International Journal of Bifurcation and Chaos, Vol. 21, No. 5 (2011) 1493–1500

DOI: 10.1142/S0218127411029069

PROOF OF SYNCHRONIZED CHAOTIC BEHAVIORS IN COUPLED MAP LATTICES*

WEN-WEI LIN† and YI-QIAN WANG‡
†*Department of Mathematics, National Chiao-Tung University, Hsinchu 30010, Taiwan*
wwlin@math.nctu.edu.tw
‡*Department of Mathematics, Nanjing University, Nanjing 210093, P. R. China*
yqwangnju@yahoo.com

Received February 3, 2010

In this paper, we consider chaotic synchronization in coupled map lattices (CMLs) with periodic boundary conditions. We give a rigorous proof of the occurrence of synchronization for 1D such CMLs with lattice size $n = 5$ for suitable parameters in the chaotic regime by Lyapunov method.

Keywords: Synchronization; coupled map lattices; chaos; Lyapunov method.

1. Introduction

Chaotic synchronization is a fundamental phenomenon in physical systems with dissipation. Experimental simulations show that chaotic subsystems in a lattice manifest synchronized chaotic behavior in time provided they are coupled with a dissipative coupling and a coefficient of this coupling is greater than some critical value. This phenomenon has been observed and well-studied in many different fields — synchronization of coupled chaotic circuits [Carroll & Pecora, 1991; Chua *et al.*, 1993; Goldsztein & Streogatz, 1995], coupled chaotic oscillators [Afraimovich *et al.*, 1997; Afraimovich & Lin, 1998; Chiu *et al.*, 1998; Chan & Chao, 1998; Ermentrout, 1985; Fujisaka & Yamada, 1983; Mirollo & Strogatz, 1990] and master-slave chaotic Lorenz equations [Pecora & Carroll, 1990; He & Vaidya, 1992], etc.

In practice, with the combination of synchronization and unpredictability, chaotic synchronization has attracted a lot of attention for its promising potential in secure communication. A secret message can be modulated on the chaotic signal of a sender, and a receiver with an identical system which is driven by the modulated signal can decrypt this message. Many encryption models based on chaotic synchronization have been proposed, see [Pecora & Carroll, 1990; Vohra *et al.*, 1992; Cuomo & Oppenheim, 1992, 1993; Wu & Chua, 1994; Heagy *et al.*, 1995; Pecora *et al.*, 1997]. More recently, communication with chaos synchronization has been demonstrated with semiconductor lasers which were synchronized over a distance of 120 km in a public fiber network in Greece, see [Argyris *et al.*, 2005].

A mathematical foundation of synchronization of these coupled systems was heavily dependent on the bounded dissipativeness, the coupling rule and the type of chaotic subsystems. Thus, the problem of coming up with a rigorous mathematical proof of chaotic synchronization for specified coupled systems appears to be attractive and important from both theoretical and practical points of view.

*This work was supported by the NNSF of China 10871090 and National Basic Research Program of China 2007CB814804.

In this paper, we consider chaotic synchronization in coupled map lattices (CMLs) which can be considered as systems of interacting maps, where the individual map is characterized not only by its internal state but also by the position in the physical space. CMLs are, in general, the intermediate between partial differential equations (PDEs) and cellular automata which form a wide class of extended dynamical systems. PDEs are usually used to describe the physical phenomenon of spatial-temporal dynamical systems. However, the analytic study of solutions of PDEs suffers from extreme difficulty with complex behavior. On the other hand, the computer simulation is utilized as an effective and powerful tool to study dynamical systems with complex behavior. In such a study the dynamical system shall be discretized in space as well as in time. This is one of the motivations to introduce new models of CMLs (see [Afraimovich & Bunimovich, 1993; Bunimovich, 1997; Bunimovich & Carlen, 1995; Giberti & Vernia, 1994; Kaneko, 1993]).

A popular model in CMLs is defined as follows:

(1) 1D lattice, for $1 \leq i \leq n$,

$$x_i(k+1) = f(x_i(k)) + c(f(x_{i-1}(k)) + f(x_{i+1}(k)) - 2f(x_i(k))) \quad (1)$$

with periodic boundary conditions $f(x_0(k)) = f(x_n(k))$ and $f(x_{n+1}(k)) = f(x_1(k))$, and

(2) 2D lattice, for $i = (i_1, i_2)$ with $1 \leq i_1, i_2 \leq n$,

$$x_i(k+1) = f(x_i(k)) + c(f(x_{i_1+1,i_2}(k)) + f(x_{i_1-1,i_2}(k)) + f(x_{i_1,i_2+1}(k)) + f(x_{i_1,i_2-1}(k)) - 4f(x_i(k))) \quad (2)$$

with $f(x_{0,i_2}(k)) = f(x_{n,i_2}(k))$, $f(x_{n+1,i_2}(k)) = f(x_{1,i_2}(k))$, $f(x_{i_1,0}(k)) = f(x_{i_1,n}(k))$ and $f(x_{i_1,n+1}(k)) = f(x_{i_1,1}(k))$, where f is a one-dimensional logistic map $x(k+1) = f(x(k)) = \gamma x(k)(1 - x(k))$ with $f : (0,1) \to (0,1)$ and $\gamma \in [\gamma_\infty, 4]$ with $\gamma_\infty \approx 3.57$. It is well known (see [Campbell, 1989; Gleick, 1987], etc.) that the map f becomes chaotic whenever γ increases from 3.57 to 4, except that γ is at a very narrow interval of periodic windows near 3.63, 3, 73 or 3.83.

