Alternating Runs of Geometrically Distributed Random Variables*

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Run statistics about a sequence of independent geometrically distributed random variables has attracted some attention recently in many areas such as applied probability, reliability, statistical process control, and computer science. In this paper, we first study the mean and variance of the number of alternating runs in a sequence of independent geometrically distributed random variables. Then, using the relation between the model of geometrically distributed random variables and the model of random permutation, we can obtain the variance in a random permutation, which is difficult to derive directly. Moreover, using the central limit theorem for dependent random variables, we can obtain the distribution of the number of alternating runs in a random permutation.

Keywords: alternating runs, geometric random variables, asymptotic properties, random permutation, central limit theorem

1. INTRODUCTION

Let $X_1, \ldots, X_n$ be n independent geometrically distributed random variables satisfying for $i \in \{1, 2, \ldots, n\}$, $\Pr[X_i = k] = pq^{k-1}$, $\forall k \in \mathbb{N}$, where $p + q = 1$. Combinatorics of independent geometrically distributed random variables has caught some attention recently in many areas including computer science due to the applications to, for example, the skip list [1-6] and probabilistic counting [7-10]. Moreover, since taking the limit $q \to 1$, with probability $1 - o(1)$, there are no equal values in $X_1, \ldots, X_n$ and each relative ordering of values is equally likely, we have that the model of geometrically distributed random variables approaches the model of random permutation [11, 12]. Therefore, various runs of geometrically distributed random variables have been investigated [11, 13-16]. For runs of consecutive equal numbers, Louchard [15], Grabner *et al.* [14], and Eryilmaz [13] studied the asymptotic properties of the number of runs using a Markov chain approach, probability generating functions, and the existing theory for m-dependent random variables (that is, the two sets $(X_1, \ldots, X_r)$ and $(X_{r+m+1}, \ldots, X_n)$ are independent), respectively. Another line of research considered ascending runs, which is a sequence of contiguous strictly increasing numbers. Louchard and Prodinger investigated the asymptotic properties of the number of ascending runs and the longest ascending runs

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using a Markov chain approach [11], where they mentioned the possibility of applying their approach to study alternating runs properties.

For a permutation, an alternating run is a sequence of consecutive ascending or descending numbers, in which the first run can be ascending or descending [17]. For example, 814965237 is considered to have 4 alternating runs: 81,149,9652, and 237. With permutation array, we can use alternating run for information hiding [18]. For information hiding, we want to hide some message in an object, such as an image data or a MIDI file, without changing the appearance of the object. Using a permutation array [19], which contains some permutations of a fixed length and satisfies the property that the distance (under some metric) of any two permutations in this array is not too large, one can incorporate the capability of error correcting into information hiding, but we can only hide few bits. Fortunately, with the help of alternating runs, we can increase the number of hiding bits and preserve the ability of error correcting, where we can simply treat the increasing run of length 2 as 1 and decreasing run of length 2 as 0.

In this paper, we will study the mean, variance and distribution of the number of alternating runs. Instead of using the approach of [11], we prove with an elementary method and the central limit theorem for dependent random variables [20]. Since in a random permutation $(Y_1Y_2\ldots Y_n)$, Y_i's are mutually dependent, it is difficult to derive the variance of the number of alternating runs. Hence, we first consider the model of geometrically distributed random variables, and then by taking the limit $q \to 1$, we can obtain the result in the model of random permutation. Since there may be some consecutive equal numbers in a sequence of independent geometrically distributed random variables, we modify the definition of alternating runs to define modified alternating runs. In short, a modified alternating run is a sequence of alternatively strictly ascending and descending numbers. Like alternating runs, each modified alternating run can begin with ascending or descending. Moreover, we also consider turning points in a sequence of numbers. In a sequence $x_1, \ldots, x_n$, we call x_i a turning point if $\max\{x_{i+1}, x_{i-1}\} < x_i$ or $\min\{x_{i+1}, x_{i-1}\} > x_i$. One can observe that the number of alternating runs in a sequence of random variables is exactly the sum of the number of modified alternating runs and the number of turning points. Therefore, to obtain the asymptotic properties of the numbers of alternating runs, we can study the asymptotic properties of the numbers of modified alternating runs and turning points.

To make the above definitions clear, we consider the following example. Let AR_n, MAR_n, and T_n denote the total number of alternating runs, modified alternating runs, and turning points in a sequence of n independent geometrically distributed random variables, respectively. Then consider the sequence: 5342221442. There are 6 alternating runs: 53, 34,42,21,14,42; 3 modified alternating runs: 5342, 214, 42; and 3 turning points: 2 turning points (3 and 4) in the modified alternating run 5342 and one turning point (1) in the modified alternating run 214. Hence, $AR_{10} = 6$, $MAR_{10} = 3$, and $T_{10} = 3$. Note that $AR_{10} = MAR_{10} + T_{10}$.

