Stein-Lovasz 定理的推廣及其應用

成果報告

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中文摘要：

Stein-Lovasz 定理在論及覆蓋相關的組合問題時，提供一種具有演算法本質的存在性充分條件；進而爲某些具有組合意義的參數提供其上界。在本論文理，我們針對 Stein - Lovasz 定理討論其推廣，給出其演算法；進而針對(d,s, out of r, z]-disjunct 矩陣、 (k,m,c,n;z) - 選子的實驗次數的上界作討論，得出其上界。

中文關鍵字：Stein-Lovasz 定理、組合覆蓋問題、組合參數的上界、Lovasz 局部引理

英文摘要：

The Stein-Lovasz theorem provides an algorithmic way to dealing with the existence of good coverings and then deriving some bounds related to some combinatorial structures. An extension of the classical Stein-Lovasz theorem is given, followed by some applications for finding upper bounds of the sizes of (d,s, out of r, z]-disjunct matries, (k,m,c,n;z) -selectors respectively and some others.

Key words: Stein-Lovasz theorem, Combinatorial covering problems, upper bounds of parameters, Lovasz Local Lemma

An Extension of Stein-Lovász Theorem and Some of its Applications

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Abstract

The Stein-Lovász theorem provides an algorithmic way to dealing with the existence of good coverings and then deriving some bounds related to some combinatorial structures. An extension of the classical Stein-Lovász theorem is given, followed by some applications for finding upper bounds of the sizes of $(d, s$ out of $r; z]$-disjunct matries, $(k, m, c, n; z)$-selectors respectively and some others.

1 Introduction

Let X be a finite set and Γ be a family of subsets of X. We denote by $H = (X, \Gamma)$ the hypergraph having X as the set of vertices and Γ as the set of hyperedges. A subset $T \subseteq X$ such that $T \bigcap E \neq \emptyset$ for any hyperedge E is called a cover of the hypergarph H. The minimum size of a cover of an hypergraph H is denoted by $\tau(H)$. An upper bound for $\tau(H)$ was given by Lovász [L75]:

$$\tau(H) \leq \frac{|X|}{\min_{E\in\Gamma} |E|}(1 + \ln \triangle),$$

where $\triangle = \max_{x\in X} |\{E : E \in \Gamma \text{ with } x \in E\}|$.

An equivalent statement in terms of the block-point incidence matrices of the corresponding hypergraphs was given by Stein [S74] independently (see Theorem 3.1). It was called the Stein-Lovász Theorem in [CLZ96] while dealing with the covering problems in coding theory. The Stein-Lovász theorem was first used in dealing with the upper bounds for the sizes of (k, m, n)-selectors [BGV05]. Inspired by this work, it was also used in dealing with the upper bound for the sizes of $(d, r; z]$-disjunct matrices [CFH08]. Some more applications can also be found in [DZR08, DZ08]. The notion of (k, m, n)-selectors was first introduced by de Bonis et al. in [BGV05], followed by a generalization to the notion of (k, m, c, n)-selectors [B08, pp.87]. A further generalization of (k, m, c, n)-selectors will be given in Section 2.

Definitions of several properties over binary matrices are considered in Section 2 including $(d, s$ out of $r; z]$-disjunct matries, $(k, m, c, n; z)$-selectors for group testing purpose, and uniform $(m, t; z)$-splitting systems, uniform $(m, t_1, t_2; z)$-separating systems, $(v, k, t; z)$-covering designs, $(v, k, t, p; z)$-lotto designs for others. Some formulas are given in Lemmas 2.1 $\sim$ 2.4 for later simplification purpose used in Section 4.

In order to dealing with the upper bounds for these binary matrices defined in Section 2, an extended Stein-Lovász Theorem is derived in Section 3. Some applications of the determination of some upper bounds of the sizes of various models are considered in Section 4. In the first half of Section 4, the extended Stein-Lovász theorem will be used in dealing the upper bound for the sizes of $(d, s$ out of $r; z]$-disjunct matrices (Theorem 4.1) and $(k, m, c, n; z)$-selectors (Theorem 4.2). It in turn provides a few upper bounds for the sizes in various models with specific parameters as shown in Corollaries 4.1.1 $\sim$ 4.2.1. Those upper bounds for the sizes of uniform splitting systems, uniform separating systems, cover designs and lotto designs are given in Theorems 4.3 $\sim$ 4.6 respectively.

Most of these bounds are roughly the same as those derived by the basic probabilistic method including the Lovász Local Lemma. Thus, due to its constructive nature, the Stein-Lovász theorem can be regarded as a de-randomized algorithm for the probabilistic methods. The relationship between the extended Stein-Lovász theorem and the Lovász Local Lemma deserve further study [DZ08].

2 Preliminaries

2.1 A few models for various purposes

A few types of binary matrices will be introduced in this section, followed by corresponding associated parameters. These families of binary matrices will be used as models for pooling designs and for many other identification purposes.

Definition 2.1
A binary matrix M of $t \times n$ order is called $(d, s$ out of $r; z]$-disjunct, $1 \leq s \leq r$, if for any d columns and any other r columns of M, there exist z row indices in which none of the d columns appear and at least s of the r columns do. The integer t is called the size of the $(d, s$ out of $r; z]$-disjunct matrix. The minimum number of rows over all $(d, s$ out of $r; z]$-disjunct matrices with n columns is denoted by $t(n, d, r, s; z]$.

Definition 2.2
For integers k, m, s and n with $1 \leq s \leq k \leq n$ and $1 \leq m \leq \binom{k}{s}$, a $t \times n$ binary matrix M is called a $(k, m, c, n; z)$-selector if any $t \times k$ submatrix of M contains at least z disjoint submatrices with each row weight exactly s, with at least m distinct rows each. The integer t is called the size of the $(k, m, c, n; z)$-selector. The minimum size over all $(k, m, c, n; z)$-selectors is denoted by $t_s(k, m, c, n; z)$.

It is interesting to remark that the notion of (k, m, n)-selectors, first introduced by de Bonis et. al. in [BGV05] and it was then generalized to the notion of (k, m, c, n)-selectors [B08, pp.87], which are equivalent to $(k, m, 1, n; 1)$-selectors and $(k, m, w, n; 1)$-selectors rsepectively. The upper bounds for the size of (k, m, n)-selectors and (k, m, c, n)-selectors were studied in [BGV05] and in [B08, pp. 87 $\sim$ 91] respectively by the Stein-Lovász theorem. The bound for the size of $(k, m, c, n; z)$-selectors will be derived by the extended Stein-Lovász theorem (Theorem 3.2) in Theorem 4.2. The following are tables of subclasses of $(d, s$ out of $r; z]$-disjunct matrices and of $(k, m, c, n; z)$-selectors respectively.

parameters	types	bounds	references
$s = r = 1,\ z = 1$	d-disjunct	$t(d, n)$	[Y02]
$s = r = 1$	d^e-disjunct		
$s = r,\ z = 1$	$(d, r]$-disjunct	$t(n, d, r]$	[Y07]
$s = r$	$(d, r; z]$-disjunct	$t(n, d, r; z]$	[CFH08]
$s = 1,\ z = 1$	(d, r)-disjunct	$t(n, d, r)$	[Y07]
$s = 1$	$(d, r; z)$-disjunct	$t(n, d, r; z)$	
$z = 1$	$(d, s$ out of $r]$-disjunct	$t(n, d, r, s]$	[Y07]
	$(d, s$ out of $r; z]$-disjunct	$t(n, d, r, s; z]$	

parameters	types	bounds	references
$c = 1,\ z = 1$	(k, m, n)-selectors	$t_s(k, m, n)$	[BGV05,Y07]
$c = 1$	$(k, m, n; z)$-selectors	$t_s(k, m, n; z)$	
$z = 1$	(k, m, c, n)-selectors	$t_s(k, m, c, n)$	[B08]
	$(k, m, c, n; z)$-selectors	$t_s(k, m, c, n; z)$	

As the relationship between various models (disjunct matrices, selectors) and nonadaptive group testing:

1. A $(d, r]$-disjunct matrix can be used to identify the up-to-d positives on the complex model (ref to Chen, Du and Hwang [CDH07]).

