Report of NSC Project on "Statistical Inferences for Confidence of Percentage γ Acceptable Products in Lot"

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Statistical Inferences for Confidence of Percentage γ Acceptable Products in Lot

1. Introduction

The tolerance interval is popularly used by manufacturer and consumer for judgement of production lots. In mass-production, the manufacturer is interesting in an interval that contains a specified (usually large) percentage of the product and he knows that unless a fixed proportion (say γ) of the production is acceptable in the sense that the items' characteristics conform to specification limits LSL and USL, he will lose money in this production. On the other hand, if this claim is not true, the consumer may loss the money. With this interest, the manufacturer and consumer want to know the following:

$$\textbf{Is there percentage } \gamma \textbf{ of acceptable measurements in a production lot?} \tag{1}$$

Statisticians try to verify this problem through two steps. To begin, suppose that we have a random sample $X = (X_1, ..., X_n)'$ from a distribution with probability density function $f_\theta(x)$ representing observations from tha same process of production. For the first step, the pioneer article by Wilks (1941) introduced a γ-content tolerance interval with confidence $1 - \alpha$ defined as a random interval $(T_1, T_2) = (t_1(X), t_2(X))$ that satisfies

$$P_\theta\{P_\theta(X_0 \in (T_1, T_2)|X) \geq \gamma\} \geq 1 - \alpha \text{ for } \theta \in \Theta$$

where X_0 represents the future observation, also from the same production process. Let (t_1, t_2) be the observation of this tolerance interval. The general rule for verifying a manufacturer's problem using the tolerance interval is as follows:

$$\begin{array}{c}\textbf{If } (t_1, t_2) \subset (LSL, USL), \textbf{the lot of product is acceptable} \\ \textbf{because we have confidence } 1-\alpha \\ \textbf{that at least } 100\gamma\% \textbf{of the population is} \\ \textbf{conforming to specification limits}\end{array} \quad (2)$$

It is desired to study the power of this classical test and, if this classical one is not satisfactory for the manufacturer, we need to investigate if there is alternative one.

2. Research Purpose

To study the appropriateness of tolerance interval in this engineering problem, we want to study the power of the γ content tolerance interval of Eisenhart et al. (1947) with test of (1.3). If this test is not promising for the manufacturer, it is desired to propose a new test for the same purpose of studying if there is γ percentage of acceptable products at confidence $1 - \alpha$. One other importance is the sample size determination problem that guarantees a low probability of acceptance of the hypothesis when the true specification limits are moderately shorter than the desired ones and a large probability of acceptance of the hypothesis when the true specification limits are moderately wider than the desired ones.

3. Literature Review

Much attention has been received for developing tolerance intervals, see For examples Wilks (1941), Wald (1943), Paulson (1943), Guttman (1970) and, for a review, Patel (1986). In general, a common effort been made in the literature is to investigate the version with minimum width, for which Eisenhart et al. (1947) constructed an approximate minimum width tolerance interval for normal random variable. This normal tolerance interval is now popularly implemented in manufacturing industries and is presented in text books of engineering statistics. The interest of this paper is to study if the tolerance interval is appropriate to deal with problem in (1) for the manufacturer and consumer.

4. Research Methods

The probability that a product to be acceptable is

$$p_{item}(\theta) = \int_{LSL}^{USL} f(x,\theta)\,dx = F_\theta(USL) - F_\theta(LSL).$$

Suppose that the lot size is known as constant k (usually a large number). For this production lot, the number of acceptable products is with binomial distribution $b(k, p_{item}(\theta))$. Then the true confidence for having proprotion γ of production lot conforming to specification limits is

$$q = \sum_{i=[k\gamma]}^{k} \binom{k}{i} p_{item}(\theta)^i (1 - p_{item}(\theta))^{k-i}.$$

The interest for a manufacturer is to test the following hypothesis:

$$H^* : q \geq q_0 \tag{3}$$

for some specified (large) value q_0.

The classical approach to test H^* is rule (2). We want to simulate the powers for the Eisenhart et al.'s tolerance interval for several combinations of specification limits. Let's set replication number m and specification limits $(LSL, USL) = (-b, b)$. The simulated power of a tolerance interval (T_1, T_2) is defined as

$$\pi = \frac{1}{m} \sum_{j=1}^{m} I((t_1^j, t_2^j) \subset (-b, b))$$

The simulated power of a tolerance interval (T_1, T_2) is defined as

$$\pi = \frac{1}{m} \sum_{j=1}^{m} I((t_1^j, t_2^j) \subset (-b, b)) \tag{4}$$

A New Test Based on Tolerance Interval

Suppose that we have an appropriate estimate, denoted by $\hat{\theta}$ of parameter θ and then we have estimated probability density function of characteristic variable X as $f_{\hat{\theta}}(x)$. The rule of the new test for hypothesis H^* of (3) is:

$$\text{Accepting } H^* \text{ if } \int_{(t_1,t_2)\cap(LSL,USL)} f_{\hat{\theta}}(x)dx \geq p_{item}.$$

With this test, the power function is

$$P_\theta\{\int_{(T_1,T_2)\cap(LSL,USL)} f_{\hat{\theta}}(x)dx \geq p_{item}\}.$$