The simplest type of synchronization of CMLs in Eq. (1) or Eq. (2) occurs in stable spatially homogeneous regimes corresponding to the existence of attractive spatially homogeneous solutions. In other words, in such cases there is a large (open) set of initial conditions such that a solution starting from an initial condition in the set becomes spatially homogeneous as discrete time k becomes very large, i.e. the coordinates of the individual maps become almost equal to each other (and are equal as $k \to \infty$). In established regimes, individual maps become indistinguishable and we observe exact perfect synchronization. Thus, it may occur that suitable coupling strength permits the existence of a spatially homogeneous solution provided all individual maps are identical.

In 1999, [Lin *et al.*, 1999] considered the synchronized chaotic behavior of Eqs. (1) and (2). They provided a complete numerical analysis for the range of parameters γ, the lattices size n and the coupling strengths c such that chaotic synchronization occurs in the corresponding 1D and 2D CMLs. Moreover, they gave a rigorous proof for chaotic synchronization in the case of 1D CMLs with lattice size $n = 2, 3$ for $\gamma \in [\gamma_\infty, 4]$ and with lattice size $n = 4$ for $\gamma \in [\gamma_\infty, 3.82] \subset [\gamma_\infty, 4]$ of the chaotic regime. As the authors know, it is the first rigorous proof on chaotic synchronization of CMLs.

In 2001, [Lin & Wang, 2002] generalized the mathematical result of [Lin *et al.*, 1999] on the case of $n = 4$ to the full chaotic interval $[\gamma_\infty, 4]$ by Lyapunov method.

On the other hand, the numerical simulations in [Lin *et al.*, 1999] also showed that to ensure the occurrence of chaotic synchronization, the larger is the lattice size n, the less is the measure of the parameter set (γ, c). Moreover, when $n \geq 12$, no chaotic synchronization behavior can be observed, which is due to the fact that the spatially homogeneous regime becomes unstable for large n.

In this paper, we will prove the occurrence of chaotic synchronization for the case $n = 5$. For this purpose, however, we cannot follow the approach in [Lin *et al.*, 1999] or [Lin & Wang, 2002] for the case $n = 4$. The reason is that CMLs with size $n = 4$ have some special property of "decoupling", which is critical in the proof of [Lin *et al.*, 1999; Lin & Wang, 2002]; in comparison, unfortunately, CMLs with size $n \geq 5$ have no such property.

In the following, by modifying the method in [Lin & Wang, 2002], we will construct a more efficient Lyapunov function for 1D CMLs (1) with size $n = 5$ to prove the occurrence of the synchronized chaotic behavior. More precisely, we will prove the following theorem:

Theorem 1. *For every parameter $\gamma \in [\gamma_\infty, 4)$ for the logistic map in the CMLs (1) with lattices size $n = 5$, there exists $\delta = \delta(\gamma) > 0$ such that for any initial values $x_i(0) \in (0,1)$, $i = 1, \ldots, 5$ and any $c \in (2/5 - \delta(\gamma), 2/5 + \delta(\gamma))$, it holds that*

$$\lim_{k\to\infty} |x_i(k) - x_j(k)| = 0$$

for $i, j = 1, 2, 3, 4, 5$.

Remark 1.1. The Lyapunov method used here can also be used to generalize the result of [Lin & Wang, 2002], i.e. we can obtain the proof of chaotic synchronization for more parameters (c, γ) than in [Lin & Wang, 2002]. Hence this modified Lyapunov method is more efficient.

Remark 1.2. The authors believe that by careful computation, it is possible to use our idea of Lyapunov method to prove the synchronized chaotic behavior for CMLs (1) with more lattice sizes.

2. Lyapunov Function and Proof of Theorem 1

In this section, we will construct a Lyapunov function to prove the synchronized chaotic behavior for 1D coupled map lattices with size $n = 5$.

