Catalan Random Walks and Flights

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The Catalan random walk is considered as a non-Gaussian but stable random walk. The stochastic equivalence, the distribution function and the stochastic equation are constructed by a ${}^nZ$ fractal theory. A general formula has been derived to reexamine the possibility of reconstructing the foundation of statistical physics.

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Recently, ultraslow Gaussian converging stochastic process and the growth rate of value distribution problem have been studied by random walks of nonequilibrium systems in physics [1] and iterated function analysis of dynamical systems in mathematics [2]. The Levy walk is the one considered as stable non-Gaussian random process. The natural question to ask, is that "are there any other non-Gaussian random processes interesting to us?" We propose that Catalan random walk is playing the role.

Levy walk is a random walk performed by visiting the same sites of Levy flight. Instantaneous jumps are responsible for the infinite variances which are not allowed, and a time sequence is introduced so that long steps are penalized. One possible resolution of the paradox of infinite variance is provided by Levy walks. Levy walks have infinite variance and can be differentiated, but this resolution applies only when a spatio-temporal coupling is present. The paradox of infinite variance may also be resolved by introducing a variance of the Levy flight which does not require the hypothesis of a spatio-temporal coupling.

In physical systems, the variance of any stationary processes is finite, so that a Gaussian behavior is expected in the absence of long range correlation. Gaussian behavior arises from the central limit theorem (CLT) which is fundamental to statistical mechanics. It states that the sum

$$Y_n \equiv \sum_{i=1}^{n} X_i$$

has a finite variance and converges to a normal (Gaussian) stochastic process as $n \to \infty$. This statement is basically under the assumption that n stochastic variables $\{X_i\}_{i=1}^{n}$ are statistically independent, identically distributed and normal [3]. On the other hand. Lévy flights which have infinite variance or a related process called Levy walks have been observed experimentally in fluid dynamics and polymers and have been used to describe subrecoil laser cooling, turbulent fluids, very stiff polymers and the spectral random walk of a single molecule embedded in a solid [4]. Especially, the storage ring design for synchrotron radiation and thermal nuclear fusion reactor design for plasma confinement are also in this process.

. In these physical systems, an unavoidable cutoff is always present. For example, in the case of a single molecule embedded in a solid, due to the minimal length between the molecules and the nearest two-level systems, a cutoff is present in the distribution of the jumps of the resonance frequency. The jump of the frequency is induced by the thermal transitions occurring in the surrounding two-level systems. Multi-level colored systems can also be obtained by the p-adic analysis of a set of functional equations [5]. A second example is turbulence: in numerical simulations performed using Lévy walks in a Boltzmann lattice gas, the jumps of the particles are limited by the finite size of the simulated system [6,7]. A third example is the irregular nerve impulse distribution calculated for two different three-state models which describe the GABA-ergic inhibitory postsynaptic currents from melanotropes of xenopus iaevis [8].

In this note, rather, another kind of random walk so called Catalan random walk has been considered [9,10]. Catalan walk is as a random walk performed by visiting the lattice sites of coordinates of the point at which it is measured. The coordinates is in the Catalan number system $\{1, 2, 5, 14, 42, \cdots\}$ which is the cardinality of the set of cluster sets. These numbers are the possible combinations of the ${}^nZ$ sets with different n, where ${}^nZ$ is the n-th hyperpower of Z, i.e. ${}^nZ \equiv \overbrace{Z^{Z^{\cdot^{\cdot^{Z}}}}}^{n}$.

There are three prototypical experiments that one may suggest in physics issues. (1) The anomalous crystal growth in dusty plasma under gravitational potential along with the $1/{}^nr, n \geq 2$, potentials. (2) The diffusion coefficient or the conductivity measurement on the $\sum c_i({}^nx_i)$ lattice sites. (3) The computational experiment to determine the universality and critical exponent of the "nonstandard map"

$$X_{i+1} = X_i + \sum C_i({}^n\sin 2\pi\theta_i)$$

$$\theta_{i+1} = \theta_i + X_{i+1}$$

The enumeration of the probability is then measured in the Catalan number system.

The afored mentioned physical systems should also exist the same paradox of infinite variance. It may also be resolved by introducing a variance of the Catalan flight which is termed the truncated Catalan flight.

However, the mathematical structure is involved. It is worth while to detail the analysis in the first place. For the sake of simplicity, the set theoretical language has been used [11]. Since the full analysis is in the complex domain, the foundation is on the two dimensional real space. This study is to provide eventually a rigorous and unified exposition of the theory on complex dynamical system.

In order to construct the stochastic equation, three mean results are presented therein. namely, the stochastic equivalence, distribution function up to stochastic periodicity and the valuation of Schwarzian derivatives.

Let $S \stackrel{stoch}{=} T$ denote the stochastic equivalence if the given random variables S and T have the same distribution.