Next, we describe our results. First, we derive the following three theorems about the means and variances of MAR_n, T_n, and AR_n in a sequence of geometrically distributed random variables, which have not (to the best of our knowledge) been discussed before.

Theorem 1 The expectation of the number of modified alternating runs for $n \geq 3$ is given by

$$E[MAR_n]=\frac{q+q^2+2q^3}{(1+q)(1+q+q^2)}n+\frac{6q^3}{(1+q)(1+q+q^2)}.$$

The variance of the number of modified alternating runs for $n = 3$ is

$$Var[MAR_3]=\frac{3q-6q^3+3q^5}{(1+q)^2(1+q+q^2)^2}.$$

Furthermore, for $n \geq 4$, $Var[MAR_n]$ is

$$\frac{q+q^2-6q^3+8q^4+15q^5+4q^6+17q^7-14q^9-11q^{10}-q^{11}-14q^{12}}{(1+q)(1+q+q^2)^2(1+q+q^2+q^3)(1+q+q^2+q^3+q^4)}\cdot n+$$
$$\frac{18q^3-26q^4-52q^5-14q^6-58q^7+6q^8+50q^9+36q^{10}-2q^{11}-42q^{12}}{(1+q)(1+q+q^2)^2(1+q+q^2+q^3)(1+q+q^2+q^3+q^4)}.$$

Theorem 2 The expected number of turning points for $n \geq 3$ is

$$E[T_n]=(n-2)\frac{q+q^2+2q^3}{(1+q)(1+q+q^2)}.$$

The variance of the number of turning points for $n = 3$ is

$$Var[T_3]=\frac{q+2q^2+4q^3+2q^4+q^5-2q^6}{(1+q)^2(1+q+q^2)^2}.$$

Furthermore, for $n \geq 4$, $Var[T_n]$ is

$$\frac{q+5q^2+14q^3+18q^4+25q^5+18q^6+15q^7-4q^8-8q^9-11q^{10}-3q^{11}-6q^{12}}{(1+q)(1+q+q^2)^2(1+q+q^2+q^3)(1+q+q^2+q^3+q^4)}\cdot n+$$
$$\frac{-2q-12q^2-34q^3-42q^4-60q^5-40q^6-32q^7+22q^8+30q^9+32q^{10}+6q^{11}+16q^{12}}{(1+q)(1+q+q^2)^2(1+q+q^2+q^3)(1+q+q^2+q^3+q^4)}.$$

Theorem 3 For $n \geq 3$, the expectation of the number of alternating runs satisfies

$$E[AR_n]=\frac{2q+2q^2}{(1+q)(1+q+q^2)}n-\frac{2q+2q^2-2q^3}{(1+q)(1+q+q^2)}.$$

The variance of the number of alternating runs for $n = 3$ is

$$Var[AR_3]=\frac{6q+2q^2-2q^3-2q^4+2q^5+2q^6}{(1+q)^2(1+q+q^2)^2},$$

while for $n \geq 4$, $Var[AR_n]$ is

$$\frac{4q+4q^2+6q^4+6q^5+2q^6+10q^7+12q^8+6q^9+8q^{10}+6q^{11}}{(1+q)(1+q+q^2)^2(1+q+q^2+q^3)(1+q+q^2+q^3+q^4)}\cdot n+$$
$$\frac{-6q-4q^2+12q^3-8q^4-8q^5+2q^6-26q^7-32q^8-12q^9-20q^{10}-16q^{11}+2q^{12}}{(1+q)(1+q+q^2)^2(1+q+q^2+q^3)(1+q+q^2+q^3+q^4)}.$$

The proofs of these theorems are in sections 2, 3, and 4, respectively. Then, as mentioned above, taking the limit $q \to 1$, we can obtain the mean and variance in a random permutation using the results of geometrically distributed random variables.

Corollary 1 In a random permutation of length n, the expectation of the number of alternating runs satisfies, for $n \geq 3$,

$$E[AR_n]=\frac{2}{3}n-\frac{1}{3}=1+\frac{2}{3}(n-2).$$

For $n = 3$, the variance of the number of alternating runs is 2/9, while for $n \geq 4$, it is

$$\frac{8}{45}n-\frac{29}{90}.$$

Note that the expectation is exactly the result for the case of random permutations [17]. Moreover, since the random variables we use to evaluate AR_n are m-dependent for some integer m, we can get the following theorem about the distribution of AR_n in a random permutation, whose proof is in section 5.