2. A (h, d)-disjunc is a necessary condition for identifying the positive set on the (d, h)-inhibitor model [BV98].

3. There exists a two-state group testing algorithm for finding up-to-d positives out of n items and that uses a number of tests equal to $t + k - 1$, where t is the size of a $(k, d + 1, n)$-selector [BGV05].

Definition 2.3
Let m and t be even integers with $2 \le t \le m$, and let z be a positive integer. An uniform $(m, t; z)$-splitting system is a pair (X, Γ) where $|X| = m$, Γ is a set of $\frac{m}{2}$–subsets of X, called blocks such that for every $T \subseteq X$ with $|T| = t$, there exist at least z blocks $B \in \Gamma$ such that $|T \bigcap B| = \frac{t}{2}$, i.e., B splits T. The system (X, Γ) is also called a $(t; z)$-splitting system. The minimum number of blocks over all $(t; z)$-splitting systems is denoted by $SP(m, t; z)$.

Definition 2.4
Let m be an even integer and let t_1, t_2 and z be positive integers with $t_1 + t_2 \le m$. An uniform $(m, t_1, t_2; z)$-separating system is a pair (X, Γ) where $|X| = m$, Γ is a set of $\frac{m}{2}$–subsets of X, called blocks such that for every $T_1, T_2 \subseteq X$, where $|T_i| = t_i$ for $i = 1, 2$ and $|T_1 \bigcap T_2| = \emptyset$, there exist at least z blocks $B \in \Gamma$ for which either $T_1 \subseteq B$, $T_2 \bigcap B = \emptyset$ or $T_2 \subseteq B$, $T_1 \bigcap B = \emptyset$, i.e., P, Q are separated by B. The system (X, Γ) is also called a $(t_1, t_2; z)$-separating system. The minimum number of blocks over all $(t_1, t_2; z)$-separating systems is denoted by $SE(m, t_1, t_2; z)$.

Definition 2.5
Let v, k, t and z be positive integers with $t \le k \le v$. A $(v, k, t; z)$-covering design is a pair (X, Γ) where $|X| = v$, Γ is a family of k-subsets of X, called blocks such that for every $T \subseteq X$ with $|T| = t$, there exist at least z blocks $B \in \Gamma$ containing T. The minimum number of blocks over all $(v, k, t; z)$-covering designs is denoted by $C(v, k, t; z)$.

Definition 2.6
Let v, k, t, p and z be positive integers with $p \le t, k \le v$. A $(v, k, t, p; z)$-lotto design is a pair (X, Γ) where $|X| = v$, Γ is a set of k-subsets of X, called blocks such that for every $T \subseteq X$ with $|T| = t$, there exist at least z blocks $B \in \Gamma$ such that $T \bigcap B \ge p$. The minimum number of blocks over all $(v, k, t, p; z)$-lotto designs is denoted by $L(v, k, t, p; z)$.

Note that when $p = t$, a $(v, k, t, t; z)$-lotto design will be reduced to a $(v, k, t; z)$-covering design. The related bounds are summarized in the following table.

types	bounds	references
uniform $(m, t; z)$-splitting systems	$SP(m, t; z)$	$z = 1$[DZR08]
uniform $(m, t_1, t_2; z)$-separating systems	$SE(m, t_1, t_2; z)$	$z = 1$[DZR08]
$(v, k, t; z)$-covering designs	$C(v, k, t; z)$	$z = 1$[DZR08]
$(v, k, t, p; z)$-lotto designs	$L(v, k, t, p; z)$	$z = 1$[DZR08]

2.2 Some basic counting results

Stein-Lovász Theorem and its extension will be used to estimate the upper bounds of the sizes for pooling designs of various models. In order to give upper bounds for the above mentioned parameters, the following results involving binomial coefficients will be involved. Lemma 2.1 will be used in showing appropriate values of w for pooling designs of various models. We need information regarding the maximum of the function

$$f(w) = \binom{n-w}{d}\binom{n-w-d}{r-s}\binom{w}{s}$$

with various r and s when dealing with possible upper bounds for t of various models. Lemmas 2.2 $\sim$ 2.4 will be used in the simplifications of the bounds $\frac{M}{v}$, and $\ln a$ respectively in the expression $\frac{M}{v}(1 + \ln a)$ found in the Stein-Lovász theorem (Theorem 3.1).

Since $e^x = \sum_{n\geq 0}\frac{x^n}{n!}$, we have $e^x \geq \frac{x^b}{b!}$ for each x, thus $e^b \geq \frac{b^b}{b!}$, i.e., $1 \leq \frac{b!e^b}{b^b}$; and hence $\binom{a}{b} = \frac{a!}{(a-b)!b!} = \frac{a(a-1)\cdots(a-b+1)}{b!} \leq \frac{a^b}{b!} \leq \frac{a^b}{b!}\cdot\frac{b!e^b}{b^b} = (\frac{ea}{b})^b$, i.e., $\binom{a}{b} \leq \frac{a^b}{b!} \leq (\frac{ea}{b})^b$.

Lemma 2.1
For any positive integers n, d, r, s with $k = d + r \leq n$ and $1 \leq s \leq r$, the function $f(w) = \binom{n-w}{d}\binom{n-w-d}{r-s}\binom{w}{s}$ gets its maximum at $w = \frac{ns-(k-s)}{k}$.

Proof. First we note that

$$\begin{aligned} f(w) &= \binom{n-w}{d}\binom{n-w-d}{r-s}\binom{w}{s} \\ &= \frac{(n-w)!}{(n-w-d)!d!}\cdot\frac{(n-w-d)!}{(n-w-d-r+s)!(r-s)!}\cdot\binom{w}{s}\cdot\frac{(d+r-s)!}{(d+r-s)!} \\ &= \binom{n-w}{d+r-s}\binom{w}{s}\binom{d+r-s}{d} = \binom{n-w}{k-s}\binom{w}{s}\binom{k-s}{d}. \end{aligned}$$

Since

$$\begin{aligned} f(w+1) &= \binom{n-(w+1)}{k-s}\binom{w+1}{s}\binom{k-s}{d} \\ &= (\frac{(n-w)-(k-s)}{n-w}\binom{n-w}{k-s})\cdot(\frac{w+1}{w+1-s}\binom{w}{s})\cdot\binom{k-s}{d} \\ &= (\frac{(n-w)-(k-s)}{n-w}\cdot\frac{w+1}{w+1-s})f(w), \end{aligned}$$

and

$$\frac{(n-w)-(k-s)}{n-w}\cdot\frac{w+1}{w+1-s} = \frac{(w+1)(n-w)-(w+1)(k-s)}{(w+1)(n-w)-s(n-w)} = 1$$

if and only if $s(n-w) = (w+1)(k-s)$, i.e., $w = \frac{ns-(k-s)}{k}$; hence

$\frac{(n-w)-(k-s)}{n-w}\cdot\frac{w+1}{w+1-s} \geq 1$ for $w \leq \frac{ns-(k-s)}{k}$ and

$\frac{(n-w)-(k-s)}{n-w}\cdot\frac{w+1}{w+1-s} \leq 1$ for $w \geq \frac{ns-(k-s)}{k}$.

As a consequence, we then have

$f(w)$ is increasing for $w \leq \frac{ns-(k-s)}{k}$, and

$f(w)$ is decreasing for $w \geq \frac{ns-(k-s)}{k}$

as required.

□

By taking $s = r = c$ in $f(w) = \binom{n-w}{d}\binom{n-w-d}{r-s}\binom{w}{s}$, we get the quadratic function $g(w) = \binom{w}{c}\binom{n-w}{k-c}$ (for selector). Hence we have the following corollary.

Corollary 2.1.1
The function $g(w) = \binom{w}{c}\binom{n-w}{k-c}$ gets its maximum at $w = \dfrac{nc-(k-c)}{k}$.