First, we introduce the following proposition:

Proposition 2.1. *For $c \in [0, 1/2]$, $3 < \gamma < 4$ and $(x_1(0), x_2(0), x_3(0), x_4(0), x_5(0))$ in $(0,1)^5$, there exists $K \in \mathbb{N}$ such that for all $k \geq K$, $(x_1(k), x_2(k), x_3(k), x_4(k), x_5(k))$ generated by (1) lie in*

$$D_0 = \left[\frac{4-\gamma}{4}, \frac{\gamma}{4}\right]^5.$$

Proof. The proof is the same as Theorem 2.3 in [Lin *et al.*, 1999] and we omit it here. ■

By Proposition 2.1, without loss of generality, we assume (1) is defined on D_0, i.e. $x_i(0) \in [(4-\gamma)/4, \gamma/4]$, $i = 1, 2, 3, 4, 5$.

We rewrite the iteration in lattice of (1) for $n = 5$ by replacing $x_i(k+1)$, $x_i(k)$ and $f(x_i)$ by $\bar{x}_i$, x_i and $f_i, i = 1, 2, 3, 4, 5$ respectively as follows:

$$\begin{aligned}
\bar{x}_1 &= f(x_1) + c(f(x_2) + f(x_5) - 2f(x_1)),\\
\bar{x}_2 &= f(x_2) + c(f(x_1) + f(x_3) - 2f(x_2)),\\
\bar{x}_3 &= f(x_3) + c(f(x_2) + f(x_4) - 2f(x_3)),\\
\bar{x}_4 &= f(x_4) + c(f(x_3) + f(x_5) - 2f(x_4)),\\
\bar{x}_5 &= f(x_5) + c(f(x_4) + f(x_1) - 2f(x_5)),
\end{aligned} \tag{3}$$

where $f(x) = \gamma x(1-x)$.

By direct computation, we have

$$\begin{aligned}
(\bar{x}_1 - \bar{x}_2)^2 &= [(1-3c)(f_1 - f_2) + c(f_5 - f_3)]^2\\
&= (1-3c)^2(f_1 - f_2)^2 + c^2(f_5 - f_3)^2\\
&\quad + 2(1-3c)c(f_1 - f_2)(f_5 - f_3),\\
(\bar{x}_1 - \bar{x}_3)^2 &= [(1-2c)(f_1 - f_3) + c(f_5 - f_4)]^2\\
&= (1-2c)^2(f_1 - f_3)^2 + c^2(f_5 - f_4)^2\\
&\quad + 2(1-2c)c(f_1 - f_3)(f_5 - f_4),\\
(\bar{x}_1 - \bar{x}_4)^2 &= [(1-2c)(f_1 - f_4) + c(f_2 - f_3)]^2\\
&= (1-2c)^2(f_1 - f_4)^2 + c^2(f_2 - f_3)^2\\
&\quad + 2(1-2c)c(f_1 - f_4)(f_2 - f_3),\\
(\bar{x}_1 - \bar{x}_5)^2 &= [(1-3c)(f_1 - f_5) + c(f_2 - f_4)]^2\\
&= (1-3c)^2(f_1 - f_5)^2 + c^2(f_2 - f_4)^2\\
&\quad + 2(1-3c)c(f_1 - f_5)(f_2 - f_4),\\
(\bar{x}_2 - \bar{x}_3)^2 &= [(1-3c)(f_2 - f_3) + c(f_1 - f_4)]^2\\
&= (1-3c)^2(f_2 - f_3)^2 + c^2(f_1 - f_4)^2\\
&\quad + 2(1-3c)c(f_2 - f_3)(f_1 - f_4),\\
(\bar{x}_2 - \bar{x}_4)^2 &= [(1-2c)(f_2 - f_4) + c(f_1 - f_5)]^2\\
&= (1-2c)^2(f_2 - f_4)^2 + c^2(f_1 - f_5)^2\\
&\quad + 2(1-2c)c(f_2 - f_4)(f_1 - f_5),\\
(\bar{x}_2 - \bar{x}_5)^2 &= [(1-2c)(f_2 - f_5) + c(f_3 - f_4)]^2\\
&= (1-2c)^2(f_2 - f_5)^2 + c^2(f_3 - f_4)^2\\
&\quad + 2(1-2c)c(f_2 - f_5)(f_3 - f_4),\\
(\bar{x}_3 - \bar{x}_4)^2 &= [(1-3c)(f_3 - f_4) + c(f_2 - f_5)]^2\\
&= (1-3c)^2(f_3 - f_4)^2 + c^2(f_2 - f_5)^2\\
&\quad + 2(1-3c)c(f_3 - f_4)(f_2 - f_5),\\
(\bar{x}_3 - \bar{x}_5)^2 &= [(1-2c)(f_3 - f_5) + c(f_2 - f_1)]^2\\
&= (1-2c)^2(f_3 - f_5)^2 + c^2(f_2 - f_1)^2\\
&\quad + 2(1-2c)c(f_3 - f_5)(f_2 - f_1),
\end{aligned}$$

$$\begin{aligned}(\overline{x}_4 - \overline{x}_5)^2 &= [(1-3c)(f_4 - f_5) + c(f_3 - f_1)]^2 \\ &= (1-3c)^2(f_4-f_5)^2 + c^2(f_3-f_1)^2 \\ &\quad + 2(1-3c)c(f_4-f_5)(f_3-f_1).\end{aligned}$$

