1. Introduction

We consider the following lattice dynamical system for unknown $u = \{u_j\}_{j\in\mathbb{Z}}$:

$$u_j' = d_{j+1}u_{j+1} + d_j u_{j-1} - (d_{j+1} + d_j)u_j + f_j(u_j),\ j \in \mathbb{Z}, \tag{1.1}$$

where $f_j \in C^{1+\alpha}[0,1]$ for some $\alpha \in (0,1)$ for $j \in \mathbb{Z}$, $f_{j+N} = f_j$ and $d_{j+N} = d_j > 0$ for all $j \in \mathbb{Z}$ for some positive integer N. The equation (1.1) can be regarded as a spatial discrete version of the following reaction-diffusion equation

$$u_t = (d(x)u_x)_x + f(x,u),$$

where $d(x)$ and $f(x,u)$ are periodic in x. In biology, let u_j denote the density of a certain species in a periodic patchy environment. Assuming the species at site j can only interact with those at the nearby sites, then the equation (1.1) describes the rate of change of density of this species at each site j. It is equal to the sum of the source $f_j(u_j)$ at site j and the fluxes $q_{j\pm1}$ from sites $j \pm 1$ to site j:

$$q_{j+1} := d_{j+1}[u_{j+1} - u_j],\ q_{j-1} := d_j[u_{j-1} - u_j],$$

where d_j, d_{j+1} are the diffusion constants. See [8, 15, 16] for more references and details.

It is trivial that for a given initial data $\{u_j(0)\} \in [0,1]$ there exists a unique solution u to (1.1) for $t \geq 0$ such that $0 \leq u_j(t) \leq 1$ for all $t \geq 0$ and $j \in \mathbb{Z}$. We are interested in the wave propagation phenomenon. In particular, we are interested in special solutions U of (1.1) for $t \in \mathbb{R}$ satisfying the following conditions:

$$U_j(t + N/c) = U_{j-N}(t),\ t \in \mathbb{R}, j \in \mathbb{Z}, \tag{1.2}$$

$$U_j(t) \to 1 \text{ as } j \to -\infty,\ \ U_j(t) \to 0 \text{ as } j \to +\infty, \quad \text{locally in } t \in \mathbb{R}, \tag{1.3}$$

for some nonzero constant c. We shall call a solution (c, U) of (1.1)-(1.3) as a traveling wave solution. The constant c is the wave speed and U is the profile. In this paper, we shall always assume that

$$f_j(0) = f_j(1) = 0 \quad \forall\ j \in \mathbb{Z}. \tag{1.4}$$

The study of traveling wave for lattice dynamical system has attracted a lot of attention for past years, see, e.g., the works [1, 2, 3, 4, 6, 7, 10, 11, 12, 13, 14, 17, 18, 19, 20]. The main concerns are existence, uniqueness, and stability of traveling waves. Typically, there are two different nonlinearities, namely, monostable and bistable cases. In the monostable case, we have

$$f_j'(1) < 0 < f_j'(0)\ \forall\ j \in \mathbb{Z}, \quad f_j(s) > 0\ \forall\ s \in (0,1), j \in \mathbb{Z}. \tag{1.5}$$

For the bistable case, we have $f_j'(0) < 0$ and $f_j'(1) < 0$ for all $j \in \mathbb{Z}$. If $N = 1$, then $f_{j+1} = f_j$ and $d_{j+1} = d_j$ for all j. This is the so-called homogeneous media case. In general, if $N > 1$, then it is called the periodic case.

In this paper, we shall focus on the periodic monostable case. We refer the reader to the work [5] and the references cited therein for the periodic bistable case. In [5], the existence, uniqueness and stability of traveling waves for periodic bistable case are studied in details.

The existence of traveling waves for monostable case in periodic media was first obtained by Hudson and Zinner [11, 12] under the extra assumption

$$f_j'(0)s - Ms^{1+\alpha} \leq f_j(s) \leq f_j'(0)s,\ \forall\ s \in [0,1], j \in \mathbb{Z}, \tag{1.6}$$

for some constants $M > 0$ and $\alpha \in (0, 1)$. Recently, one of the authors and Hamel [10] gave a different approach to prove the existence of traveling waves for all speeds $c \geq c^*$ for some positive minimal speed c^*. Moreover, it is also shown in [10] that the condition $c \geq c^*$ is not only a sufficient condition but also a necessary condition for the existence of traveling waves.

For reader's convenience, we recall some properties of traveling wave from [10]. Let (c, U) be a traveling wave solution of (1.1)-(1.3) with $c \neq 0$. Then we have $0 < U_j(t) < 1$ for all $(j, t) \in \mathbb{Z} \times \mathbb{R}$; $U_j(t) \to 0$ as $t \to -\infty$; $U_j(t) \to 1$ as $t \to \infty$; $U_j'(t) > 0$ for all $t \in \mathbb{R}$ and $U_j'(t) \to 0$ as $t \to \pm\infty$.