Lemma 2.2
For any positive integers n, d, r, s with $k = d + r \le n$ and $1 \le s \le r$,

$$\frac{n(n-1)\cdots(n-s+1)(n-s)(n-s-1)\cdots(n-r-d+1)}{n(n-\frac{k}{s})\cdots(n-k+\frac{k}{s})\cdot n\cdot(n-\frac{k}{k-s})\cdots(n-k+\frac{k}{k-s})} \le 1.$$

Proof. Without loss of generality, let $s \le k - s$ and thus $1 \le \frac{k}{k-s} \le 2 \le \frac{k}{s}$. Moreover, we note that the left hand side is

$$\begin{aligned}&\frac{n(n-1)\cdots(n-s+1)(n-s)(n-s-1)\cdots(n-r-d+1)}{n(n-\frac{k}{s})\cdots(n-k+\frac{k}{s})\cdot n\cdot(n-\frac{k}{k-s})\cdots(n-k+\frac{k}{k-s})}\\ =&\frac{\prod\limits_{0\le i\le r+d-1}(n-i)}{\prod\limits_{0\le i\le s-1}(n-i\cdot\dfrac{k}{s})\prod\limits_{0\le j\le k-s-1}(n-j\cdot\dfrac{k}{k-s})}.\end{aligned}$$

To prove this inequality, we will rearrange the terms in the denominator so that $\dfrac{n-t}{n-f(t)} \le 1$ for each t with $0 \le t \le r + d - 1$, i.e., we will give a bijection

$$f : \{0, 1, ..., r + d - 1\} \to \{i \cdot \frac{k}{s} \mid 0 \le i \le s - 1\} \bigcup \{j \cdot \frac{k}{k-s} \mid 0 \le j \le k - s - 1\}$$

with the property that $f(t) \le t$ for each t. Note that the element 0 will be counted twice as $0 \cdot \frac{k}{s}$ and $0 \cdot \frac{k}{k-s}$ respectively in the range of the function f. Note also that if $i \cdot \frac{k}{s} = i + i \cdot \frac{k}{k-s} > t$, then $\frac{s}{k-s} < \frac{i}{t-i}$ and hence $j \cdot \frac{k}{k-s} = j(1 + \frac{s}{k-s}) < j(1 + \frac{i}{t-i}) = j(\frac{i}{t-i}) = j(\frac{i}{j})$, where $i + j = t$.

A such function f is defined recursively as follows. For $t = 0, 1, 2$, let $f(0) = 0_{\frac{k}{s}}$, $f(1) = 0_{\frac{k}{k-s}}$, $f(2) = \frac{k}{k-s}$. For $3 \le t \le r + d - 1$, let i (resp. j) be the smallest positive integers such that $i \cdot \frac{k}{s}$ (resp. $j \cdot \frac{k}{k-s}$)$\notin \{f(0), f(1), ..., f(t - 1)\}$ if they exist, it follows that $t = i + j$.

1. Let $f(t) = i \cdot \frac{k}{s}$ if $i \cdot \frac{k}{s} \le t$.
2. Otherwise, we define $f(t) = j \cdot \frac{k}{k-s}$.
3. Finally, suppose t is large and there is no such i, note that

   $$\frac{n-s}{n} \le \frac{n-s-1}{n-\frac{k}{k-s}} \le \cdots \le \frac{n-r-d+1}{n-k+\frac{k}{k-s}} \text{ and } \frac{n-r-d+1}{n-k+\frac{k}{k-s}} < 1,$$

   we have $\dfrac{n-t}{n-(t-s)\frac{k}{k-s}} \le 1$ for all $s \le t \le r + d - 1$, we define $f(t) = (t - s)\frac{k}{k-s}$.

Cearly, the above defined function is 1-1, onto, and $f(t) \le t$ for all $0 \le t \le r + d - 1$ as required. $\square$

Lemma 2.3
For any positive integers n, d, r, s with $k = d + r \le n$ and $1 \le s \le r$, let $n' \ge n$ be the smallest positive integer such that $w = \dfrac{n's}{k}$ is an integer, then

$$\frac{\binom{n'}{d}\binom{n'-d}{r}}{\binom{n'-w}{d}\binom{n'-w-d}{r-s}\binom{w}{s}} \le \frac{(\frac{k}{s})^s(\frac{k}{k-s})^{k-s}}{\binom{r}{s}}.$$

Proof.

$$\frac{\binom{n'}{d}\binom{n'-d}{r}}{\binom{n'-w}{d}\binom{n'-w-d}{r-s}\binom{w}{s}}$$

$$=\frac{\frac{(n')!}{(n'-d)!d!}\cdot\frac{(n'-d)!}{(n'-d-r)!r!}}{\frac{(n'-w)!}{(n'-w-d)!d!}\cdot\frac{(n'-w-d)!}{(n'-w-d-r+s)!(r-s)!}\cdot\frac{w!}{(w-s)!s!}}$$

$$=\frac{\frac{1}{r!}n'(n'-1)\cdots(n'-s+1)(n'-s)(n'-s-1)\cdots(n-d-r+1)}{\frac{1}{(r-s)!s!}w(w-1)\cdots(w-s+1)(n'-w)(n'-w-1)\cdots(n'-w-d-r+s+1)}$$

$$=\frac{1}{\binom{r}{s}}\cdot\frac{n'(n'-1)\cdots(n'-s+1)(n'-s)(n'-s-1)\cdots(n-d-r+1)}{\frac{n's}{k}(\frac{n's}{k}-1)\cdots(\frac{n's}{k}-s+1)(n'-\frac{n's}{k})(n'-\frac{n's}{k}-1)\cdots(n'-\frac{n's}{k}-d-r+s+1)}$$

$$=\frac{1}{\binom{r}{s}}\cdot\frac{1}{(\frac{s}{k})^s(\frac{k-s}{k})^{k-s}}\cdot\frac{n'(n'-1)\cdots(n'-s+1)(n'-s)(n'-s-1)\cdots(n-d-r+1)}{n'(n'-\frac{k}{s})\cdots(n'-k+\frac{k}{s})\cdot n'\cdot(n'-\frac{k}{k-s})\cdots(n'-k+\frac{k}{k-s})}$$

$$=\frac{(\frac{k}{s})^s(\frac{k}{k-s})^{k-s}}{\binom{r}{s}}\text{(by Lemma 2.2).}$$

□

Lemma 2.4
For any positive integers n, d, r, s with $k = d + r \le n$ and $1 \le s \le r$, let $n' \ge n$ be the smallest positive integer such that $w = \dfrac{n's}{k}$ is an integer, then

$$\ln(\binom{n'-w}{d}\binom{n'-w-d}{r-s}\binom{w}{s}) < k[1+\ln(\frac{n}{k}+1)] + \ln\binom{k-s}{d}.$$

Proof. First we note that $n' < n + k$ for such n'. Since $\binom{a}{b} \le (\frac{ea}{b})^b$, we have

$$\binom{n'-w}{d}\binom{n'-w-d}{r-s}\binom{w}{s}$$

$$=\frac{(n'-w)!}{(n'-w-d)!d!}\cdot\frac{(n'-w-d)!}{(n'-w-d-r+s)!(r-s)!}\cdot\binom{w}{s}\cdot\frac{(d+r-s)!}{(d+r-s)!}$$

$$=\binom{n'-w}{d+r-s}\binom{w}{s}\binom{d+r-s}{d} = \binom{n'-w}{k-s}\binom{w}{s}\binom{k-s}{d}$$

$$\le(\frac{e(n'-\frac{n's}{k})}{k-s})^{k-s}\cdot(\frac{e(\frac{n's}{k})}{s})^s\cdot\binom{k-s}{d} = (e\cdot\frac{n'}{k})^{k-s}\cdot(e\frac{n'}{k})^s\cdot\binom{k-s}{d}$$

$$= e^k(\frac{n'}{k})^k \cdot \binom{k-s}{d} < e^k(\frac{n+k}{k})^k \cdot \binom{k-s}{d} = e^k(\frac{n}{k}+1)^k \cdot \binom{k-s}{d}.$$

Therefore,

$$\ln(\binom{n'-w}{d}\binom{n'-w-d}{r-s}\binom{w}{s})$$

$$< \ln(e^k(\frac{n}{k}+1)^k \cdot \binom{k-s}{d}) = k[1+\ln(\frac{n}{k}+1)] + \ln\binom{k-s}{d}.$$