Theorem 4 In a random permutation of length n, the random variable $n^{-1/2}(AR_n - E[AR_n])$ has a limiting normal distribution with mean 0 and variance $A = 4/45$ as $n \to \infty$.

2. TOTAL NUMBER OF MODIFIED ALTERNATING RUNS

Throughout this paper, let $X_1, \ldots, X_n$ be n independent geometrically distributed random variables satisfying for $i \in \{1, 2, \ldots, n\}$,

$$\Pr[X_i = k] = pq^{k-1},\ \forall k \in \mathbb{N},\ \text{where } p + q = 1.$$

In this section, we prove Theorem 1 which is about the mean and variance of the total number of modified alternating runs in $X_1, \ldots, X_n$. Let MAR_n denote the total number of modified alternating runs in $X_1, \ldots, X_n$. Since each modified alternating run must end with repeated numbers, we observe that MAR_n is one more than the number of consecutive identical substrings except for the cases of $X_1 = X_2$ and $X_{n-1} = X_n$. Take the sequences 55542331 and 1233 as examples. We find that there are 2 modified alternating runs in the sequence 55542331: 5423,31, while the number of repeated subsequence is exactly 2. Moreover, there is 1 modified alternating run in the sequence 1233: 123, while the number of repeated subsequence is exactly 1. Hence, to count the number of repeated substrings and deal with the case of $X_{n-1} = X_n$, we define a binary random variable xi_i for each $i \in \{1, 2, \ldots, n-2\}$ by

$$\xi_i = \begin{cases} 1, & \text{if } X_i = X_{i+1} \neq X_{i+2}; \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, we define an additional random variable

$$\varsigma_1 = \begin{cases} -1, & \text{if } X_i = X_2; \\ 0, & \text{otherwise;} \end{cases}$$

for the case of $X_1 = X_2$. Then, we have

$$MAR_n = 1 + \sum_{i=1}^{n-2} \xi_i + \varsigma_1. \tag{1}$$

To evaluate the expectation of MAR_n, we need to obtain $E[\xi_i]$, for each i and $E[\varsigma_1]$. For each $i \in \{1, \ldots, n-2\}$, since X_i's are independent, then by the formula for a geometric series, the expectation of ξ_i is[1]

$$\begin{aligned} E[\xi_i] &= \Pr[X_i = X_{i+1} \neq X_{i+2}] = \sum_{k \in \mathrm{N}} \Pr[X_i = X_{i+1} = k] \Pr[X_{i+2} \neq k] \\ &= \sum_{k \in \mathrm{N}} (pq^{k-1})^2 \cdot (1 - pq^{k-1}) = \sum_{k \in \mathrm{N}} (p^2 q^{2k-2} - p^3 q^{3k-3}) \\ &= \frac{p^2}{1-q^2} - \frac{p^3}{1-q^3} = \frac{q + q^2 + 2q^3}{(1+q)(1+q+q^2)}. \end{aligned}$$

On the other hand, the expectation of ς_1 is

$$E[\varsigma_1] = -1 \cdot \Pr[X_i = X_2] = -1 \cdot \sum_{k \in \mathrm{N}} (pq^{k-1})^2 = \frac{-p}{1+q}.$$

Therefore, by the linearity of expectation and Eq. (1), we obtain that the expectation of MAR_n is

$$E[MAR_n] = 1 + \sum_{i=1}^{n-2} E[\xi_i] + E[\varsigma_1] = \frac{q + q^2 - 2q^3}{(1+q)(1+q+q^2)} n + \frac{6q^3}{(1+q)(1+q+q^2)}.$$

Next, we derive the variance of MAR_n. Since the variance of MAR_n is given by

$$Var[MAR_n] = \sum_{i=1}^{n-2} Var[\xi_i] + Var[\varsigma_1] + 2\left[\sum_{1 \le i < j \le n-2} cov(\xi_i,\ \xi_j) + \sum_{i=1}^{n-2} cov(\xi_i,\ \varsigma_1)\right], \tag{2}$$

we consider the following values: $Var[\xi_i]$, $Var[\varsigma_1]$, $cov(\xi_i, \xi_j)$, and $cov(\xi_i, \varsigma_1)$ for any i and $j > i$. For each $i \in \{1, 2, \ldots, n-2\}$, the variance of ξ_i is

[1] Throughout this paper, we use matlab to expand the polynomials.

$$Var[\xi_i] = E[(\xi_i)^2] - (E[\xi_i])^2 = \frac{q + 2q^2 + 2q^4 + q^5 - 6q^6}{(1+q)^2(1+q+q^2)^2}. \tag{3}$$