Clearly ${}^nZ$ the n-th hyperpower of Z is the extreme growth rate for statistical systems. As an example, in ${}^2Z$ stochasticity, let $t \in C$, X_t and $X_{t+\frac{1}{{}^nZ}}$ be independent variates. p may be considered as a prime number so that one can define random variable $X_t + \frac{1}{{}^np}$ on which the growth rate in time is being estimated. Then we have

$$(p^p)^{(p^p)} \prod_{i=0}^{p^p-1} X_{t+\frac{1}{p^p}} \stackrel{stoch}{=} X^{p^p}_{(p^p)t} \tag{1}$$

with

$$(p^p)^{(p^p)t-1}\Gamma(t)\left(t\frac{1}{p^p}\right)\frac{1}{\quad} = \Gamma\Big((p^p)t\Big)\Gamma\left(\frac{1}{p^p}\right)$$

and for independent Γ multivariates $X_t, X_{t+\frac{1}{p}}, \ldots, X_{t+\frac{p^p-1}{p^p}}$

$$(p^p)^{(p^p)} \prod_{i=0}^{p^p-1} X_{t+\frac{i}{p^p}} \stackrel{stoch}{=} \Gamma\Big((p^p)t\Big) X^{p^p}_{(p^p)t} \tag{2}$$

with

$$(p^p)^{(p^p)t-\frac{1}{2}} \prod_{i=0}^{p^p-1} \Gamma\left(t+\frac{i}{p^p}\right) = (2\pi)^{((p^p)-1)/2}\Gamma\Big((p^p)t\Big).$$

It can be derived as follows. Since

$$p^p \log(X_{(p^p)t}) \stackrel{stoch}{=} -(p^p)\gamma + c + \sum_{i=0}^{p^p-1}\sum_{j=0}^{\infty}\left[\frac{1}{j+1} - \frac{Y_{(p^p)j+i}}{j+\frac{i}{p^p}}\right],$$

where γ is the Euler constant (3)

$$\stackrel{stoch}{=} c + \sum_{i=0}^{p^p-1} \log\left(X_{t+\frac{i}{p^p}}\right)$$

with independent summands on the right. By taking the expectation, one can show that the univeral constant in the Eq. (3) is $e^c = (p^p)^{(p^p)}$.

For $t > 0$

$$\begin{aligned} e^c \prod_{i=0}^{p^p-1} \left[t + \frac{i}{p^p}\right] &= e^c E\left\{\prod_{i=0}^{p^p-1} X_{t+\frac{i}{p^p}}\right\} \\ &= E\left\{X^{p^p}_{(p^p)t}\right\} \\ &= \Gamma\Big((p^p)t + p^p\Big)/\Gamma\Big((p^p)t\Big) \\ &= \prod_{i=0}^{p^p-1}\Big((p^p)t + i\Big) \end{aligned} \tag{4}$$

Then, applying the Stirling's formula to both sides of the Eq. **(1)** and setting $t=\frac{1}{p^p}$, again, take the expectation of the same equation to obtain

$$E\left\{\left[(p^p)^{(p^p)}\right]^r \prod_{i=1}^{p^p} X_{i/p^p}\right\} = E\{X_1^{(p^p)r}\}$$

and

$$(p^p)^{(p^p)r} \prod_{i=1}^{p^p} \Gamma\left(r+\frac{i}{p^p}\right) = b\Gamma\left((p^p)r+1\right) \tag{5}$$

for all $r>0$ when $r\to\infty$. Applying the equality $\Gamma(x+1) = x\Gamma(x)$, the normalization constant b arrives,

$$b = (2\pi)^{(p^p-1)/2}(p^p)^{-1/2}$$

$$(p^p)^{(p^p)t-\frac{1}{2}} \prod_{i=0}^{p^p-1} \Gamma\left(t+\frac{i}{p^p}\right) = (2\pi)^{((p^p)-1)/2}\Gamma\left((p^p)t\right).$$

A function is said to be transcendentally transcendental if it satisfies no algebraic differential equation. The gamma function is proved to be transcendentally transcendental [12]. Here, we show that it indeed satisfies a functional equation if the random variable is chosen to be $X_{t+\frac{1}{n_p}}$.

It is important to observe that ${}^3Z$ can have two expressions $(Z^Z)^Z$ or $Z^{(Z^Z)}$. Then, the result follows immediately,

$$\left((p^p)^p\right)^{((p^p)^p)} \prod_{i=0}^{(p^p)^p-1} X_{t+\frac{i}{(p^p)^p}} \overset{stoch}{=} \Gamma((p^p)^p t) X^{(p^p)^p}_{((p^p)^p t)} \tag{6}$$

with

$$\left((p^p)^p\right)^{((p^p)^p)t-\frac{1}{2}} \prod_{i=0}^{(p^p)^p-1} \Gamma\left(t + \frac{i}{p^p}\right) = (2\pi)^{((p^p)^p-1)/2}\Gamma\left((p^p)^p t\right)$$

or

$$[p^{(p^p)}]^{[p^{(p^p)}]} \prod_{i=0}^{p^{(p^p)}-1} X_{t+\frac{i}{p^{(p^p)}}} \overset{stoch}{=} \Gamma(p^{(p^p)}t) X^{p^{(p^p)}}_{(p^{(p^p)}t)} \tag{7}$$

with

$$[p^{(p^p)}]^{[p^{(p^p)}]t-\frac{1}{2}} \prod_{i=0}^{p^{(p^p)}-1} \Gamma\left(+\frac{i}{p^{(p^p)}}\right) = (2\pi)^{((p^{(p^p)})-1)/2}\Gamma(p^{(p^p)}t).$$

It gives two initial categories to describe the universality of ${}^3Z$-system, so that the infinite-${}^nZ$-system universality can be obtained [13].