□

The substitutions of w for various subclasses are summarized in the following table for later references:

types	parameters		
d-disjunct	$s=r=1,\ z=1$	$w=\frac{n-d}{k}$	$w=\frac{n'}{k}$
d^e-disjunct	$s=r=1$	$w=\frac{n-d}{k}$	$w=\frac{n'}{k}$
$(d,r]$-disjunct	$s=r,\ z=1$	$w=\frac{nr-d}{k}$	$w=\frac{n'r}{k}$
$(d,r;z]$-disjunct	$s=r$	$w=\frac{nr-d}{k}$	$w=\frac{n'r}{k}$
(d,r)-disjunct	$s=1,\ z=1$	$w=\frac{n-(k-1)}{k}$	$w=\frac{n'}{k}$
$(d,r;z)$-disjunct	$s=1$	$w=\frac{n-(k-1)}{k}$	$w=\frac{n'}{k}$
$(d,s$ out of $r]$-disjunct	$z=1$	$w=\frac{ns-(k-s)}{k}$	$w=\frac{n's}{k}$
$(d,s$ out of $r;z]$-disjunct		$w=\frac{ns-(k-s)}{k}$	$w=\frac{n's}{k}$

types	parameters		
(k,m,n)-selectors	$c=1,\ z=1$	$w=\frac{n-(k-1)}{k}$	$w=\frac{n'}{k}$
$(k,m,n;z)$-selectors	$c=1$	$w=\frac{n-(k-1)}{k}$	$w=\frac{n'}{k}$
(k,m,c,n)-selectors	$z=1$	$w=\frac{nc-(k-c)}{k}$	$w=\frac{n'c}{k}$
$(k,m,c,n;z)$-selectors		$w=\frac{nc-(k-c)}{k}$	$w=\frac{n'c}{k}$

3 The Stein-Lovász Theorem and its extension

We now introduce the Stein-Lovász Theorem as follows. The proof is included for completeness [DZR08]. The Stein-Lovász theorem can be further extended from rows with weight at least 1 to the case of rows with weight at least $z \geq 1$. The bound can be further improved when M is a matrix with constant row weight and column weight as well, i.e., in the language of hypergraphs, uniform and regular.

Theorem 3.1 [DZR08] Let A be a (0,1) matrix with N rows and M columns. Assume that each row contains at least v ones, and each column at most a ones. Then there exists an $N \times K$ submatrix C with

$$K \leq (\frac{N}{a}) + (\frac{M}{v}) \ln a \leq (\frac{M}{v})(1 + \ln a),$$

such that C does not contain an all-zero row.

Proof. A constructive approach for producing C is presented. Let $A_a = A$. We begin by picking the maximal number K_a of columns from A_a, whose supports are pairwise disjoint and each column having a ones (perhaps, $K_a = 0$). Discarding these columns and all rows incident to one of them, we are left with a $k_a \times (M - K_a)$ matrix A_{a-1}, where $k_a = N - aK_a$. Clearly, the columns of A_{a-1} have at most $a-1$ ones (indeed, otherwise such a column could be added to the previously discarded set, contradicting its maximality). Now we remove from A_{a-1} a maximal number K_{a-1} of columns having $a - 1$ ones and whose supports are pairwise disjoint, thus getting a $k_{a-1} \times (M - K_a - K_{a-1})$ matrix A_{a-2}, where $k_{a-1} = N - aK_a - (a-1)K_{a-1}$. The process will terminate after at most a steps. The union of the columns of the discarded sets form the desired submatrix C with $K = \sum_{i=1}^{a} K_i$.

The first step of the algorithm gives $k_a = N - aK_a$, which we rewrite, setting $k_{a+1} = N$, as $K_a = \frac{k_{a+1} - k_a}{a}$. Analogously, $K_i = \frac{k_{i+1} - k_i}{i}, i = 1, ..., a$. Now we derive an upper bound for k_i by counting the number of ones in A_{i-1} in two ways: every row of A_{i-1} contains at least v ones, and every column at most $i - 1$ ones, thus

$$vk_i \leq (i-1)(M - K_a - \cdots - K_{i+1}) \leq (i-1)M.$$

Furthermore,

$$K = \sum_{i=1}^{a} K_i = \sum_{i=1}^{a} \frac{k_{i+1} - k_i}{i} = \frac{k_{a+1}}{a} + \frac{k_a}{a(a-1)} + \frac{k_{a-1}}{(a-1)(a-2)} + \cdots + \frac{k_2}{1 \times 2} - k_1$$

$$\leq (N/a) + (M/v)(1/a + 1/(a-1) + \cdots + 1/2),$$

thus giving the result.

□

The greedy procedure as shown in the proof constructs the desired submatrix one column at a time ([DZ08, pp.4]), and hence the algorithm below follows.

Algorithm: STEIN-LOVÁSZ(A)
input: A is an $N \times M$ matrix, each column has at most a ones, each row has at least v ones
  $C \leftarrow \emptyset$
  while A has at least one row
  do find a column c in A having maximum weight
  delete all rows of A that contain a "1" in column c
  delete column c from A
output: Returns a submatrix of A with no all-zero row

At each state, a new column is added to the submatrix that maximizes the number of "new" rows that are yet uncovered. When all rows are covered, the algorithm stops. It seems quite interesting that we can use the Stein-Lovász Theorem to derive bounds for some combinatorial array [DZ08, pp.4]. If the entries of 1 occur in a binary matrix of order $N \times M$ as uniformly as possible, for example, v-uniform and a-regular, an upper bound for K is found so that the submatrices of order $N \times K$ will share similar property too.

Theorem 3.2 (Extension of The Stein-Lovász Theorem)
Let A be a (0,1) matrix of order $N \times M$, and let a, v, z be positive integers. Assume that each row contains exactly v ones, and each column exactly a ones. Then there exists a submatrix C of order $N \times K$ with

$$K \leq \frac{z(z+1)}{2}(\frac{M}{v})(1+\ln a),$$

such that each row of C has weight at least z. More specifically, if the matrix is v-uniform and a-regular, the upper bound can be reduced to

$$K \leq z(\frac{M}{v})(1+\ln a).$$

The strategy for the proof of Theorem 3.2 is as follows:

1. Use the Stein-Lovász Theorem to obtain a submatrix C_1 with each row weight at least 1.
2. Choose some columns in the matrix $A \backslash C_1$ to combine with the submatrix C_1 to form a submatrix C_2 with each row weight at least 2.
3. Choose some columns in the matrix $A \backslash C_2$ to combine with the submatrix C_2 to form a submatrix C_3 with each row weight at least 3.
4. Step by step, and finally we obtain the desired submatrix $C = C_z$ with each row weight at least z.

Note that this upper bound makes sense only if $\dfrac{z(z+1)}{2}(\frac{M}{v})(1+\ln a) < M$, i.e., $\dfrac{z(z+1)}{2} < \dfrac{v}{1+\ln a}$ in general, or if $z(\frac{M}{v})(1+\ln a) < M$, i.e., $z < \dfrac{v}{1+\ln a}$ for the case of uniform and regular.

Proof. A constructive approach for producing C is presented. Let $A_1 = A$. By the Stein-Lovász Theorem, there exists an $N \times M_1$ submatrix $C_1 (= B_1^{'} = B_1)$ of A_1 with $M_1 \leq \frac{M}{v}(1+\ln a)$ such that each row of C_1 has weight at least 1.

The algorithm used in the proof of the Stein-Lovász Theorem shows that some rows of C_1 have weight exactly 1. Let R_1 be the set of indices of those rows and $|R_1| = r_1$. Let A_2 be the submatrix of order $r_1 \times (M - M_1)$ obtained from A_1 by deleting the submatrix C_1 and the i-th row, $i \notin R_1$ as well. Then each row of A_2 contains at least $v-1$ ones, and each column at most a ones. Again, by the Stein-Lovsz Theorem, there exists an $r_1 \times M_2$ submatrix B_2 with $M_2 \leq \frac{M-M_1}{v-1}(1+\ln a)$ such that each row of $B_2^{'}$ has weight at least 1. Let B_2 be the matrix of order $N \times M_2$ obtained from $B_2^{'}$ by adding the i-th row, $i \notin R_1$. Let C_2 be the matrix of order $N \times (M_1 + M_2)$ obtained by the union of B_1 and B_2. Then C_2 is a submatrix of A, and each row of C_2 has weight at least 2.