In addition, the variance of ς_1 is

$$Var[\varsigma_1] = E[(\varsigma_1)^2] - (E[\varsigma_1])^2 = \frac{2q - 2q^2}{(1+q)^2}. \tag{4}$$

Then, we proceed to bound the covariance. Since for $i, j \in \{1, 2, \ldots, n-2\}$ with $|i-j| \geq 3$, ξ_i and ξ_j are independent, we know $cov(\xi_i, \xi_j) = 0$. Moreover, for $i \geq 3$, ξ_i and ς_1 are also independent, that is, $cov(\xi_i, \varsigma_1) = 0$. It remains to evaluate $cov(\xi_i, \xi_{i+1})$, $cov(\xi_i, \xi_{i+2})$, $cov(\xi_1, \varsigma_1)$, and $cov(\xi_2, \varsigma_1)$ for any i. For $i \in \{1, 2, \ldots, n-3\}$, the covariance of ξ_i and ξ_{i+1} is

$$\begin{aligned} cov(\xi_i, \xi_{i+1}) &= E[\xi_i \xi_{i+1}] - E[\xi_i]E[\xi_{i+1}] \\ &= \Pr[X_i = X_{i+1} \neq X_{i+2} \text{ and } X_{i+1} = X_{i+2} \neq X_{i+3}] - (E[\xi_i])^2 \\ &= -(E[\xi_i])^2 \\ &= -\frac{q^2 + 2q^3 - 3q^4 - 4q^5 + 4q^6}{(1+q)^2(1+q+q^2)^2}, \end{aligned} \tag{5}$$

and similarly, for $i \in \{1, 2, \ldots, n-4\}$, the covariance of ξ_i and ξ_{i+2} is

$$\begin{aligned} & cov(\xi_i, \xi_{i+2}) \\ &= E[\xi_i \xi_{i+2}] - E[\xi_i]E[\xi_{i+2}] \\ &= \Pr[X_i = X_{i+1} \neq X_{i+2} \text{ and } X_{i+2} = X_{i+3} \neq X_{i+4}] - (E[\xi_i])^2 \\ &= \sum_{k \in \mathbb{N}} \Pr[X_i = X_{i+1} = k] \sum_{h \in \mathbb{N}; h \neq k} \Pr[X_{i+2} = X_{i+3} = h] \Pr[X_{i+4} \neq h] - (E[\xi_i])^2 \\ &= \sum_k (pq^{k-1})^2 \sum_{h \neq k} (pq^{h-1})^2 (1 - pq^{h-1}) - (E[\xi_i])^2 \\ &= \sum_k (pq^{k-1})^2 \left[\frac{p^2}{1-q^2} - \frac{p^3}{1-q^3} - p^2 q^{2k-2} + p^3 q^{3k-3} \right] - (E[\xi_i])^2 \\ &= -\frac{2q^3 - q^4 - 2q^5 + q^6 - 2q^7 + q^8 + 4q^9 - 2q^{11}}{(1+q)(1+q+q^2)^2(1+q+q^2+q^3)(1+q+q^2+q^3+q^4)}. \end{aligned} \tag{6}$$

On the other hand, the covariance of ξ_1 and ς_1 is

$$\begin{aligned} cov(\xi_1, \varsigma_1) &= E[\xi_1 \varsigma_1] - E[\xi_1]E[\varsigma_1] \\ &= -\Pr[X_1 = X_2 \neq X_3 \text{ and } X_1 = X_2] - E[\xi_1]E[\varsigma_1] \\ &= -\sum_{k \in \mathbb{N}} \Pr[X_1 = X_2 = k] \Pr[X_3 \neq k] - E[\xi_1]E[\varsigma_1] \\ &= -\frac{2q^2 + 2q^3 - 4q^4}{(1+q)^2(1+q+q^2)^2}, \end{aligned} \tag{7}$$

and the covariance of ξ_2 and ς_1 is

$$
\begin{aligned}
cov(\xi_2,\varsigma_1) &= E[\xi_2\varsigma_1]-E[\xi_2]E[\varsigma_1] \\
&= -\ \Pr[X_2 = X_3 \neq X_4 \text{ and } X_1 = X_2] - E[\xi_2]E[\varsigma_1] \\
&= -\sum_{k\in\mathbb{N}} \Pr[X_1 = X_2 = X_3 = k]\Pr[X_4 \neq k] - E[\xi_2]E[\varsigma_1] \\
&= -\ \frac{2q^3-4q^4+2q^5}{(1+q)(1+q+q^2)(1+q+q^2+q^3)}.
\end{aligned} \tag{8}
$$

Substituting (3), (4), and (7) into (2), we can obtain the variance of MAR_3, and substituting (3), (4), (5), (6), (7), and (8) into (2), we can obtain the variance of MAR_n for $n \geq 4$. This completes the proof of Theorem 1.

