew-symmetric pencil $\mathcal{M}_c - \lambda\mathcal{L}_c$ corresponding to the eigenvalue matrix pair (S_c, E_c) with $\sigma(S_c, E_c) \subseteq \mathbb{C}_-$, where $\mathcal{U}_c$ is the column space of $U_c \in \mathbb{R}^{(2n+m)\times n}$ which satisfies $\mathcal{M}_c U_c E_c = \mathcal{L}_c U_c S_c$.

We assume that the matrix pencil $\mathcal{M}_c - \lambda\mathcal{L}_c$ has no eigenvalues on the imaginary axis. The generalized eigenvalues of $(\mathcal{M}_c, \mathcal{L}_c)$ can be arranged by

$$\lambda_1, \ldots, \lambda_{2n}; \bar{\lambda}_1, \ldots, \bar{\lambda}_{2n}; \underbrace{\infty, \ldots, \infty}_{m},$$

where $\lambda_i \in \mathbb{C}_-$, for $1 \leqslant i \leqslant 2n$. The m trivial infinity eigenvalues are from the nonsingularity of R_c.

With

$$U_c = \begin{bmatrix} X_c E_c \\ I_n \\ K_c \end{bmatrix} \begin{matrix} \}n \\ \}n, \\ \}m \end{matrix}$$

X_c is the c-stabilizing solution of GCARE (4.2) and K_c is the optimal controller for (4.3b) [11].

In order to utilize the PDA to compute an orthogonal basis $V = \left[V_1^\top,\ V_2^\top,\ V_3^\top\right]^\top$ for $\mathcal{U}_c$ with $V_1, V_2 \in \mathbb{R}^{n\times n}$, we consider the Cayley transformation

$$\mathcal{A}_0^\top - \lambda\mathcal{A}_0 = (\mathcal{M}_c + \mathcal{L}_c) - \lambda(\mathcal{M}_c - \mathcal{L}_c),$$

where

$$\mathcal{A}_0 = \mathcal{M}_c - \mathcal{L}_c = \begin{bmatrix} 0 & A_c - E_c & B_c \\ A_c^\top + E_c^\top & M_c & N_c \\ B_c^\top & N_c^\top & R_c \end{bmatrix}. \tag{4.5}$$

Then the c-stabilizing solution X_c for GCARE (4.2) can be obtained by $X_c = V_1 V_2^{-1} E_c^{-1}$.

To measure the accuracy of the computed solution $\widetilde{X}_c$ for the GCARE, we use the "normalized" residual (NR_c)

$$\mathrm{NR}_c \equiv \frac{\left\| A_c^\top \widetilde{X}_c E_c + E_c^\top \widetilde{X}_c A_c - \left(N_c + E_c^\top \widetilde{X}_c B_c\right) R_c^{-1} \left(N_c + E_c^\top \widetilde{X}_c B_c\right)^\top + M_c \right\|}{\|A_c^\top \widetilde{X}_c E_c\| + \|E_c^\top \widetilde{X}_c A_c\| + \|(N_c + E_c^\top \widetilde{X}_c B_c) R_c^{-1} \left(N_c + E_c^\top \widetilde{X}_c B_c\right)^\top\| + \|M_c\|}.$$

In the following example, we compare the residuals NR_c of $\widetilde{X}_c$ computed by `care` in MATLAB, Newton's method (NTM) [1] and the PDA.

Example 4.2 (Example 4.3 of [1]). Let $n = m = 8$. Given

$$A_c = \mathrm{diag}\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}\right), \quad R_c = I_8, \quad E_c = I_8,$$
$$G_c \equiv B_c R_c^{-1} B_c^\top = \mathrm{trid}(1, 2, 1) + e_8 e_1^\top + e_1 e_8^\top,$$
$$M_c = 0_8, \quad N_c = 0_8,$$

where trid(1, 2, 1) is a 8×8 tridiagonal matrix with the sub-, main- and super-diagonal elements being 1, 2 and 1, respectively.