Similarly, there exist some rows of C_2 that have weight exactly 2. Let R_2 be the set of indices of those rows and $|R_2| = r_2$. Continue in this way, we have:

For $2 \leq i \leq z$,

1. A_i is a matrix of order $r_{i-1} \times (M - \sum_{j=1}^{i-1} M_j)$, and each row contains at least $v-(i-1)$ ones, and each column at most a ones.
2. $B_i^{'}$ is an $r_{i-1} \times M_i$ submatrix of A_i with $M_i \leq \dfrac{M - \sum_{j=1}^{i-1} M_j}{v-(i-1)}(1+\ln a)$, and each row has weight at least 1.

For $1 \le i \le z$,

3. B_i is a matrix of order $N \times M_i$.

4. C_i is an $N \times \sum_{j=1}^{i} M_j$ submatrix of A, and each row has weight at least i.

Hence, $C = C_z$ is the submatrix required, that is,

$$
\begin{aligned}
K &= \sum_{j=1}^{z} M_j = M_1 + M_2 + \cdots + M_z \\
&\le \frac{M}{v}(1+\ln a) + \frac{M - M_1}{v-1}(1+\ln a) + \cdots + \frac{M - \sum_{j=1}^{z-1} M_j}{v-(z-1)}(1+\ln a) \\
&\le \frac{M}{v}(1+\ln a) + \frac{M}{v-1}(1+\ln a) + \cdots + \frac{M}{v-(z-1)}(1+\ln a) \\
&= M(1+\ln a)(\frac{1}{v} + \frac{1}{v-1} + \cdots + \frac{1}{v-(z-1)}) \\
&\le M(1+\ln a)(\frac{1}{v} + \frac{2}{v} + \cdots + \frac{z}{v}) \\
&= \frac{z(z+1)}{2}(\frac{M}{v})(1+\ln a),
\end{aligned}
$$

thus gives the result.

More specifically, for the case of uniform and regular, using similar argument as above with a minor modification. First we note that $Nv = Ma$ by counting the number of entries 1 in A in two ways. For $2 \le i \le z$, A_i is a matrix of order $r_{i-1} \times (M - \sum_{j=1}^{i-1} M_j)$, and each row contains exactly $v-(i-1)$ ones, and each column at most a ones. Moreover, a lower bound for $\sum_{j=1}^{i-1} M_j$ is derived by counting the number of 1s in the submatrix C_{i-1} in two ways; each row of C_{i-1} contains at least $i-1$ ones, and each column exactly a ones, thus $N(i-1) \le (\sum_{j=1}^{i-1} M_j)a$, and hence $\frac{M}{v}(i-1) \le \sum_{j=1}^{i-1} M_j$ for $2 \le i \le z$. Furthermore,

$$
\begin{aligned}
K &= \sum_{j=1}^{z} M_j = M_1 + M_2 + \cdots + M_z \\
&\le \frac{M}{v}(1+\ln a) + \frac{M - M_1}{v-1}(1+\ln a) + \cdots + \frac{M - \sum_{j=1}^{z-1} M_j}{v-(z-1)}(1+\ln a) \\
&\le \frac{M}{v}(1+\ln a) + \frac{M - \frac{M}{v}}{v-1}(1+\ln a) + \cdots + \frac{M - (z-1)\cdot\frac{M}{v}}{v-(z-1)}(1+\ln a) \\
&= \frac{M}{v}(1+\ln a) + \frac{M}{v}(1+\ln a) + \cdots + \frac{M}{v}(1+\ln a) \\
&= z(\frac{M}{v})(1+\ln a),
\end{aligned}
$$

thus gives the result.

$\square$

For the case of uniform and regular, Theorem 3.2 is restated in the language of hypergraphs in the following corollary. A z-cover is a subset C of X such that $|C \bigcap E| \geq z$ for any hyperedges E. Let $\tau_z(H)$ be the minimum size over all z-covers of the hypergraph H.

Corollary 3.3 Let $H = (X, \Gamma)$ be a v-uniform and a-regular hypergraph with vertex set X and edge set Γ, then $\tau_z(H) \leq z\frac{|X|}{v}(1 + \ln a)$.

We conjecture that $\tau_z(H) \leq z\tau_1(H)$ holds for hypergraphs which are uniform and regular. However, it need not be true in general as shown in the following example. For the hypergraph with where $X = \{1, 2, 3, ..., 8\}$ and $\Gamma = \{\{1, 2, 3\}, \{4, 5, 6\}, \{1, 7, 8\}\}$. It is easy to see that $\{1, 4\}$ is a 1-cover with minimum size, hence $\tau_1(H) = 2$. Similarly, $\{1, 2, 4, 5, 7\}$ is a 2-cover with minimum size, hence $\tau_2(H) = 5$. This shows that $\tau_2(H) = 5 > 2 \cdot 2 = 2\tau_1(H)$.

4 Some Applications of the extended Stein-Lovász Theorem

The Stein-Lovász theorem was first used in dealing with the upper bound for the sizes in the model of (k, m, n)-selecters [BGV05]. Inspired by this work, it was also used in dealing with the upper bound for the sizes of $(d, r; z]$-disjunct matrices [CFH08]. In the first half of Section 4, the extended Stein-Lovász theorem will be used in dealing the upper bound for the sizes of $(d, s$ out of $r; z]$-disjunct matrices (Theorem 4.1) and $(k, m, c, n; z)$-selectors (Theorem 4.2). It in turn provides a few upper bounds for the sizes in various models with specific parameters as shown in Corollaries 4.1.1 $\sim$ 4.2.1. Those upper bounds for the sizes of uniform splitting systems, uniform separating systems, cover designs and lotto designs are given in Theorems 4.3 $\sim$ 4.6 respectively.

4.1 bounds for various models of pooling designs

Upper bounds for the sizes of d-disjunct matrices, $(d, r]$-disjunct matrices, (d, r)-disjunct matrices and $(d, s$ out of $r]$-disjunct matrices were given in [Y02, Y07] Lovász local lemma. Recall that $t(n, d, r, s; z]$ is the minimum of the sizes of $(d, s$ out of $r; z]$-disjunct matrices.

Theorem 4.1
For any positive integers n, d, r, s and z, with $1 \leq s \leq r$, if $k = d + r \leq n$, then

$$t(n, d, r, s; z] < \frac{z(\frac{k}{s})^s(\frac{k}{k-s})^{k-s}}{\binom{r}{s}}\{1 + k[1 + \ln(\frac{n}{k} + 1)] + \ln\binom{k-s}{d}\}.$$

Proof. For $s \leq w \leq n - d$, let A be the binary matrix of order $[\binom{n}{d}\binom{n-d}{r}] \times \binom{n}{w}$ with rows indexed by $\{(D, R) \mid D \in \binom{[n]}{d}, R \in \binom{[n]}{r}$ with $D \bigcap R$ empty$\}$, and the columns indexed by $V = \{v \mid v \in \{0, 1\}^n, wt(v) = w\}$. The entry of A at the row indexed by the pair (D, R) and the column indexed by the vector $v \in V$ is 1 if the entries of v over D are all zero and at least s entries of v over R are one; and 0 otherwise.