3. TOTAL NUMBER OF TURNING POINTS

In this section, we prove Theorem 2, which is about the properties of the total number of turning points in $X_1, \ldots, X_n$. Let T_n be the total number of turning points in $X_1, \ldots, X_n$. Recall that x_i is a turning point in a sequence $x_1, \ldots, x_n$, exactly when $\max\{x_{i+1}, x_{i-1}\} < x_i$ or $\min\{x_{i+1}, x_{i-1}\} > x_i$. Hence, for each $i \in \{1, 2, \ldots, n-2\}$, define a binary random variable η_i by

$$
\eta_i = \begin{cases} 1, & \text{if } (X_i < X_{i+1} \text{ and } X_{i+1} > X_{i+2}) \text{ or } (X_i > X_{i+1} \text{ and } X_{i+1} < X_{i+2}); \\ 0, & \text{otherwise,} \end{cases}
$$

and it is clear that

$$
T_n = \sum_{i=1}^{n-2} \eta_i. \tag{9}
$$

To simplify the evaluation of expectation and variance of T_n, we introduce the following handy lemma first.

Lemma 1 For any $i = 1, \ldots, n-3$ and $s \in \mathbb{N}$,

1. $\Pr[X_i > s] = q^s$.
2. $\Pr[X_i < s] = 1 - q^{s-1}$.
3. $\Pr[X_i < s \text{ and } X_{i+1} > X_i] = \Pr[X_{i+1} < s \text{ and } X_i > X_{i+1}] = \dfrac{q(1-q^{2s-2})}{1+q}$.
4. $\Pr[X_i > s \text{ and } X_{i+1} < X_i] = \Pr[X_{i+1} > s \text{ and } X_i < X_{i+1}] = q^s - q^{2s}/(1+q)$.

Proof: By the formula for a geometric series and the fact $p = 1 - q$, we have

$$
\Pr[X_i > s] = \sum_{t=s+1}^{\infty} \Pr[X_i = t] = \sum_{t=s+1}^{\infty} pq^{t-1} = \frac{p\cdot q^s}{1-q} = q^s,
$$

and

$\Pr[X_i < s] = 1 - \Pr[X_i > s - 1] = 1 - q^{s-1}$.

Moreover, since X_i and X_{i+1} are independent, we get

$$\Pr[X_i < s \text{ and } X_{i+1} > X_i] = \sum_{t=1}^{s-1} \Pr[X_i = t]\Pr[X_{i+1} > t] = \sum_{t=1}^{s-1} pq^{t-1} \cdot q^t$$
$$= \frac{pq(1-q^{2s-2})}{1-q^2} = \frac{q(1-q^{2s-2})}{1+q}.$$

On the other hand, we have

$$\Pr[X_{i+1} < s \text{ and } X_i > X_{i+1}] = \sum_{t=1}^{s-1} \Pr[X_{i+1} = t]\Pr[X_i > t] = \sum_{t=1}^{s-1} pq^{t-1} \cdot q^t$$
$$= \Pr[X_i < s \text{ and } X_{i+1} > X_i] = \frac{q(1-q^{2s-2})}{1+q}.$$

Similarly, we obtain

$$\Pr[X_i > s \text{ and } X_{i+1} < X_i] = \Pr[X_{i+1} > s \text{ and } X_i < X_{i+1}]$$
$$= \sum_{t=s+1}^{\infty} \Pr[X_i = t]\Pr[X_{i+1} < t] = \sum_{t=s+1}^{\infty} pq^{t-1} \cdot (1-q^{t-1})$$
$$= p \cdot \left[\frac{q^s}{1-q} - \frac{q^{2s}}{1-q^2}\right] = \frac{q(1-q^{2s})}{1+q}. \quad \square$$

Now we derive the expectation of T_n. For each $i \in \{1, \ldots, n-2\}$, since X_i's are independent, then by Lemma 1, we obtain $E[\eta_i]$ as

$$\Pr[X_i < X_{i+1} \text{ and } X_{i+1} > X_{i+2}] + \Pr[X_i > X_{i+1} \text{ and } X_{i+1} < X_{i+2}]$$
$$= \sum_{s \in \mathrm{N}} \Pr[X_{i+1} = s]\Pr[X_i < s]\Pr[X_{i+2} < s] + \sum_{s \in \mathrm{N}} \Pr[X_{i+1} = s]\Pr[X_i > s]\Pr[X_{i+2} > s]$$
$$= \sum_{s \in \mathrm{N}} pq^{s-1} \cdot (1-q^{s-1})^2 + \sum_{s \in \mathrm{N}} pq^{s-1} \cdot (q^s)^2$$
$$= \frac{q + q^2 + 2q^3}{(1+q)(1+q+q^2)}.$$