Especially, when $c = 2/5$, it follows that

$$\sum_{i,j=1}^{5}(\overline{x}_i - \overline{x}_j)^2 = \frac{1}{5}\sum_{i,j=1}^{5}(f_i - f_j)^2. \tag{4}$$

Define a Lyapunov function for (3) as follows

$$L(x_1, \ldots, x_5) = \frac{\sum_{i,j=1}^{5}(x_i - x_j)^2}{\sum_{i=1}^{5} x_i\left(5 - \sum_{i=1}^{5} x_i\right)}, \quad (x_1, \ldots, x_5) \in D_0. \tag{5}$$

Note that from the condition $(x_1, \ldots, x_5) \in D_0$, the function $L(x_1, \ldots, x_5)$ is well defined. Obviously, L is positive and continuous on D_0 and satisfies that $L(x_1, \ldots, x_5) = 0 \Leftrightarrow x_i = x_j$, $i, j = 1, 2, 3, 4, 5$.

The following result is critical for this paper:

Theorem 2. *Assume* $\gamma \in [\gamma_\infty, 4)$. *Then there exist* $\lambda(\gamma) \in (0, 1)$ *and small* $\delta = \delta(\gamma) > 0$ *independent of* $(x_1, \ldots, x_5) \in D_0$ *such that for each* $c \in (2/5 - \delta(\gamma), 2/5 + \delta(\gamma))$, *it holds that*

$$L(\overline{x}_1, \ldots, \overline{x}_5) \le \lambda(\gamma)L(x_1, \ldots, x_5). \tag{6}$$

Proof of Theorem 1. Theorem 2 shows that for $\gamma \in [\gamma_\infty, 4]$, there exist $0 < \lambda(\gamma) < 1$ and $\delta = \delta(\gamma) > 0$ such that

$$L(x_1(k), \ldots, x_5(k)) \le \lambda(\gamma)^k \cdot L(x_1(0), \ldots, x_5(0))$$

for $c \in (2/5 - \delta(\gamma), 2/5 + \delta(\gamma)), (x_1(0), \ldots, x_5(0)) \in D_0$ and $k \in \mathbb{N}$. It implies $L(x_1(k), \ldots, x_5(k)) \to 0$ as $k \to \infty$. Thus $|x_i(k) - x_j(k)| \to 0$ as $k \to \infty$, $i, j = 1, 2, 3, 4, 5$. This completes the proof. ■

3. Proof of Theorem 2

In this section, we will prove Theorem 2.

In fact, we only need to prove Theorem 2 for the case $c = 2/5$. Then by continuation and the compactness of D_0, we can prove the existence of a small interval $(2/5 - \delta(\gamma), 2/5 + \delta(\gamma))$ such that the same conclusion holds true. Thus, without loss of generality, we assume $c = 2/5$.

Substituting (4) into (5), we have that for $c = 2/5$,

$$L(\overline{x}_1, \ldots, \overline{x}_5) = \frac{\sum_{i,j=1}^{5}\frac{(f_i - f_j)^2}{5}}{\sum_{i=1}^{5} f_i\left(5 - \sum_{i=1}^{5} f_i\right)}. \tag{7}$$

In the following, to illustrate the idea of the proof, we first prove the assertion for a special case:

$$\max_{1 \le i,j \le 5}|x_i - x_j| < \epsilon_0, \tag{8}$$

where ϵ_0 is a small positive number determined later. Then we will deal with the general case.

Proof of the special case. From Eq. (8), we have

$$\begin{aligned}\min_{1 \le i,j \le 5}|1 - x_i - x_j| &\le \max_{1 \le i,j \le 5}|1 - x_i - x_j| \\ &\le \min_{1 \le i \le 5}|1 - 2x_i| + 2\epsilon_0.\end{aligned} \tag{9}$$

Consequently, by the definition of f_i and f_j, it is easily seen that

$$\begin{aligned}(f_i - f_j)^2 &= \gamma^2(1 - x_i - x_j)^2(x_i - x_j)^2 \\ &\le \gamma^2 \cdot \left(\max_{1 \le i,j \le 5}|1 - x_i - x_j|\right)^2 (x_i - x_j)^2 \\ &\le \gamma^2 \cdot \left(\left(\min_{1 \le i \le 5}|1 - 2x_i|\right)^2 + c_1\epsilon_0\right) \\ &\quad \times (x_i - x_j)^2,\end{aligned}$$

where $c_1 \le 4$ is a constant independent of ϵ_0.