Observe that each row of A has weight $\sum_{j=s}^{\min(r,w)}\binom{r}{j}\binom{n-(d+r)}{w-j}$, and each column of A has weight $\binom{n-w}{d}\sum_{j=s}^{\min(r,w)}\binom{n-w-d}{r-j}\binom{w}{j}$. By the extension of the Stein-Lovász Theorem, there exists a submatrix M of A of order $[\binom{n}{d}\binom{n-d}{r}] \times t$ with each row weight at least z, where

$$t \le \frac{z\binom{n}{w}}{\sum_{j=s}^{\min(r,w)} \binom{r}{j}\binom{n-(d+r)}{w-j}} \{1+\ln[\binom{n-w}{d}\sum_{j=s}^{\min(r,w)}\binom{n-w-d}{r-j}\binom{w}{j}]\}$$
$$= \frac{z\binom{n}{d}\binom{n-d}{r}}{\binom{n-w}{d}\sum_{j=s}^{\min(r,w)}\binom{n-w-d}{r-j}\binom{w}{j}} \{1+\ln[\binom{n-w}{d}\sum_{j=s}^{\min(r,w)}\binom{n-w-d}{r-j}\binom{w}{j}]\}.$$

Note that the equality is obtained by counting the number of 1s in A in two ways. It is straightforward to show that the columns of M form a $(d, s$ out of $r; z]$-disjunct matrix of order $t \times n$. We then have

$$t(n,d,r,s;z] \le \frac{z\binom{n}{d}\binom{n-d}{r}}{\binom{n-w}{d}\sum_{j=s}^{\min(r,w)}\binom{n-w-d}{r-j}\binom{w}{j}} \{1+\ln[\binom{n-w}{d}\sum_{j=s}^{\min(r,w)}\binom{n-w-d}{r-j}\binom{w}{j}]\}$$
$$\le \frac{z\binom{n}{d}\binom{n-d}{r}}{\binom{n-w}{d}\binom{n-w-d}{r-s}\binom{w}{s}} \{1+\ln[\binom{n-w}{d}\binom{n-w-d}{r-s}\binom{w}{s}]\} (\text{only take } j=s).$$

For given positive integers n, d, r and s, let $n' \ge n$ be the smallest positive integer such that $\frac{n's}{k}$ is an integer. We have

$$\frac{\binom{n'}{d}\binom{n'-d}{r}}{\binom{n'-w}{d}\binom{n'-w-d}{r-s}\binom{w}{s}} \le \frac{(\frac{k}{s})^s(\frac{k}{k-s})^{k-s}}{\binom{r}{s}}$$

by Lemma 2.3, and

$$\ln(\binom{n'-w}{d}\binom{n'-w-d}{r-s}\binom{w}{s}) < k[1+\ln(\frac{n}{k}+1)] + \ln\binom{k-s}{d}$$

by Lemma 2.4. Therefore,

$$t(n,d,r,s;z] \le t(n',d,r,s;z] \le \frac{z\binom{n'}{d}\binom{n'-d}{r}}{\binom{n'-w}{d}\binom{n'-w-d}{r-s}\binom{w}{s}} \{1+\ln[\binom{n'-w}{d}\binom{n'-w-d}{r-s}\binom{w}{s}]\}$$
$$< \frac{z(\frac{k}{s})^s(\frac{k}{k-s})^{k-s}}{\binom{r}{s}} \{1+k[1+\ln(\frac{n}{k}+1)]+\ln\binom{k-s}{d}\}$$

as required.

$\square$

As a consequence, upper bounds for the sizes of various situations are summarized in the following corollary.

Corollary 4.1.1

1. If $k = d + 1 \leq n$, then

$$t(d, n) = t(n, d, 1, 1; 1] < k(\frac{k}{d})^d\{1 + k[1 + \ln(\frac{n}{k} + 1)]\}.$$

2. If $k = d + r \leq n$, then

$$t(n, d, r] = t(n, d, r, r; 1] < (\frac{k}{r})^r(\frac{k}{d})^d\{1 + k[1 + \ln(\frac{n}{k} + 1)]\}.$$
$$t(n, d, r; z] = t(n, d, r, r; z] < z(\frac{k}{r})^r(\frac{k}{d})^d\{1 + k[1 + \ln(\frac{n}{k} + 1)]\}.\text{[CFH08]}$$
$$t(n, d, r) = t(n, d, r, 1; 1] < \frac{k}{r}(1 + \frac{1}{k-1})^{k-1}\{1 + k[1 + \ln(\frac{n}{k} + 1)] + \ln\binom{k-1}{d}\}.$$
$$t(n, d, r; z) = t(n, d, r, 1; z] < z\frac{k}{r}(1 + \frac{1}{k-1})^{k-1}\{1 + k[1 + \ln(\frac{n}{k} + 1)] + \ln\binom{k-1}{d}\}.$$
$$t(n, d, r, s] = t(n, d, r, s; 1] < \frac{(\frac{k}{s})^s(\frac{k}{k-s})^{k-s}}{\binom{r}{s}}\{1 + k[1 + \ln(\frac{n}{k} + 1)] + \ln\binom{k-s}{d}\}.$$

The notion of (k, m, n)-selectors was first introduced by de Bonis et. al. in [BGV05] , and it was then generalized to the notion of (k, m, w, n)-selectors [B08, pp.87]. It is interesting to remark that the notions of (k, m, n)-selecters and (k, m, w, n)-selectors are equivalent to $(k, m, 1, n; 1)$-selectors and $(k, m, w, n; 1)$-selectors respectively.

An upper bound for the size of (k, m, n)-selectors was also given in [Y07, 4.4.1] by Lovász local lemma.

Using similar arguments in [BGV05] and [B08] with a minor modification, the upper bound of the size of $(k, m, c, n; z)$-selectors is given below.

Theorem 4.2

For $a = \binom{k}{c}$,

$$t_s(k, m, c, n; z) < \frac{(a - m + 1)(z - 1) + 1}{a - m + 1}(\frac{k}{c})^c(1 + \frac{1}{k-c})^{k-c}\{1 + k[1 + \ln(\frac{n}{k} + 1)] + \ln\binom{a-1}{a-m}\}.$$

Proof. For $c \leq w \leq n - k + c$, let $X = \{x \in \{0, 1\}^n \mid wt(x) = w\}$ and $U = \{u \in \{0, 1\}^k \mid wt(u) = c\}$. Moreover, for any $A \subseteq U$ of size r, $r = 1, ..., \binom{k}{c}$, and any set $S \in \binom{[n]}{k}$, define $E_{A,S} = \{x \in X : x|_S \in A\}$.

Let $a = \binom{k}{c}$ and M be a (0,1) matrix of order $[\binom{a}{a-m+1}\binom{n}{k}] \times \binom{n}{w}$ with rows indexed by the sets in $\Gamma = \{E_{A,S} \subseteq X \mid A \subseteq U$ with $|U| = a, |A| = a - m + 1, S \in \binom{[n]}{k}\}$, and the columns indexed by the vectors in $X = \{x \in \{0, 1\}^n \mid wt(x) = w\}$. The entry of M at the row indexed by the set $E_{A,S}$ and the column indexed by the vector $x \in X$ is 1 if $x \in E_{A,S}$; and 0 otherwise.

Observe that each row of M has weight $\binom{a-m+1}{1}\binom{n-k}{w-c}$, and each column of M has weight $\binom{w}{c}\binom{n-k}{w-c}\binom{a-1}{(a-m+1)-1}$. By the extension of the Stein-Lovász Theorem, there exists a submatrix M of M of order $[\binom{a}{a-m+1}\binom{n}{k}] \times t$ with each row weight at least $(a - m + 1)(z - 1) + 1$, where

$$\begin{aligned} t &\leq \frac{[(a - m + 1)(z - 1) + 1]\binom{n}{w}}{(a - m + 1)\binom{n-k}{w-c}}\{1 + \ln[\binom{w}{c}\binom{n-k}{w-c}\binom{a-1}{a-m}]\} \\ &= \frac{[(a - m + 1)(z - 1) + 1]\binom{a}{a-m+1}\binom{n}{k}}{\binom{w}{c}\binom{n-w}{k-c}\binom{a-1}{a-m}}\{1 + \ln[\binom{w}{c}\binom{n-k}{w-c}\binom{a-1}{a-m}]\}. \end{aligned}$$

Note that the equality is obtained by counting the number of 1s in M in two ways.

It suffices to show that the matrix M^* of order $t\times n$ formed by the columns of M is a $(k,m,c,n;z)$-selector, that is, any submatrix of k arbitrary columns of M^* contains at least z disjoint submatrices with each row weight exactly c, with at least m distinct rows each.