In general, the periodic stochasticity still preserves the Γ distribution. With the aid the Catalan table and the iterative scheme, one can resolve the paradox of infinite variance. Consequently,
if

$$f(z) = {}^n z \equiv {}^n p,\ f^k(z) \equiv \underbrace{f \cdot f \cdot f \cdots f({}^n p)}_{k},\ k \in Z^+,$$

then the stochastic equivalence is in the form

$$(f^k({}^n p))^{(f^k({}^n p))} \prod_{i=0}^{f^k({}^n p)-1} X_{t+\frac{i}{f^k({}^n p)}} \overset{stoch}{=} X^{f^k({}^n p)}_{(f^k({}^n p))t} \tag{8}$$

with

$$= (f^k({}^n p))^{(f^k({}^n p))t-\frac{1}{2}} \prod_{i=0}^{f^k({}^n p)-1} \Gamma\left(t + \frac{i}{f^k({}^n p)}\right)$$
$$= (2\pi)^{((f^k({}^n p))-1)/2}\Gamma((f^k({}^n p))t).$$

It may interpretate the density distributions in mass, velocity and photon, neutrino, which are intensively measured by several observatories [14,15]. Moreover, the singularities of the gamma function will give the exact location in space-time where the birth of new stars, even our universe, can occur and reoccur.

The two dimensional space-time structure is conveniently expressed by complex numbers. The two initial categories to describe the universality and transcendency of " Z-system can also be analyzed by the classical theory on automorphic forms [16,17].

Let $z \in$ C, the Schwarzian derivative is denoted by S_f,

$$S_f = \frac{2'''}{f'} - \frac{f''}{3}\left(_{f'}\right)^2$$

If $f(z) = {}^2 z$, then $S_f = -\frac{1}{2}(\ln z + 1)^2 - \frac{1}{z^2(\ln z + 1)} - \frac{3}{2}\frac{1}{z^2(\ln z + 1)^2}$
If $f(z) = {}^3 z$, then $S_f = -\frac{1}{2}(2z \ln z + z)^2 - \frac{2}{z(2z \ln z + z)} - \frac{3}{2}\frac{(21nz + 3)^2}{(2z \ln z + z)^2}$

$$\text{or} \quad S_f = -\frac{1}{2}(z^z)^2\left((\text{n } z)^2 + \ln z + \frac{1}{z}\right)^2 - \frac{1}{2}(\ln z + 1)^2 + \frac{1}{z}$$
$$- \frac{\left(\frac{2}{z}(\ln z)^2 + \frac{3}{z}\ \text{In } z + \frac{1}{z} - \frac{2}{z^2} - \frac{1}{z^3}\right)}{\left((\ln z)^2 + \ln z + \frac{1}{z}\right)} - \frac{3}{2}\left[\frac{2\frac{1}{z}\ln z + \frac{1}{z} - \frac{1}{z^2}}{\left((\ln z)^2 + \ln z + \frac{1}{z}\right)}\right]^2$$

It is well known [18,19] that if there exist two linearly indepentent solutions $u_1(z)$ and $u_2(z)$ of the differential equation $u'' + \frac{1}{2}S_f u = 0$, then $f(z) = u_1/u_2$. Moreover, the Cayley identity shows the equality $S_{w\circ f} = S_w f'' + S_f$ and the fact $S_{w\circ f} = S_w$ if w is an automorphic transformation. Therefore, the stochastic equivalent means that the random variables share the same differential equation by the automorphic transformation. Here, $W(z) \equiv (w \circ f)(z) = \frac{a(^n z)+b}{c(^n z)+d}$ will give the stochastic equivalent of Catalan walks. Note that $S_W(^n z)$ for integer n is an invariant which is independent of a, b, c, d especially.

Here there is anothor side in the probabilistic theory point of view. If Z_i belongs to the probability measure space, then the Boltzmann-Gibbs-Kolmogorov-Shannon-Sinai entropy is $S_q = -\sum_i \ln^n Z_i$ with $n = 2$. For arbitrary n with such Γ variates the generalized entropy will be achieved and reconstructed by the above analysis.

In conclusion, starting from the preliminary results on statistical physics with Catalan random walks and flights, we hope to extend the quantum field theory under hypertranscendental action [20-22] and to find more paradigms to verify the theory in the foreseeable future.

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