Similarly, we have

$$\begin{aligned}&\left|\frac{\sum_{i=1}^{5} f_i}{5} \cdot \left(1 - \frac{\sum_{i=1}^{5} f_i}{5}\right)\right| \\ &\quad \ge f(\overline{x})(1 - f(\overline{x})) + \gamma \max_{1 \le i \le 5}(x_i - \overline{x})^2 \\ &\quad \ge f(\overline{x})(1 - f(\overline{x})) - c_3\epsilon_0^2,\end{aligned}$$

where $\overline{x} = (x_1 + x_2 + x_3 + x_4 + x_5)/5$ and $c_3 \le \gamma$ are two constants independent of ϵ_0.

Hence, from Eq. (7), we have

$$\begin{aligned}
L(\bar{x}_1,\ldots,\bar{x}_5) &\le \frac{\gamma^2\left(\min\limits_{1\le i\le 5}|1-2x_i|^2+c_1\epsilon_0\right)\sum\limits_{i,j=1}^{5}(x_i-x_j)^2}{5\sum\limits_{i=1}^{5}f_i\left(5-\sum\limits_{i=1}^{5}f_i\right)} \\
&\le \frac{\gamma^2\left(\min\limits_{1\le i\le 5}|1-2x_i|^2+c_1\epsilon_0\right)\sum\limits_{i,j=1}^{5}(x_i-x_j)^2}{125(f(\bar{x})(1-f(\bar{x}))-c_3\epsilon_0^2)} \\
&= \frac{\gamma^2\left(\min\limits_{1\le i\le 5}|1-2x_i|^2+c_1\epsilon_0\right)\sum\limits_{i=1}^{5}x_i\cdot\left(5-\sum\limits_{i=1}^{5}x_i\right)}{125(f(\bar{x})(1-f(\bar{x}))-c_3\epsilon_0^2)}\cdot L(x_1,\ldots,x_5) \\
&\le \frac{\gamma^2\left(\min\limits_{1\le i\le 5}|1-2x_i|^2+c_1\epsilon_0\right)\cdot\max\limits_{1\le i,j\le 5}x_i(1-x_j)}{5(f(\bar{x})(1-f(\bar{x}))-c_3\epsilon_0^2)}\cdot L(x_1,\ldots,x_5) \\
&\le \max_{1-\gamma/4\le x\le\gamma/4}\frac{\gamma^2|1-2x|^2x(1-x)}{5f(x)(1-f(x))}\cdot L(x_1,\ldots,x_5)+c_4(\epsilon_0)L(x_1,\ldots,x_5),
\end{aligned}$$

where $c_4(\epsilon_0)$ is a rational function of ϵ_0 satisfying $c_4(\epsilon_0)\to 0$ as $\epsilon_0\to 0$.

Since

$$\begin{aligned}
&\max_{1-\gamma/4\le x\le\gamma/4}\frac{\gamma^2|1-2x|^2x(1-x)}{5f(x)(1-f(x))} \\
&\quad= \max_{1-\gamma/4\le x\le\gamma/4}\frac{\gamma(1-2x)^2}{5(1-\gamma x(1-x))}\le\frac{\gamma}{5},
\end{aligned}$$

we obtain

$$L(\bar{x}_1,\ldots,\bar{x}_5)\le\left(\frac{\gamma}{5}+c_4(\epsilon_0)\right)L(x_1,\ldots,x_5).$$

Obviously, if ϵ_0 is small enough, it holds that $\gamma/5+c_4(\epsilon_0)<1$. Hence, we complete the proof for the special case. ■

Proof of the general case. It is easily seen that

$$\begin{aligned}
&\frac{L(\bar{x}_1,\ldots,\bar{x}_5)}{L(x_1,\ldots,x_5)} \\
&\quad= \frac{\sum\limits_{i=1}^{5}x_i\left(5-\sum\limits_{i=1}^{5}x_i\right)\cdot\sum\limits_{i,j=1}^{5}\frac{(f_i-f_j)^2}{5}}{\sum\limits_{i=1}^{5}f_i\left(5-\sum\limits_{i=1}^{5}f_i\right)\cdot\sum\limits_{i,j=1}^{5}(x_i-x_j)^2}.
\end{aligned} \tag{10}$$

Let

$$F_1=\sum_{i=1}^{5}x_i\cdot\left(5-\sum_{i=1}^{5}x_i\right),\quad F_2=\sum_{i,j=1}^{5}\frac{(f_i-f_j)^2}{5},$$

$$G_1=\sum_{i=1}^{5}f_i\cdot\left(5-\sum_{i=1}^{5}f_i\right),\quad G_2=\sum_{i,j=1}^{5}(x_i-x_j)^2,$$

and

$$H(x_1,\ldots,x_5)=\frac{F_1\cdot F_2}{G_1\cdot G_2}=\frac{F}{G}.$$

In the following, we will analyze the maximum value of the function H on the domain D_0.