Let $x_1,x_2,...,x_t$ be the t rows of M^* and let $T=\{x_1,x_2,...,x_t\}$. Suppose contradictorily that there exists a set $S\in\binom{[n]}{k}$ such that the submatrix $M^*|_S$ of M^* contains at most $z-1$ disjoint submatrices with each row weight exactly c, with at least m distinct rows each. Moreover, $M^*|_S$ contains another disjoint submatrix with at most $m-1$ distinct rows with weight exactly c. Let $u_{j_1},u_{j_2},...,u_{j_q}$ be such rows, with $q\le m-1$; let A be any subset of $U\backslash\{u_{j_1},u_{j_2},...,u_{j_q}\}$ of cardinality $|A|=a-m+1$, then we have $|T\bigcap E_{A,S}|\le(a-m+1)(z-1)+1$, contradicting the fact that M is a matrix of order $[\binom{a}{a-m+1}\binom{n}{k}]\times t$ with each row weight at least $(a-m+1)(z-1)+1$. Hence we have

$$t_s(k,m,c,n;z)\le\frac{[(a-m+1)(z-1)+1]\binom{a}{a-m+1}\binom{n}{k}}{\binom{w}{c}\binom{n-w}{k-c}\binom{a-1}{a-m}}\{1+\ln[\binom{w}{c}\binom{n-k}{w-c}\binom{a-1}{a-m}]\}.$$

This bound can be simplified further as shown below. Let $n'\ge n$ be the smallest positive integer such that $w=\dfrac{n'c}{k}$ is an integer. We have

$$\begin{aligned}\frac{\binom{a}{a-m+1}\binom{n'}{k}}{\binom{w}{c}\binom{n'-w}{k-c}\binom{a-1}{a-m}}&=\frac{\frac{a!}{(m-1)!(a-m+1)!}}{\frac{(a-1)!}{(m-1)!(a-m)!}}\cdot\frac{\binom{n'}{k}}{\binom{w}{c}\binom{n'-w}{k-c}}=\frac{1}{a-m+1}\cdot\frac{a\binom{n'}{k}}{\binom{w}{c}\binom{n'-w}{k-c}}\\&=\frac{1}{a-m+1}\cdot\frac{\binom{k}{c}\binom{n'}{k}}{\binom{w}{c}\binom{n'-w}{k-c}}\le\frac{1}{a-m+1}(\frac{k}{s})^s(1+\frac{1}{k-s})^{k-s}\end{aligned}$$

by taking $s=r=c$ in Lemma 2.3, and

$$\ln[\binom{w}{c}\binom{n'-k}{w-c}\binom{a-1}{a-m}]=\ln[\binom{w}{c}\binom{n'-k}{w-c}]+\ln\binom{a-1}{a-m}<k[1+\ln(\frac{n}{k}+1)]+\ln\binom{a-1}{a-m}$$

by taking $s=r=c$ in Lemma 2.4. Therefore, we have

$$\begin{aligned}t_s(k,m,c,n;z)\le&t_s(k,m,c,n';z)\\\le&\frac{[(a-m+1)(z-1)+1]\binom{a}{a-m+1}\binom{n'}{k}}{\binom{w}{c}\binom{n'-w}{k-c}\binom{a-1}{a-m}}\{1+\ln[\binom{w}{c}\binom{n'-k}{w-c}\binom{a-1}{a-m}]\}\\<&\frac{(a-m+1)(z-1)+1}{a-m+1}(\frac{k}{s})^s(1+\frac{1}{k-s})^{k-s}\{1+k[1+\ln(\frac{n}{k}+1)]+\ln\binom{a-1}{a-m}\}\end{aligned}$$

as required.

□

Corollary 4.2.1

1. $$\begin{aligned}t_s(k,m,n)&=t_s(k,m,1,n;1)\\&<\frac{k}{k-m+1}(1+\frac{1}{k-1})^{k-1}\{1+k[1+\ln(\frac{n}{k}+1)]+\ln\binom{k-1}{k-m}\}.\end{aligned}$$
2. $$\begin{aligned}t_s(k,m,n;z)&=t_s(k,m,1,n;z)\\&<\frac{(k-m+1)(z-1)+1}{k-m+1}(1+\frac{1}{k-1})^{k-1}\{1+k[1+\ln(\frac{n}{k}+1)]+\ln\binom{k-1}{k-m}\}.\end{aligned}$$
3. $$\begin{aligned}t_s(k,m,c,n)&=t_s(k,m,c,n;1)\\&<\frac{1}{a-m+1}(\frac{k}{s})^s(1+\frac{1}{k-s})^{k-s}\{1+k[1+\ln(\frac{n}{k}+1)]+\ln\binom{a-1}{a-m}\}.\end{aligned}$$

Moreover, we will give a better, but nonconstructive upper bound for the size of $(k, m, n; z)$-selectors in the following.

Lemma A $(k, m, n; z)$-selector is $(m-1, k-m+1; z)$-disjunct.

Proof. Suppose not. Then there exist k columns $C_1, C_2, ..., C_k$ such that

$$|\bigcup_{i=m}^{k} C_i \setminus \bigcup_{i=1}^{m-1} C_i| \leq z-1,$$

that is, there exist at most $z-1$ rows with row weight 1 such that each of them hits exactly one of the columns $C_m, C_{m+1}, ..., C_k$. Hence there exist at most $z-1$ disjoint $m \times k$ submatrices of the identity matrix I_k, it contradicts that M is a $(k, m, n; z)$-selector.

□

By the above lemma, an upper bound for the sizes the sizes of $(k, m, n; z)$-selectors can be obtained from that of $(m-1, k-m+1; z)$-disjunct matrices. Hence a nonconstructive upper bound (3.10, 36/42) of the sizes of $(k, m, n; z)$-selectors is given below. Put $(d, r) = (m-1, k-(m-1))$ in $t(n, d, r; z)$ as shown in Corollary 4.1.1, we have

$$t_s(k, m, n; z) < z\frac{k}{k-m+1}(1+\frac{1}{k-1})^{k-1}\{1+k[1+\ln(\frac{n}{k}+1)]+\ln\binom{k-1}{m-1}\}.$$

For the case $z = 1$, i.e.,

$$t_s(k, m, n; 1) < \frac{k}{k-m+1}(1+\frac{1}{k-1})^{k-1}\{1+k[1+\ln(\frac{n}{k}+1)]+\ln\binom{k-1}{m-1}\}$$

was given in Corollary 4.2.1 followed by a constructive proof in term of the Stein-Lovász theorem.

Some upper bounds for $t(d, n)$, $t(n, d, r]$, $(n, d, r; z]$, $t(n, d, r)$, $t(n, d, r, s]$, $t_s(k, m, n)$ are survey in the following.

1. d-disjunct
   $t(d, n) \leq (d+1)(1+\frac{1}{d})^d\{1+\ln[(d+1)(\binom{n}{d+1}-\binom{n-d-1}{d+1})]\}$ [Y02]
2.
3. $(d, r]$-disjunct
   $t(n, d, r] \leq (1+\frac{d}{r})^r(1+\frac{r}{d})^d\{1+\ln[\binom{n}{d}\binom{n-d}{r}-\binom{n-(d+r)}{d}\binom{n-(d+r)-d}{r}]\}$ [Y07]
4. $(d, r; z]$-disjunct
   $t(n, d, r; z] < z(1+\frac{d}{r})^r(1\frac{r}{d})^d\{1+k[1+\ln(\frac{n}{k}+1)]\}$, $k = d+r$ [CFH08]
5. (d, r)-disjunct
   $t(n, d, r) \leq (1+\frac{d}{r})(1+\frac{r}{d})^d\{1+\ln[\binom{n}{d}\binom{n-d}{r}-\binom{n-(d+r)}{d}\binom{n-(d+r)-d}{r}]\}$ [Y07]
6.
7. $(d, s$ out of $r]$-disjunct
   $t(n, d, r, s] \leq \frac{1+\ln[\binom{n}{d}\binom{n-d}{r}-\binom{n-(d+r)}{d}\binom{n-(d+r)-d}{r}]}{f_{d,r,s}(p)}$ for all $0 < p < 1$,
   where $f_{d,r,s}(p) = (1-p)^d[1-\sum_{i=0}^{s-1}\binom{r}{i}p^i(1-p)^{r-i}]$ [Y07]

8.