Then, by the linearity of expectation and the representation of Eq. (9), we have the expectation of T_n as

$$E[T_n] = \sum_{i=1}^{n-2} E[\eta_i] = (n-2)\frac{q + q^2 + 2q^3}{(1+q)(1+q+q^2)}.$$

Next, we evaluate the variance of T_n. For each $i \in \{1, 2, \ldots, n-2\}$, the variance of η_i is

$$Var[\eta_i] = E[(\eta_i)^2] - (E[\eta_i])^2 = \frac{q + 2q^2 + 4q^3 + 2q^4 + q^5 - 2q^6}{(1+q)^2(1+q+q^2)^2}. \tag{10}$$

Since for $i, j \in \{1, \ldots, n-2\}$ with $|i-j| \geq 3$, η_i and η_j are independent, we know $cov(\eta_i, \eta_j) = 0$. In addition, for $i \in \{1, 2, \ldots, n-3\}$, by Lemma 1, the expectation of $\eta_i \eta_{i+1}$ is

$$\begin{aligned}
E[\eta_i \eta_{i+1}] &= \Pr[(X_i < X_{i+1}) \text{ and } (X_{i+1} > X_{i+2}) \text{ and } (X_{i+2} < X_{i+3})] + \\
&\quad \Pr[(X_i > X_{i+1}) \text{ and } (X_{i+1} < X_{i+2}) \text{ and } (X_{i+2} > X_{i+3})] \\
&= \sum_{s \in \mathrm{N}} \Pr[X_{i+1} = s]\Pr[X_i < s]\Pr[X_{i+2} < s \text{ and } X_{i+3} > X_{i+2}] + \\
&\quad \sum_{s \in \mathrm{N}} \Pr[X_{i+1} = s]\Pr[X_i > s]\Pr[X_{i+2} > s \text{ and } X_{i+3} < X_{i+2}] \\
&= \sum_{s \in \mathrm{N}} pq^{s-1} \cdot (1 - q^{s-1}) \cdot \left(\frac{q(1-q^{2s-2})}{1+q} \right) + \sum_{s \in \mathrm{N}} pq^{s-1} \cdot q^s \cdot \left(q^s - \frac{q^{2s}}{1+q} \right) \\
&= \frac{2(q^2 + q^3 + q^4 + q^5 + q^6)}{(1+q)(1+q+q^2)(1+q+q^2+q^3)},
\end{aligned}$$

and therefore the covariance of η_i and η_{i+1} is

$$\begin{aligned}
cov(\eta_i, \eta_{i+1}) &= E[\eta_i \eta_{i+1}] - E[\eta_i]E[\eta_{i+1}] \\
&= \frac{2(q^2 + q^3 + q^4 + q^5 + q^6)}{(1+q)(1+q+q^2)(1+q+q^2+q^3)} - \left(\frac{q + q^2 + 2q^3}{(1+q)(1+q+q^2)} \right)^2 \\
&= \frac{q^2 + 2q^3 - 3q^6 - 2q^8)}{(1+q)(1+q+q^2)^2(1+q+q^2+q^3)}.
\end{aligned} \tag{11}$$

Similarly, for $i \in \{1, 2, \ldots, n-4\}$, by Lemma 1, the expectation of $\eta_i \eta_{i+2}$ is