It is readily seen that $X_c = 0$ and $\sigma(A_c - G_c X_c) = \{-1, 0, \pm i, \pm 2i\}$ with purely imaginary eigenvalues having linear elementary divisors. We apply the NTM method to GCARE (4.2) with $X_0 = I_8$, and apply the PDA to

$$\mathcal{A}_0 = \begin{bmatrix} G_c & A_c - E_c \\ A_c^\top + E_c^\top & M_c \end{bmatrix},$$

which is a degenrate form of (4.5) with $N_c = 0$. The tolerance τ in the NTM and the PDA is chosen to be 10^{-10}. The numerical results are given in Table 4.2.

Table 4.2
Results for Example 3.

	care	NTM	PDA
NR_c	$*$	5.25×10^{-10}	6.61×10^{-10}
Iter. no.	-	10	27

From Table 4.2, `care` in MATLAB dose not work because of the existence of the purely imaginary eigenvalues. We see that the NTM and the PDA almost have the same accuracy. Both methods have linear convergence rate 1/2, but the PDA requires much more iterative steps. However, the PDA only needs to compute a LU-factorization in each step, and NTM is accelerated by some modified technique [1] which needs to solve a more expensive Sylvester equation in each step.

4.3. GDARE

In this subsection, we are interested in finding the d-semi-stabilizing solution of the generalized discrete-time algebraic Riccati equation (GDARE)

$$A_d^\top X_d A_d - E_d^\top X_d E_d - \left(N_d + A_d^\top X_d B_d\right)\left(R_d + B_d^\top X_d B_d\right)^{-1}\left(N_d + A_d^\top X_d B_d\right)^\top + M_d = 0, \tag{4.6}$$

which solves the discrete-time linear-quadratic control problem

$$\min_{u_k} \frac{1}{2}\sum_{k=0}^{\infty} \begin{bmatrix} x_k \\ u_k \end{bmatrix}^\top \begin{bmatrix} M_d & N_d \\ N_d^\top & R_d \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix} \tag{4.7a}$$

subject to the singular descriptor linear system

$$E_d x_{k+1} = A_d x_k + B_d u_k, \quad x_0 = x^0, \tag{4.7b}$$

where $E_d, A_d, M_d = M_d^\top, X_d = X_d^\top \in \mathbb{R}^{n\times n}, B_d, N_d \in \mathbb{R}^{n\times m}$ and $R_d = R_d^\top \in \mathbb{R}^{m\times m}$ with E_d being singular. Furthermore, the d-semi-stabilizing closed-loop matrix pencil of (4.7b) is given by $A_d + B_d K_d - \lambda E_d$ with the $\sigma(A_d + B_d K_d, E_d) \subseteq \mathbb{D}_1 \cup \{\infty\}$, where

$$K_d \equiv -\left(R_d + B_d^\top X_d B_d\right)^{-1}\left(B_d^\top X_d A_d + N_d^\top\right).$$

Let

$$\mathcal{M}_d - \lambda \mathcal{L}_d \equiv \begin{bmatrix} 0 & A_d & B_d \\ E_d^\top & M_d & N_d \\ 0 & N_d^\top & R_d \end{bmatrix} - \lambda \begin{bmatrix} 0 & E_d & 0 \\ A_d^\top & 0 & 0 \\ B_d^\top & 0 & 0 \end{bmatrix}.$$

One common approach to solve (4.6) is to compute the n-dimensional, d-semi-stable invariant subspace $\mathcal{U}_d$ of the matrix pencil $\mathcal{M}_d - \lambda\mathcal{L}_d$ corresponding to the eigenvalue matrix pair (S_d, E_d) with $\sigma(S_d, E_d) \subseteq \mathbb{D}_1 \cup \{\infty\}$, where $\mathcal{U}_d$ is the column space of $U_d \in \mathbb{R}^{(2n+m)\times n}$ which satisfies $\mathcal{M}_d U_d E_d = \mathcal{L}_d U_d S_d$.

With

$$U_d = \begin{bmatrix} X_d E_d \\ I_n \\ K_d \end{bmatrix} \begin{matrix} \}n \\ \}n, \\ \}m \end{matrix}$$

X_d is the d-semi-stabilizing solution of GDARE (4.6) and K_d is the optimal controller for (4.7b) [11].