Consider that we have a random sample $X_1, ..., X_n$ drawn from the distribution $N(\mu, \sigma^2)$. Let $\hat{\mu} = \bar{x}$ and $\hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \hat{\mu})^2$. The rule for testing hypothesis H_0 is:

$$\text{Accepting } H_0 \text{ if } \int_{(t_1,t_2)\cap(LSL,USL)} \phi_{\hat{\mu},\hat{\sigma}}(x)dx \geq p_{item},$$

and the empirical power of tolerance interval (T_1, T_2) is

$$\pi_{Spe} = \frac{1}{m}\sum_{j=1}^{m} I(\int_{(t_1^j,t_2^j)\cap(LSL,USL)} \phi_{\hat{\mu},\hat{\sigma}}(x)dx \geq p_{item}).$$

5. Results and Discussions

Power for classical test

To study (4), suppose that the random sample $X_1, ..., X_n$ is drawn from normal distribution $N(\mu, \sigma)$ where both μ and σ are unknown. The general form of a prediction interval for a future normal random variable is of the form

$$(\bar{X} - m^* s, \bar{X} + m^* s)$$

where the $100(1-\alpha)\%$ confidence interval (prediction interval) is the form with $m^* = t_{1-\frac{\alpha}{2}}(n-1)\sqrt{1+\frac{1}{n}}$ and where $t_{1-\frac{\alpha}{2}}(n-1)$ represents the $1-\frac{\alpha}{2}$th quantile of the central t-distribution with degrees of freedom. We select values m^* corresponding with $\gamma = 0.9, 1-\alpha = 0.95$ from the table developed in Eisenhart et al. (1947).

With replication $m = 100,000$, we generate random sample of size n from distribution $N(0,1)$. Let $\bar{X}_j$ and S_j^2 be te sample mean and sample variance for jth sample. We compute this tolerance interval and study its powers of (3.2) with several sample size $n = 20, 30, 50$ and various values b where $b = 1.7184, 1.6686$ and 1.6525 corresponds, respectively, to specification limits such that their true confidences are identical to $1-\alpha = 0.95$. Tables for simulated results are skipped.

We have several comments drawn from the simulated results:

(a) As expected, the power of the tolerance interval is incresing when the specification limits are wider indicating increasing in p_{item}. For $b \geq 1.7184$ with $k = 1,000$, the corresponding confidence $q \geq q_0 = 0.95$, we see that the larger the sample size the more the chance (probability) to accept H^*.

(b) When $b = 1.7184$ for $k = 1,000$, the process does guarantee confidence 0.95 with percentage 0.9 of acceptable products. However, the simulated power values are $0.0257, 0.0289, 0.0466$, respectively, for sample sizes $n = 20, 30, 50$. These revealed little chance to observe that the lots are already $\gamma = 0.9$ percentage of acceptable products at confidence 0.95. Hence, the test of (2.3) is not satisfactory in lossing benefits for the manufacturer.

Results for new test

We conduct a simulation with the same design set in Section 3 to study the power function of this new test. The simulated results for lot sizes, $k = 1,000, 10,000, 100,000$ are done but the results are skipped. However, the tolerance intervals considered including the shortest version (STL) and the version (CITL) developed by Huang, Chen and Welsh (2007).

We have several comments drawn from the results:

(a) Monotone power values are as our expectation. However, the power values are relatively higher than them based on test of (2.4). Hence, there arelarger probabilities in all settings of specification limits and sample sizes for accepting H^*.

(b) There is no significant differences in the performance between the shortest tolerance interval and the version of Huang, Chen and Welsh.

(c) When b is the value (1.7184 for $k = 1,000$, 1.6686 for $k = 10,000$ and 1.6525 for $k = 100,000$) that the true confidence is identical to 0.95 the simulated power values are all close to 0.48. This is interesting indicating that this new test is more capable in our purpose.

6. Judgements for Research Results

The efficiencies of the proposed techniques and the drawbacks of the classical test are introduced in section of results and discussions. Some special points are listed below:

(a) This new test is more powerful than the classical test.

(b) The sample size determination provides an useful technique in achieving the required power.

(c) There is need for developing theoretical results to support this new technique of test.

7. References

Huang, J.-Y., Chen,L.-A. and Welsh, A. (2007). The mode and its application to control charts and tolerance intervals. Submitted to *Statistics and Probability Letter* for publication (In revision).

Eisenhart, C., Hastay, M. W. and Wallis, W. A. (1947). *Techniques of Statistical Analysis.* McGraw-Hill Book Company: New York.

Guttman, I. (1970). *Statistical tolerance regions: Classical and Baysian.* Griffin: London.

Patel, J.K. (1986). Tolerance limits - A review. *Communications in statistics - Theory and Methods*, 15, 2719-2762.

Paulson, E. (1943). A note on tolerance limits. *Annals of Mathematical Statistics*, 14, 90-93.

Poulsen, O.M., Holst, E. & Christensen, J.M. (1997). Calculation and appli cation of coverage intervals for biological reference values (technical repo rt). *Pure and Applied Chemistry*, 69, 1601-1611.

Wald, A. (1943). An extension of Wilks' method for setting tolerance limits. *Annals of Mathematical Statistics*, 14, 45-55.

Wilks, S. S. (1941). Determination of sample sizes for setting tolerance limits. *Annals of Mathematical Statistics*, 12, 91-96.