$$\begin{aligned}
&E[\eta_i \eta_{i+2}] \\
&= \Pr[(X_i < X_{i+1}) \text{ and } (X_{i+1} > X_{i+2} > X_{i+3}) \text{ and } (X_{i+3} < X_{i+4})] \\
&\quad + \Pr[(X_i < X_{i+1}) \text{ and } (X_{i+1} > X_{i+2}) \text{ and } (X_{i+2} < X_{i+3}) \text{ and } (X_{i+3} > X_{i+4})] \\
&\quad + \Pr[(X_i > X_{i+1}) \text{ and } (X_{i+1} < X_{i+2} < X_{i+3}) \text{ and } (X_{i+3} > X_{i+4})] \\
&\quad + \Pr[(X_i > X_{i+1}) \text{ and } (X_{i+1} < X_{i+2}) \text{ and } (X_{i+2} > X_{i+3}) \text{ and } (X_{i+3} < X_{i+4})] \\
&= \sum_{s \in \mathrm{N}} \Pr[X_{i+3} = s]\Pr[X_{i+4} > s] \sum_{t=s+1}^{\infty} (\Pr[X_{i+2} = t]\Pr[X_{i+1} > t \text{ and } X_i < X_{i+1}]) \\
&\quad + \sum_{s \in \mathrm{N}} \Pr[X_{i+3} = s]\Pr[X_{i+4} < s] \sum_{t=1}^{s-1} (\Pr[X_{i+2} = t]\Pr[X_{i+1} > t \text{ and } X_i < X_{i+1}]) \\
&\quad + \sum_{s \in \mathrm{N}} \Pr[X_{i+1} = s]\Pr[X_i > s] \sum_{t=s+1}^{\infty} (\Pr[X_{i+2} = t]\Pr[X_{i+3} > t \text{ and } X_{i+4} < X_{i+3}]) \\
&\quad + \sum_{s \in \mathrm{N}} \Pr[X_{i+1} = s]\Pr[X_i > s] \sum_{t=s+1}^{\infty} (\Pr[X_{i+2} = t]\Pr[X_{i+3} < t \text{ and } X_{i+4} > X_{i+3}])
\end{aligned}$$

$$= \sum_{s\in\mathbb{N}} pq^{s-1}\cdot q^s \sum_{t=s+1}^{\infty}\left(pq^{t-1}\cdot\left(q^t - \frac{q^{2t}}{1+q}\right)\right)$$
$$+ \sum_{s\in\mathbb{N}} pq^{s-1}(1-q^{s-1})\sum_{t=1}^{s-1}\left(pq^{t-1}\cdot\left(q^t - \frac{q^{2t}}{1+q}\right)\right)$$
$$+ \sum_{s\in\mathbb{N}} pq^{s-1}\cdot q^s \sum_{t=s+1}^{\infty}\left(pq^{t-1}\cdot\left(q^t - \frac{q^{2t}}{1+q}\right)\right)$$
$$+ \sum_{s\in\mathbb{N}} pq^{s-1}\cdot q^s \sum_{t=s+1}^{\infty}\left(pq^{t-1}\cdot\left(\frac{q(1-q^{2t-2})}{1+q}\right)\right)$$
$$= \frac{q^2+2q^3+6q^4+8q^5+11q^6+9q^7+8q^8+5q^9+4q^{10}}{(1+q)(1+q+q^2)(1+q+q^2+q^3)(1+q+q^2+q^3+q^4)},$$

which leads to

$$cov(\eta_i, \eta_{i+2})$$
$$= E[\eta_i\eta_{i+2}] - E[\eta_i]E[\eta_{i+2}]$$
$$= \frac{q^5+q^6+q^7-q^8-q^9+q^{11}}{(1+q)(1+q+q^2)^2(1+q+q^2+q^3)(1+q+q^2+q^3+q^4)}. \tag{12}$$

Hence, by Eq. (10), we can obtain the variance of T_3, and by Eqs. (10)-(12), we can obtain the variance of T_n for $n \geq 4$. This completes the proof of Theorem 2.

4. TOTAL NUMBER OF ALTERNATING RUNS

In this section, we prove Theorem 3, that is, we consider the total number of alternating runs in a sequence of n independent random variables, which is exactly the sum of the number of modified alternating runs and the number of turning points. Let AR_n denote the number of alternating runs in a sequence of n independent geometrically distributed random variables $X_1, \ldots, X_n$. Therefore,

$$AR_n = MAR_n + T_n.$$

By the linearity of expectation along with Theorems 1 and 2, we have the expectation of AR_n as

$$E[AR_n] = E[MAR_n] + E[T_n]$$
$$= \frac{q+q^2-2q^3}{(1+q)(1+q+q^2)}n + \frac{6q^3}{(1+q)(1+q+q^2)} + (n-2)\frac{q+q^2+2q^3}{(1+q)(1+q+q^2)}$$
$$= \frac{2q+2q^2}{(1+q)(1+q+q^2)}n - \frac{2q+2q^2-2q^3}{(1+q)(1+q+q^2)}.$$