Assume that the matrix pencil $\mathcal{M}_d - \lambda\mathcal{L}_d$ has no eigenvalues on the unit circle, r_d =nullity (E_d) and $\text{ind}_\infty(A_d, E_d) \leqslant 1$. From [11] we see that

$$\sigma(\mathcal{M}_d, \mathcal{L}_d) = \sigma(A_d + B_d K_d, E_d) \cup \sigma\left(E_d^\top, (A_d + B_d K_d)^\top\right) \cup \sigma(0_m, I_m). \tag{4.8}$$

So, the generalized eigenvalues of $(\mathcal{M}_d, \mathcal{L}_d)$ corresponding to (4.8) can be arranged by

Table 4.3
Correspondence among λ_d, μ and λ.

λ_d	$0 < \|\lambda_d\| < 1$	$\|\lambda_d\| > 1$	0	∞	m trivial ∞
μ	$\mathrm{Re}(\mu) < 0$	$\mathrm{Re}(\mu) > 0$	-1	1	m trivial ∞
λ	$\lambda = \lambda_d$	$\lambda = \lambda_d$	0	∞	m trivial -1

$$\{\lambda_1, \ldots, \lambda_{n-r_d}, \underbrace{\infty, \ldots, \infty}_{r_d}\} \cup \left\{\underbrace{0, \ldots, 0}_{r_d}, \lambda_1^{-1}, \ldots, \lambda_{n-r_d}^{-1}\right\} \cup \{\underbrace{\infty, \ldots, \infty}_{m}\}, \tag{4.9}$$

where $\lambda_i \in \mathbb{D}_1$ (can possibly be zero) , $i = 1, \ldots, n - r_d$. For convenience, we apply the convention that 0 and ∞ are mutually reciprocal. The r_d infinity and r_d zero eigenvalues in (4.9) are from the assumption $r_d = \mathrm{nullity}(E_d)$. The last trivial m infinity eigenvalues are from the last m columns of $\mathcal{L}_d$. In fact, $(A_d + B_dK_d, E_d)$ is an eigenvalue matrix pair associated with the d-semi-stable invariant subspace $\mathcal{U}_d$.

We now introduce an elegant transformation between the coefficient matrices of the GDARE (4.6) and GCARE (4.2) proposed by [11]. We define

$$f_W : (E_d, A_d, B_d, M_d, N_d, R_d) \to (E_c, A_c, B_c, M_c, N_c, R_c),$$

where the matrices $E_c, A_c, B_c, M_c, N_c, R_c$ satisfy

$$\begin{bmatrix} E_c & 0 \\ A_c & B_c \end{bmatrix} = \chi \begin{bmatrix} E_d & 0 \\ A_d & B_d \end{bmatrix} W_d^\top = \frac{1}{\sqrt{2}} \begin{bmatrix} A_d + E_d & B_d \\ A_d - E_d & B_d \end{bmatrix} W_d^\top, \tag{4.10a}$$

$$\begin{bmatrix} M_c & N_c \\ N_c^\top & R_c \end{bmatrix} = W_d \begin{bmatrix} M_d & N_d \\ N_d^\top & R_d \end{bmatrix} W_d^\top, \tag{4.10b}$$

in which $\chi = \frac{1}{\sqrt{2}}\begin{bmatrix} I_n & I_n \\ -I_n & I_n \end{bmatrix}$, and $[A_d + E_d \quad B_d] = [H \quad 0]W_d$ is the QR-factorization with W_d being orthogonal and H being lower triangular.

By the important property of f_W in [11], it is assumed that $\mathrm{rank}(A_d + E_d \quad B_d) = n$ and $(\mathcal{M}_d, \mathcal{L}_d)$ has no eigenvalue "-1". Thus, the coefficient matrix tuple $(E_c, A_c, B_c, M_c, N_c, R_c)$ corresponds to a GCARE (4.2) with E_c and R_c being nonsingular. Furthermore, the GDARE (4.6) and the GCARE (4.2) share the same stabilizing solutions, i.e., $X_d = X_c$.