Obviously, we have

$$\frac{\partial H}{\partial x_i}=\frac{F_{x_i}G-FG_{x_i}}{G^2},\quad i=1,2,3,4,5.$$

Here F_{x_i} and G_{x_i} denote the partial derivatives of F and G with respect to x_i, respectively, $i=1,2,3,4,5$.

Let $\partial H/\partial x_i=0$, $i=1,2,3,4,5$, then we have

$$F_{x_i}G-FG_{x_i}=0,\quad i=1,\ldots,5. \tag{11}$$

By direct computation, we obtain

$$F'_{x_1} = \frac{1}{5}\left(5 - 2\sum_{i=1}^{5} x_i\right) F_2 + 2F_1\left(5f_1 - \sum_{i=1}^{5} f_i\right) f'_1,$$

$$G'_{x_1} = \left(5 - 2\sum_{i=1}^{5} f_i\right) G_2 f'_1 + 2G_1\left(5x_1 - \sum_{i=1}^{5} x_i\right), \tag{12}$$

$$F'_{x_2} = \frac{1}{5}\left(5 - 2\sum_{i=1}^{5} x_i\right) F_2 + 2F_1\left(5f_2 - \sum_{i=1}^{5} f_i\right) f'_2,$$

$$G'_{x_2} = \left(5 - 2\sum_{i=1}^{5} f_i\right) G_2 f'_2 + 2G_1\left(5x_2 - \sum_{i=1}^{5} x_i\right), \tag{13}$$

$$F'_{x_3} = \frac{1}{5}\left(5 - 2\sum_{i=1}^{5} x_i\right) F_2 + 2F_1\left(5f_3 - \sum_{i=1}^{5} f_i\right) f'_3,$$

$$G'_{x_3} = \left(5 - 2\sum_{i=1}^{5} f_i\right) G_2 f'_3 + 2G_1\left(5x_3 - \sum_{i=1}^{5} x_i\right), \tag{14}$$

$$F'_{x_4} = \frac{1}{5}\left(5 - 2\sum_{i=1}^{5} x_i\right) F_2 + 2F_1\left(5f_4 - \sum_{i=1}^{5} f_i\right) f'_4,$$

$$G'_{x_4} = \left(5 - 2\sum_{i=1}^{5} f_i\right) G_2 f'_4 + 2G_1\left(5x_4 - \sum_{i=1}^{5} x_i\right), \tag{15}$$

$$F'_{x_5} = \frac{1}{5}\left(5 - 2\sum_{i=1}^{5} x_i\right) F_2 + 2F_1\left(5f_5 - \sum_{i=1}^{5} f_i\right) f'_5,$$

$$G'_{x_5} = \left(5 - 2\sum_{i=1}^{5} f_i\right) G_2 f'_5 + 2G_1\left(5x_5 - \sum_{i=1}^{5} x_i\right). \tag{16}$$

Lemma 3.1. *For Eqs. (11), one of the following equations must hold true*:

(i) $x_{i_1} = x_{i_2}$,
(ii) $x_{i_1} = x_{i_3}$,
(iii) $x_{i_2} = x_{i_3}$,
(iv) $x_{i_1} + x_{i_2} + x_{i_3} = 3/2$,

where $i_1, i_2, i_3 = 1, 2, 3, 4, 5, i_1 \neq i_2, i_1 \neq i_3, i_2 \neq i_3$.

Proof. From Eqs. (11)–(13), we have

$$2F_1 G\left[(5f_1 f'_1 - 5f_2 f'_2) - \sum_{i=1}^{5} f_i \cdot (f'_1 - f'_2)\right]$$
$$= F\left[\left(5 - 2\sum_{i=1}^{5} f_i\right) G_2(f'_1 - f'_2) + 10G_1(x_1 - x_2)\right],$$

which is equivalent to

$$2F_1 G\left[5\gamma^2 k(x_1, x_2) + 2\gamma \sum_{i=1}^{5} f_i\right](x_1 - x_2)$$
$$= F\left[-2\gamma\left(5 - 2\sum_{i=1}^{5} f_i\right) G_2 + 10G_1\right](x_1 - x_2),$$

where $k(x, y) = 1 - 3(x + y) + 2(x^2 + xy + y^2)$. Similarly, from (11)–(16), we have

$$2F_1 G\left[5\gamma^2 k(x_j, x_k) + 2\gamma \sum_{i=1}^{5} f_i\right](x_j - x_k)$$
$$= F\left[-2\gamma\left(5 - 2\sum_{i=1}^{5} f_i\right) G_2 + 10G_1\right]$$
$$\times (x_j - x_k), \quad j, k = 1, 2, 3, 4, 5. \tag{17}$$