The aim of this paper is to study the uniqueness and stability of traveling waves in the periodic monostable case. Hence we shall always assume that (1.4), (1.5) and (1.6) hold.

Recall from [10] that for each $\lambda \in \mathbb{R}$ there exists a unique $v = \{v_j\}$ with $\max_{j\in\mathbb{Z}} v_j = 1$ and $v_{j+N} = v_j > 0$ for all $j \in \mathbb{Z}$ such that

$$M(\lambda)v_j = d_{j+1}e^{-\lambda}v_{j+1} + d_j e^{\lambda} v_{j-1} - (d_{j+1} + d_j)v_j + f_j'(0)v_j \tag{1.7}$$

for all $j \in \mathbb{Z}$, where $M(\lambda)$ is the largest eigenvalue of (1.7). Moreover, there exists $\lambda^* > 0$ such that $c^* = M(\lambda^*)/\lambda^*$ and the mapping $c = M(\lambda)/\lambda : (0, \lambda^*) \mapsto c \in (c^*, \infty)$ is strictly decreasing.

We shall focus our attention on those traveling waves (c, U), $c > c^*$, of (1.1)-(1.3) satisfying

$$\lim_{j-ct\to\infty} \frac{U_j(t)}{e^{-\lambda(j-ct)}v_j} = 1, \tag{1.8}$$

for some $\lambda > 0$ such that $M(\lambda) = c\lambda$ and $\{v_j\}$ is the unique eigenvector of (1.7) corresponding to λ such that $\max_{j\in\mathbb{Z}} v_j = 1$ and $v_{j+N} = v_j > 0$ for all $j \in \mathbb{Z}$.

We now state our stability theorem as follows.

Theorem 1.1. *Suppose that there exists a traveling wave (c, U) with $c > c^*$ such that (1.8) holds for some $\lambda \in (0, \lambda^*)$. Let u be the solution of (1.1) for $t \geq 0$ with the initial value $\{u_j(0)\}$ satisfying*

$$0 \leq u_j(0) \leq 1, \quad u_j(0) \leq e^{-\lambda \cdot j} v_j \quad \forall j \in \mathbb{Z}, \tag{1.9}$$

$$\liminf_{j\to-\infty} u_j(0) > 0, \quad \lim_{j\to\infty} \frac{u_j(0)}{e^{-\lambda \cdot j} v_j} = 1. \tag{1.10}$$

Then

$$\lim_{t\to\infty} \sup_j \{|[u_j(t)/U_j(t)] - 1|\} = 0.$$

The proof of Theorem 1.1 is based on a method in [3] with some nontrivial modifications. In [3], a lattice dynamical system in homogeneous media is studied. There the proof of stability theorem is through a related continuum equation by extending the spatial variable from $j \in \mathbb{Z}$ to $x \in \mathbb{R}$. But, here we shall only use the original equation (1.1) to prove the stability theorem. Moreover, there is only one wave profile for the homogeneous case in [3]. In our periodic lattice dynamical system, there are N wave profiles. This makes the stability analysis more complicated. To overcome this difficulty, we introduce the following transformation

$$W_j(x) := U_j([j - x]/c), \quad \text{equivalently} \quad U_j(t) = W_j(j - ct), \tag{1.11}$$

which is very useful in the periodic framework. Indeed, this transformation is reminiscent of a similar transformation in the case of partial differential equation (cf. [9]).

By adapting a method used in [4], we have the following uniqueness theorem.

Theorem 1.2. *Suppose that* (c, U) *and* $(c, \overline{U})$ *are two traveling wave solutions of (1.1)-(1.3) such that*

$$\lim_{j-ct\to\infty} \frac{U_j(t)}{e^{-\lambda(j-ct)}v_j} = h, \quad \lim_{j-ct\to\infty} \frac{\overline{U}_j(t)}{e^{-\lambda(j-ct)}v_j} = \bar{h} \tag{1.12}$$

for some positive constants λ, h *and* $\bar{h}$ *such that* $M(\lambda) = c\lambda$, *where* $\{v_j\}$ *is the eigenvector of (1.7) corresponding to* λ *such that* $v_j = v_{j+N} > 0$ *for all* j *and* $\max\{v_j\} = 1$. *Then there exists* $\xi \in \mathbb{R}$ *such that* $U_j(t) = \overline{U}_j(t + \xi)$ *for all* $j \in \mathbb{Z}$, $t \in \mathbb{R}$.

This paper is organized as follows. We shall give the proof of Theorem 1.1 in Section 2. The proof of Theorem 1.2 is given in Section 3. In this paper, we shall use both functions U_j and W_j define in (1.11) alternatively from time to time.

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