We construct $(\mathcal{M}_c, \mathcal{L}_c)$ by $(E_c, A_c, B_c, M_c, N_c, R_c)$ as in (4.4) which satisfies

$$\mathcal{M}_c + \mathcal{L}_c = W^{-1}\mathcal{M}_d W, \tag{4.11}$$

where $W \equiv \mathrm{diag}(\sqrt{2}I_n, W_d^\top)$. Let

$$(\mathcal{A}_0^\top, \mathcal{A}_0) = (\mathcal{M}_c + \mathcal{L}_c, \mathcal{M}_c - \mathcal{L}_c) \tag{4.12}$$

be the Cayley transformation of $(\mathcal{M}_c, \mathcal{L}_c)$. From (4.10b) and (4.11), we see that the eigenvalues $\lambda_d \in \sigma(\mathcal{M}_d, \mathcal{L}_d)$, $\mu \in \sigma(\mathcal{M}_c, \mathcal{L}_c)$ and $\lambda \in \sigma(\mathcal{A}_0^\top, \mathcal{A}_0)$ satisfy the relationship in Table 4.3, in which $\mu = (\lambda - 1)(\lambda + 1)^{-1}$. From Table 4.3, we see that the key property of the transformation $\lambda_d \to \lambda$ is to transform m trivial infinity eigenvalues to m trivial -1 while preserving other eigenvalues (including nontrivial ∞) unchanged.

In the following, we use the PDA and the special structure of $(\mathcal{M}_d, \mathcal{L}_d)$ for the computation of the d-semi-stabilizing solution X_d of GDARE (4.6).

Firstly, we apply the PDA to the matrix $\mathcal{A}_0$ until convergence to $\mathcal{A}_k$. Then we compute the orthogonal bases $\mathcal{N}_r$ and $\mathcal{N}_\ell \in \mathbb{R}^{(2n+m)\times n}$ for the right and left null spaces of $\mathcal{A}_k^\top$; i.e.,

$$\mathcal{A}_k^\top \mathcal{N}_r = 0, \quad \mathcal{A}_k \mathcal{N}_\ell = 0, \tag{4.13}$$

which form orthogonal bases for the d-stable invariant subspaces of $\left(\mathcal{A}_0^\top, \mathcal{A}_0\right)$ and $\left(\mathcal{A}_0, \mathcal{A}_0^\top\right)$, respectively.

We then compute the QR-factorization $\mathcal{A}_0\mathcal{N}_r = Q_1R_1$, where Q_1 is orthogonal and R_1 is upper triangular. Next compute

$$S = Q_s^\top \mathcal{A}_0^\top \mathcal{N}_r, \;\; T = Q_s^\top \mathcal{A}_0\mathcal{N}_r, \tag{4.14}$$

where $Q_s = Q_1(:, 1:n)$. We see that (S, T) forms the d-stable eigenvalue matrix pair of $\left(\mathcal{A}_0^\top, \mathcal{A}_0\right)$ associated with $\mathcal{N}_r$, and T is clearly nonsingular.

We would like to separate the invariant subspaces of $\left(\mathcal{A}_0^\top, \mathcal{A}_0\right)$ corresponding to the zero and nonzero d-stable eigenvalues. Let $G = T^{-1}S$. By Van Dooren's algorithm [10], there is an orthogonal matrix $\Phi \in \mathbb{R}^{n\times n}$ such that

$$\Phi G \Phi^\top = \begin{bmatrix} G_{11} & G_{12} \\ 0 & G_{22} \end{bmatrix},$$

where $G_{11} \in \mathbb{R}^{s\times s}$ with $\sigma(G_{11}) = \left\{\lambda \in \sigma\left(\mathcal{A}_0^\top, \mathcal{A}_0\right) \mid 0 < |\lambda| < 1\right\}$ and $G_{22} \in \mathbb{R}^{(n-s)\times(n-s)}$ with $\sigma(G_{22}) = \{0\}$. Since $\sigma(G_{11}) \bigcap \sigma(G_{22}) = \phi$, there is a $\Psi = \begin{bmatrix} I_s & \Psi_{12} \\ 0 & I_{n-s} \end{bmatrix}$ such that

$$\Psi^{-1}\Phi^\top G \Phi \Psi = \begin{bmatrix} G_{11} & 0 \\ 0 & G_{22} \end{bmatrix},$$

where Ψ_{12} solves the Sylvester equation $G_{11}\Psi_{12} - \Psi_{12}G_{22} = G_{12}$ uniquely. Then

$$V_0 = \mathcal{N}_r\Phi\Psi(:, s+1:n), \quad \widehat{V}_s = \mathcal{N}_r\Phi\Psi(:, 1:s) \tag{4.15}$$

span the invariant subspaces of $(\mathcal{A}_0^\top, \mathcal{A}_0)$ corresponding to the zero and nonzero d-stable eigenvalues, respectively.