Obviously, Eq. (17) implies that either

$$(x_{i_1} - x_{i_2})(x_{i_1} - x_{i_3}) = 0$$

or

$$k(x_{i_1}, x_{i_2}) = k(x_{i_1}, x_{i_3}).$$

If the former holds true, then we have proved this lemma. Otherwise, then the latter holds true, which is equivalent to

$$\left(\frac{3}{2} - x_{i_1} - x_{i_2} - x_{i_3}\right)(x_{i_2} - x_{i_3}) = 0.$$

It completes the proof. ■

Claim. If (11) holds true, then x_1, x_2, x_3, x_4, x_5 satisfies one of the following statements:

(i) $x_1 = x_3 = x_4 = x_5$.
(ii) $x_1 = x_3 = x_4,\ x_2 = x_5$.
(iii) $x_1 = x_4 = x_5,\ x_1 + x_2 + x_3 = 3/2$.
(iv) $x_1 = x_4,\ x_2 = x_5,\ x_1 + x_2 + x_3 = 3/2$.

Proof. Otherwise, without loss of generality, we assume that x_1, x_2, x_3, x_4 are different from each other. Then from Lemma 3.1, we have $x_1 + x_2 + x_3 = 3/2$ and $x_1 + x_2 + x_4 = 3/2$, which implies $x_3 = x_4$. It contradicts with the assumption. ■

For case (i), we rewrite the function H as follows:

$$H = \frac{F}{G} = \frac{\gamma^2(4x_1 + x_2)(5 - 4x_1 - x_2)(1 - x_1 - x_2)^2}{(4f_1 + f_2)(5 - 4f_1 - f_2)}.$$

It is easy to verify that

$$F'_{x_1} = \frac{4}{5}(5 - 2(4x_1 + x_2))(1 - x_1 - x_2)^2 - \frac{2}{5}(4x_1 + x_2)(5 - 4x_1 - x_2)(1 - x_1 - x_2),$$

$$G'_{x_1} = 4(5 - 2(4f_1 + f_2))f'_1,$$

$$F'_{x_2} = \frac{1}{5}(5 - 2(4x_1 + x_2))(1 - x_1 - x_2)^2 - \frac{2}{5}(4x_1 + x_2)(5 - 4x_1 - x_2)(1 - x_1 - x_2),$$

$$G'_{x_2} = (5 - 2(4f_1 + f_2))f'_2.$$

Let

$$F'_{x_1}G = FG'_{x_1}, \quad F'_{x_2}G = FG'_{G_2}.$$

Which implies

$$(F'_{x_1} - 4F'_{x_2})G = F(G'_{x_1} - 4G'_{x_2}).$$

Since

$$F'_{x_1} - 4F'_{x_2} = \frac{6}{5}(4x_1 + x_2)(5 - 4x_1 - x_2)(1 - x_1 - x_2)$$

and

$$G'_{x_1} - 4G'_{x_2} = 4(5 - 2(4f_1 + f_2))(f'_1 - f'_2),$$

we obtain

$$3(4f_1 + f_2)(5 - 4f_1 - f_2) = -4(5 - 2(4f_1 + f_2))(f_1 - f_2),$$

which is equivalent to

$$16(f_1 - f_1^2) = f_2 - f_2^2. \tag{18}$$

However, since we assume that f is defined on the interval $[1 - \gamma/4, \gamma/4]$, it is easily seen that

$$\min_{1-\gamma/4 \le x \le \gamma/4} (f(x) - f(x)^2) \ge 0.925 \times 0.075, \quad \text{and}$$

$$\max_{1-\gamma/4 \le x \le \gamma/4} (f(x) - f(x)^2) = \frac{1}{4}.$$

It shows that Eq. (18) is impossible. Thus case (i) is excluded.

We can exclude cases (ii)–(iv) in the same way.

Thus maximum values of H can be attained only on the boundary point set of D_0, i.e.

$$x_i = 1 - \frac{\gamma}{4} \quad \text{or} \quad \frac{\gamma}{4}, \qquad i \in \{1, 2, 3, 4, 5\}.$$

Now the situation for the boundary point set is simpler than the inner point set since there are less variables and we can deal with it in a similar way as above. Thus we omit the details. This completes the proof of Theorem 2. ■

Acknowledgment

The second author thanks the hospitality of the Center for Theoretical Science, Taiwan and Prof. Song-Sun Lin of National Chiao Tung University, Taiwan.

References

Afraimovich, V. S. & Bunimovich, L. A. [1993] "Simplest structures in coupled map lattices and their stabilities," *Rand. Comput. Dyn.* **1**, 423–444.

Afraimovich, V. S., Chow, S. N. & Hale, J. K. [1997] "Synchronization in lattices of coupled oscillators," *Physica D* **103**, 445–451.

Afraimovich, V. S. & Lin, W. W. [1998] "Synchronization in lattices of coupled oscillators with Neumann/periodic boundary conditions," *Dyn. Stab. Syst.* **13**, 237–264.