9. (k, m, n)-selector

$$t_s(k, m, n) \leq \frac{ek^2}{k-m+1} \ln \frac{n}{k} + \frac{ek(2k-1)}{k-m+1} \text{ [BGV05, by SLT]}$$

$$t_s(k, m, n) \leq \frac{m}{\binom{k}{m} m!} [k(1 + \frac{1}{k-1})^{k-1}]\{1 + \ln[\binom{n}{k} - \binom{n-k}{k}]\} \text{ [Y07, by LLL]}$$

10.

11. (k, m, w, n)-selector

$t_s(k, m, w, n) \leq$ [B08]

12.

4.2 Splitting systems, separating systems, covering designs and lotto designs

An upper bound for $SP(m, t)$ was given in [DSW04] by Lovász local lemma, and upper bounds for $SP(m, t)$, $C(v, k, t)$, $L(v, k, t, p)$ were given in [DZR08] by the Stein-Lovász theorem.

Upper bounds for $SP(m, t; z)$, $SE(m, t_1, t_2; z)$, $C(v, k, t; z)$ and $L(v, k, t, p; z)$ will be derived by using the extension of the Stein-Lovász theorem in the following theorems 4.3 $\sim$ 4.6.

Theorem 4.3

$$SP(m, t; z) \leq \frac{z\binom{m}{\frac{m}{2}}}{\binom{t}{\frac{t}{2}}\binom{m-t}{\frac{m}{2}-\frac{t}{2}}}\{1 + \ln[\binom{\frac{m}{2}}{\frac{t}{2}}^2]\}.$$

Proof. Let A be the binary matrix of order $\binom{m}{t} \times \binom{m}{\frac{m}{2}}$ with rows and columns indexed by $\{T \mid T \in \binom{[m]}{t}\}$ and $\Gamma = \{B \mid B \in \binom{[m]}{\frac{m}{2}}\}$ respectively. The entry of A at the row indexed by the T and the column indexed by the vector $B \in \Gamma$ is 1 if B splits T; and 0 otherwise.

Observe that each row of A has weight $v = \binom{t}{\frac{t}{2}}\binom{m-t}{\frac{m}{2}-\frac{t}{2}}$, and each column of A has weight $a = \binom{\frac{m}{2}}{\frac{t}{2}}\binom{\frac{m}{2}}{\frac{t}{2}} = \binom{\frac{m}{2}}{\frac{t}{2}}^2$. By the extension of the Stein-Lovász Theorem, there exists a submatrix M of A of order $[\binom{m}{t} + \binom{m}{t_1}\binom{m-t_1}{t_2}] \times N$ with each row weight at least z, where

$$N \leq \frac{z\binom{m}{\frac{m}{2}}}{v}\{1 + \ln a\}.$$

It is straightforward to show that the columns of M form an uniform $(m, t; z)$-splitting system with N blocks, as required.

□

In the following theorem we only discuss results where $t_1 \neq t_2$. The cases when $t_1 = t_2$ can be handled in a similar way.

Theorem 4.4

$$SE(m, t_1, t_2; z) \leq \frac{z\binom{m}{\frac{m}{2}}}{2\binom{m-(t_1+t_2)}{\frac{m}{2}-t_1}}\{1 + \ln[2\binom{\frac{m}{2}}{t_1}\binom{\frac{m}{2}}{t_2}]\}.$$

Proof. Let A be the binary matrix of order $[\binom{m}{t_1}\binom{m-t_1}{t_2}] \times \binom{m}{\frac{m}{2}}$ with rows and columns indexed by $\{(T_1, T_2) \mid T_1 \in \binom{[m]}{t_1}, T_2 \in \binom{[m]}{t_2}$ with empty $T_1 \bigcap T_2\}$ and $\Gamma = \{B \mid B \in \binom{[m]}{\frac{m}{2}}\}$ respectively. The entry of A at the row indexed by the pair (T_1, T_2) and the column indexed by the vector $B \in \Gamma$ is 1 if B separates the pair (T_1, T_2); and 0 otherwise.

Observe that each row of A has weight $v = \binom{m-(t_1+t_2)}{\frac{m}{2}-t_1} + \binom{m-(t_1+t_2)}{\frac{m}{2}-t_2} = 2\binom{m-(t_1+t_2)}{\frac{m}{2}-t_1}$, and each column of A has weight $a = \binom{\frac{m}{2}}{t_1}\binom{\frac{m}{2}}{t_2} + \binom{\frac{m}{2}}{t_2}\binom{\frac{m}{2}}{t_1} = 2\binom{\frac{m}{2}}{t_1}\binom{\frac{m}{2}}{t_2}$. By the extension of the Stein-Lovász Theorem, there exists a submatrix M of A of order $[\binom{m}{t_1}\binom{m-t_1}{t_2}] \times N$ with each row weight at least z, where

$$N \leq \frac{z\binom{m}{\frac{m}{2}}}{v}\{1 + \ln a\}.$$

It is straightforward to show that the columns of M form an uniform $(m, t_1, t_2; z)$-splitting system with N blocks, as required.

□

Theorem 4.5

$$C(v, k, t; z) \leq \frac{z\binom{v}{t}}{\binom{k}{t}}\{1 + \in \binom{k}{t}\}.$$

Proof. Let A be the binary matrix of order $\binom{v}{t} \times \binom{v}{k}$ with rows and columns indexed by $\{T \mid T \in \binom{[v]}{t}\}$ and $\Gamma = \{B \mid B \in \binom{[v]}{k}\}$ respectively. The entry of A at the row indexed by the T and the column indexed by the vector $B \in \Gamma$ is 1 if $T \subseteq B$; and 0 otherwise.

Observe that each row of A has weight $\binom{v-t}{k-t}$, and each column of A has weight $\binom{k}{t}$. By the extension of the Stein-Lovász Theorem, there exists a submatrix M of A of order $\binom{v}{t} \times N$ with each row weight at least z, where

$$N \leq \frac{z\binom{v}{k}}{\binom{v-t}{k-t}}\{1 + \ln \binom{k}{t}\} = \frac{z\binom{v}{t}}{\binom{k}{t}}\{1 + \ln \binom{k}{t}\}.$$

Note that the equality is obtained by counting the number of 1s in A in two ways. It is straightforward to show that the columns of M form an $(v, k, t; z)$-covering design with N blocks, as required.

□

Theorem 4.6

$$L(v, k, t, p; z) \leq \frac{z\binom{v}{k}}{\sum_{i=p}^{\min(t,k)} \binom{t}{i}\binom{v-t}{k-i}}\{1 + \ln[\sum_{i=p}^{\min(t,k)} \binom{k}{i}\binom{v-k}{t-i}]\}.$$

Proof. Let A be the binary matrix of order $\binom{v}{t} \times \binom{v}{k}$ with rows and columns indexed by $\{T \mid T \in \binom{[v]}{t}\}$ and $\Gamma = \{B \mid B \in \binom{[v]}{k}\}$ respectively. The entry of A at the row indexed by the T and the column indexed by the vector $B \in \Gamma$ is 1 if $|T \bigcap B| \geq p$; and 0 otherwise.

Observe that each row of A has weight $\sum_{i=p}^{\min(t,k)} \binom{t}{i}\binom{v-t}{k-i}$, and each of its column has weight $\sum_{i=p}^{\min(t,k)} \binom{k}{i}\binom{v-k}{t-i}$. By the extension of the Stein-Lovász Theorem, there exists a submatrix M of A of order $\binom{v}{t} \times N$ with each row weight at least z, where

$$N \leq \frac{z\binom{v}{k}}{\sum_{i=p}^{\min(t,k)} \binom{t}{i}\binom{v-t}{k-i}}\{1 + \ln[\sum_{i=p}^{\min(t,k)} \binom{k}{i}\binom{v-k}{t-i}]\}.$$

It is straightforward to show that the columns of M form an $(v, k, t, p; z)$-lotto design with N blocks, as required.

□

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