Let $\hat{\zeta}_\ell$ spans the left null space of E_d. Then $\zeta_\ell = \left[\hat{\zeta}_\ell^\top, 0\right]^\top \in \mathbb{R}^{(2n+m)\times r_d}$ contains the r_d eigenvectors of $(\mathcal{M}_d, \mathcal{L}_d)$ corresponding to the trivial zeros. From the transformation (4.11), we see that $W^{-1}\zeta_\ell$ contains the r_d eigenvectors of $\left(\mathcal{A}_0^\top, \mathcal{A}_0\right)$ corresponding to trivial zeros. Now, we want to extract $W^{-1}\zeta_\ell$ from span$\{V_0\}$.

Compute the QR-factorization $\begin{bmatrix} W^{-1}\zeta_\ell & V_0 \end{bmatrix} = Q_0R_0$, where Q_0 is orthogonal and R_0 is upper triangular. Let

$$\widehat{V}_0 = Q_0(:, r_d+1:n-s), \tag{4.16}$$

which forms the eigenvectors of $\left(\mathcal{A}_0^\top, \mathcal{A}_0\right)$ corresponding to zero eigenvalues of (S_d, E_d).

We will next find the invariant space U_∞ of $\left(\mathcal{A}_0^\top, \mathcal{A}_0\right)$ corresponding to the infinity eigenvalues. Compute the QR-factorization $\mathcal{A}_0\mathcal{N}_\ell = Q_\infty R_\infty$, where Q_∞ is orthogonal and R_∞ is upper triangular. Let

$$\mathcal{N}_\infty = \mathcal{N}_\ell Q_\infty(:, s+1:n) \equiv \begin{bmatrix} \mathcal{N}_{\infty,1} \\ \mathcal{N}_{\infty,2} \\ \mathcal{N}_{\infty,3} \end{bmatrix} \begin{matrix} \}n \\ \}n, \\ \}m \end{matrix} \quad V_s = \begin{bmatrix} \widehat{V}_s & \widehat{V}_0 \end{bmatrix} \equiv \begin{bmatrix} V_{s1} \\ V_{s2} \\ V_{s3} \end{bmatrix} \begin{matrix} \}n \\ \}n. \\ \}m \end{matrix}$$

From the Cayley transform, there is a full rank matrix $Z \in \mathbb{R}^{(n-s)\times r_d}$ so that

$$V = \begin{bmatrix} V_{s1} & \mathcal{N}_{\infty,1}Z \\ V_{s2} & \mathcal{N}_{\infty,2}Z \\ V_{s3} & \mathcal{N}_{\infty,3}Z \end{bmatrix} \equiv \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \tag{4.17}$$

is a basis of an invariant subspace of $\left(\mathcal{A}_0^\top, \mathcal{A}_0\right)$, satisfying Span$\{V\} = $ Span$\left\{\begin{bmatrix} X_cE_c \\ I_n \\ K_c \end{bmatrix}\right\}$.