Argyris, A., Syvridis, D., Larger, L., Annovazzi-Lodi, V., Colet, P., Fischer, I., Garcia-Ojalvo, J., Mirasso, C. R., Pesquera, L. & Shore, K. A. [2005] "Chaos-based communications at high bit rates using commercial fibre-optic links," *Nature* **438**, 343–346.

Bunimovich, L. A. & Carlen, E. A. [1995] "On the problem of stability in lattice dynamical systems," *J. Diff. Eq.* **123**, 213–229.

Bunimovich, L. A. [1997] "Coupled map lattices: Some topological and ergodic properties," *Physica D* **103**, 1–17.

Campbell, D. [1989] "An introduction to nonlinear dynamics," *Lectures in the Sciences of Complexity*, ed. Stein, D. L. (Addison-Wesley, Reading, MA).

Carroll, T. L. & Pecora, L. M. [1991] "Synchronizing nonautonomous chaotic circuits," *Circuits Syst.* **38**, 435–456.

Chan, W. C. C. & Chao, Y. D. [1998] "Synchronization of coupled forced oscillators," *J. Math. Anal. Appl.* **218**, 97–116.

Chiu, C. H. , Lin, W. W. & Wang, C. S. [1998] "Synchronization in a lattice of coupled Van der Pol system," *Int. J. Bifurcation and Chaos* **8**, 2353–2373.

Chua, L. O., Itoh, M., Kosarev, L. & Eckert, K. [1993] "Chaos synchronization in Chua's circuits," *J. Circuits Syst. Comp.* **3**, 93–108.

Cuomo, K. M. & Oppenheim, A. V. [1992] "Synchronized chaotic circuits and systems for communications," Electr. TR. MIT Res. Lab.#575.

Cuomo, K. M. & Oppenheim, A. V. [1993] "Circuit implementation of synchronized chaos, with applications to communications," *Phys. Rev. Lett.* **71**, 65–68.

Ermentrout, G. B. [1985] "Synchronization in a pool of mutually coupled oscillators with random frequencies," *J. Math. Biol.* **22**, 1–9.

Fujisaka, H. & Yamada, T. [1983] "Stability theory of synchronized motion in coupled oscillator systems," *Prog. Theor. Phys.* **69**, 32–47.

Giberti, C. & Vernia, C. [1994] "Periodic behavior in 1D and 2D coupled map lattices of small size," *Chaos* **4**, 651–664.

Gleick, J. [1987] *Chaos*: *Making a New Science* (Viking, NY).

Goldsztein, G. & Strogatz, S. H. [1995] "Stability of synchronization in networks of digital phase-locked loops," *Int. J. Bifurcation and Chaos* **5**, 983–990.

He, R. & Vaidya, P. G. [1992] "Analysis and synthesis of synchronous periodic and chaotic systems," *Phys. Rev. A* **46**, 7387–7392.

Heagy, J. F., Carroll, T. L. & Pecora, L. M. [1995] "Synchronization with application to communication," *Phys. Rev. Lett.* **74**, 5028–5031.

Hénon, M. [1976] "A two-dimensional mapping with a strange attractor," *Commun. Math. Phys.* **50**, 69–77.

Kaneko, K. (ed.) [1993] *Theory and Applications of Coupled Map Lattices* (Wiley, NY).

Lin, W. W, Peng, C. C. & Wang, C. S. [1999] "Synchronization in coupled map lattices with periodic boundary condition," *Int. J. Bifurcation and Chaos* **9**, 1635–1652.

Lin, W. & Wang, Y. [2002] "Synchronized chaos in coupled map lattices with periodic boundary condition," *SIAM J. Appl. Dyn. Syst.* **1**, 175–189.

Mirollo, R. E. & Strogatz, S. H. [1990] "Synchronization of pulse-coupled biological oscillators," *SIAM J. Appl. Math.* **50**, 1645–1662.

Pecora, L. M. & Carroll, T. L. [1990] "Synchronization in chaotic systems," *Phys. Rev. Lett.* **64**, 821–824.

Pecora, L. M., Carroll, T. L., Johnson, G. A., Mar, D. J. & Heagy, J. F. [1997] "Fundamentals of synchronization in chaotic systems, concept and applications," *Chaos* **6**, 262–276.

Rulle, D. [1977] "Application conservant une measure absolument continue par rapport á dx sur $[0, 1]$," *Commun. Math. Phys.* **55**, 47–52.

Smith, J. [1968] *Mathematical Ideas in Biology* (Cambridge Press, Cambridge).

Vohra, S., Spano, M., Shlesinger, M., Pecora, L. & Ditto, W. [1992] *Proc. First Experimental Chaos Conf.* (World Scientific, Singapore).

Wu, C. W. & Chua, L. O. [1994] "A unified framework for synchronizations and control of dynamical systems," *Int. J. Bifurcation and Chaos* **4**, 979–988.