To determine Z, (4.17) and the fact $X_c = X_c^\top$ imply

$$\begin{bmatrix} V_{s2}^\top \\ Z^\top \mathcal{N}_{\infty,2}^\top \end{bmatrix} E_c^\top \begin{bmatrix} V_{s1} & \mathcal{N}_{\infty,1}Z \end{bmatrix} = \begin{bmatrix} V_{s1}^\top \\ Z^\top \mathcal{N}_{\infty,1}^\top \end{bmatrix} E_c \begin{bmatrix} V_{s2} & \mathcal{N}_{\infty,2}Z \end{bmatrix}.$$

That is,

$$V_{s2}^\top E_c^\top V_{s1} = V_{s1}^\top E_c V_{s2}, \tag{4.18a}$$

$$Z^\top \mathcal{N}_{\infty,2}^\top E_c^\top V_{s1} = Z^\top \mathcal{N}_{\infty,1}^\top E_c V_{s2}, \tag{4.18b}$$

$$V_{s2}^\top E_c^\top \mathcal{N}_{\infty,1} Z = V_{s1}^\top E_c \mathcal{N}_{\infty,2} Z, \tag{4.18c}$$

$$Z^\top \mathcal{N}_{\infty,2}^\top E_c^\top \mathcal{N}_{\infty,1} Z = Z^\top \mathcal{N}_{\infty,1}^\top E_c \mathcal{N}_{\infty,2} Z. \tag{4.18d}$$

Since E_c is nonsingular, from the isotropic property of $\begin{bmatrix} V_{s1} \\ V_{s2} \end{bmatrix}$ and $\begin{bmatrix} N_{\infty 1} \\ N_{\infty 2} \end{bmatrix}$, (4.18a) and (4.18d) hold automatically. Since (4.18b) is the transpose of (4.18c), the matrix Z is solved by finding the basis of Null $\left(V_{s2}^\top E_c^\top \mathcal{N}_{\infty,1} - V_{s1}^\top E_c \mathcal{N}_{\infty,2} \right)$.

Finally, we have the d-semi-stabilizing solution X_d for GDARE (4.6) can be obtained by

$$X_d = X_c = V_1 V_2^{-1} E_c^{-1}. \tag{4.19}$$

We summarize the computational steps (4.12)–(4.19) for X_d in Algorithm 4.1.

Algorithm 4.1 (for GDARE (4.6)).

Input: *$E_d, A_d, B_d, M_d, N_d, R_d$; τ (a small tolerance);*
Output: *The d-semi-stabilizing solution X_d of (4.6).*

1. *Construct $\mathcal{A}_0$ via (4.12).*
2. *Apply PDA to $\left(\mathcal{A}_0^\top, \mathcal{A}_0\right)$ until* dist $(Null(\mathcal{A}_k), Null(\mathcal{A}_{k-1})) < \tau$.
3. *Compute bases $\mathcal{N}_r, \mathcal{N}_\ell$ for the right and left null spaces of $\mathcal{A}_k^\top$ as in (4.13).*
4. *Compute bases V_0, $\widehat{V}_s$ for d-stable invariant subspaces of $\left(\mathcal{A}_0^\top, \mathcal{A}_0\right)$ as in (4.15).*
5. *Compute eigenvectors $\widehat{V}_0$ of $\left(\mathcal{A}_0^\top, \mathcal{A}_0\right)$ corresponding to zeros as in (4.16).*
6. *Determine Z by (4.18c).*
7. *Compute $X_d = V_1 V_2^{-1} E_c^{-1}$ as in (4.19).*

In the following, we apply Algorithm 4.1 for a discrete-time descriptor system with E_d being singular. To measure the accuracy of the computed solution $\widetilde{X}_d$ for the GDARE, we use the "normalized" residual:

$$\mathrm{NR}_d \equiv \frac{\left\| A_d^\top \widetilde{X}_d A_d - E_d^\top \widetilde{X}_d E_d - \left(N_d + A_d^\top \widetilde{X}_d B_d\right)\left(R_d + B_d^\top \widetilde{X}_d B_d\right)^{-1}\left(N_d + A_d^\top \widetilde{X}_d B_d\right)^\top + M_d \right\|}{\left\| A_d^\top \widetilde{X}_d A_d \right\| + \left\| E_d^\top \widetilde{X}_d E_d \right\| + \left\| \left(N_d + A_d^\top \widetilde{X}_d B_d\right)\left(R_d + B_d^\top \widetilde{X}_d B_d\right)^{-1}\left(N_d + A_d^\top \widetilde{X}_d B_d^\top\right)^\top \right\| + \|M_d\|}.$$

Example 4.3. With $n = 10$, $m = 6$, we let $A_d = \mathrm{rand}(n)$, $B_d = \mathrm{rand}(n, m)$, $N_d = \mathrm{rand}(n, m)$. Construct

$$E_d = E_{d1} E_{d2}^\top, \quad R_d = R_{d1} + R_{d1}^\top, \quad M_d = M_{d1} + M_{d1}^\top,$$

where $E_{d1} = \mathrm{rand}(n, n-3)$, $E_{d2} = \mathrm{rand}(n, n-3)$, $R_{d1} = \mathrm{rand}(m)$ and $M_{d1} = \mathrm{rand}(n)$. We check that nullity$(E_d) = 3$ and the algebraic multiplicity of the zero eigenvalue of $(\mathcal{M}_d, \mathcal{L}_d)$ is also 3. Algorithm 4.1 gives

$$\mathrm{NR}_d = 1.472e - 015.$$

In this example, the PDA in Step 2 converges quadratically. In addition, Steps 4 and 5 are not required, and Z in Step 6 is chosen to be the obvious I_3.

Example 4.4. With $n = 20$, $m = 15$, we let

$$\widetilde{A}_d = [0_{n,4} \quad \mathrm{rand}(n, n-4)], \quad B_d = \mathrm{rand}(n, m), \; \widetilde{N}_d^\top = [0_{m,4} \quad \mathrm{rand}(m, n-4)],$$

$$\widetilde{E}_d = E_{d1} E_{d2}^\top, \quad \widetilde{M}_d = \mathrm{diag}\left(0_4, M_{d1} + M_{d1}^\top\right), \quad R_d = R_{d1} + R_{d1}^\top,$$

where $E_{d1} = \text{rand}(n, n-2), E_{d2} = \text{rand}(n, n-2), R_{d1} = \text{rand}(m)$ and $M_{d1} = \text{rand}(n-4)$. Construct

$$(A_d, N_d, E_d, M_d) = \left(\widetilde{A}_d Q, Q^\top \widetilde{N}_d, Q^\top \widetilde{E}_d, Q^\top \widetilde{M}_d Q\right),$$

where Q is an arbitrarily orthogonal matrix. We check that nullity$(E_d) = 2$ and the algebraic multiplicity of the zero eigenvalue of $(\mathcal{M}_d, \mathcal{L}_d)$ is 6. Then Algorithm 4.1 gives

$$\text{NR}_d = 7.501e - 015.$$

In this example, the PDA in Step 2 converges quadratically. The eigenvectors $\widehat{V}_0 \in \mathbb{R}^{55\times 4}$ computed by (4.16) in Step 5 actually correspond to the 4 zero eigenvalues of $(A_d + B_d K_d, E_d)$, and Z in Step 6 is a nontrivial 6×2-matrix of full rank as computed by (4.18c).

5. Concluding remarks

In this paper, we have developed the palindromic doubling algorithm (PDA) for solving the palindromic generalized eigenvalue problem (PGEP) $A^*x = \lambda Ax$ structurally. We prove quadratic convergence and linear convergence with rate 1/2 of the PDA, when (A^*, A) has no unimodular eigenvalues and has unimodular eigenvalues with partial multiplicities two (one or two for eigenvalue 1), respectively. Algorithm 4.1 is specially developed for the computation of the d-semi-stabilizing solution of the generalized discrete-time algebraic Riccati equation (GDARE) for the singular descriptor linear system. It is the first structure-preserving algorithm for *singular* descriptor systems.

Our numerical experience indicates that the PDA is not necessarily better than other specialist algorithms (if exist) for solving the original problem, without linearizing the associated palindromic matrix polynomials. Such specialist algorithms may be able to better utilize the finer structures of the original problems. Our numerical examples showed selected applications for which the PDA was better or when no specialist structure-preserving algorithms exist. For future work, research will be conducted on how the finer structures can be fully utilized for individual applications. For a general PGEP without finer structures, the PDA is the only structure-preserving algorithm which performs reasonably efficiently. Consequently, the “good" vibrations from “good" linearizations [5,6] can always be computed using the PDA, in the absence of better methods. Of course, numerical solutions from the PDA or other methods may be refined using the finer structures in the original problems, if feasible.

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