Next, we derive the variance of AR_n. Since for any $i, j \in \{1, \ldots, n-2\}$ with $|i-j| \geq 3$, ξ_i are independent of η_j, and ς_1 is independent of η_k for any $k \geq 3$, we have

$$
\begin{aligned}
&Var[AR_n] \\
&= Var\left[1+\sum_{i=1}^{n-2}\xi_i + \varsigma_1 + \sum_{i=1}^{n-2}\eta_i\right] \\
&= \sum_{i=1}^{n-2}Var[\xi_i] + Var[\varsigma_1] + \sum_{i=1}^{n-2}Var[\eta_i] + 2\left[\sum_{i=1}^{n-3}cov(\xi_i,\ \xi_{i+1}) + \sum_{i=1}^{n-4}cov(\xi_i,\ \xi_{i+2}) + \right. \\
&\quad cov(\xi_1,\ \varsigma_1) + cov(\xi_2,\ \varsigma_1) + \sum_{i=1}^{n-3}cov(\eta_i,\ \eta_{i+1}) + \sum_{i=1}^{n-4}cov(\eta_i,\ \eta_{i+2}) + cov(\eta_1,\ \varsigma_1) + \\
&\quad cov(\eta_2,\ \varsigma_1) + \sum_{i=1}^{n-2}cov(\xi_i,\ \eta_i) + \sum_{i=1}^{n-3}cov(\xi_i,\ \eta_{i+1}) + \sum_{i=1}^{n-4}cov(\xi_i,\ \eta_{i+2}) + \\
&\quad \left.\sum_{i=1}^{n-3}cov(\xi_{i+1},\ \eta_i) + \sum_{i=1}^{n-4}cov(\xi_{i+2},\ \eta_i)\right] \\
&= Var[MAR_n] + Var[T_n] + 2\left[cov(\eta_1,\ \varsigma_1) + cov(\eta_2,\ \varsigma_1) + \sum_{i=1}^{n-2}cov(\xi_i,\ \eta_i) + \right. \\
&\quad \left.\sum_{i=1}^{n-3}cov(\xi_i,\ \eta_{i+1}) + \sum_{i=1}^{n-4}cov(\xi_i,\ \eta_{i+2}) + \sum_{i=1}^{n-3}cov(\xi_{i+1},\ \eta_i) + \sum_{i=1}^{n-4}cov(\xi_{i+2},\ \eta_i)\right].
\end{aligned} \tag{13}
$$

Hence, by Theorems 1 and 2, it remains to evaluate $cov(\eta_1, \varsigma_1)$, $cov(\eta_2, \varsigma_1)$, $cov(\xi_i, \eta_i)$, $cov(\xi_i, \eta_{i+1})$, $cov(\xi_i, \eta_{i+2})$, $cov(\xi_{i+1}, \eta_i)$, and $cov(\xi_{i+2}, \eta_i)$ for any i.

The covariance of η_1 and ς_1 is

$$
\begin{aligned}
&cov(\eta_1,\varsigma_1) = E[\eta_1\varsigma_1] - E[\eta_1]E[\varsigma_1] \\
&= -\ \Pr[((X_1 < X_2 \text{ and } X_2 > X_3) \text{ or } (X_1 > X_2 \text{ and } X_2 < X_3)) \text{ and } X_1 = X_2] - E[\eta_1]E[\varsigma_1] \\
&= -\ E[\eta_1]E[\varsigma_1] \\
&= \frac{q+q^3-2q^4}{(1+q)^2(1+q+q^2)},
\end{aligned} \tag{14}
$$

and by Lemma 1, the covariance of η_2 and ς_1 is

$$
\begin{aligned}
&cov(\eta_2,\varsigma_1) \\
&= E[\eta_2\varsigma_1] - E[\eta_2]E[\varsigma_1] \\
&= -\Pr[((X_2 < X_3 \text{ and } X_3 > X_4) \text{ or } (X_2 > X_3 \text{ and } X_3 < X_4)) \text{ and } X_1 = X_2] - E[\eta_2]E[\varsigma_1] \\
&= -\Pr[X_1 = X_2 < X_3 \text{ and } X_3 > X_4] - \Pr[X_1 = X_2 > X_3 \text{ and } X_3 < X_4] - E[\eta_2]E[\varsigma_1] \\
&= -\sum_{s\in\mathbb{N}}\Pr[X_1 = X_2 = s]\cdot(\Pr[s < X_3 \text{ and } X_3 > X_4] + \Pr[s > X_3 \text{ and } X_3 < X_4]) - E[\eta_2]E[\varsigma_1] \\
&= -\sum_{s\in\mathbb{N}}(pq^{s-1})^2\cdot\left(q^s - \frac{q^{2s}}{1+q} + \frac{q(1-q^{2s-2})}{1+q}\right) - E[\eta_2]E[\varsigma_1] \\
&= \frac{q^3-2q^4+q^5}{(1+q)(1+q+q^2)(1+q+q^2+q^3)}.
\end{aligned} \tag{15}
$$

On the other hand, for $i \in \{1, \ldots, n-2\}$, the covariance of ξ_i and η_i is

$$
\begin{aligned}
&cov(\xi_i, \eta_i) \\
&= E[\xi_i \eta_i] - E[\xi_i]E[\eta_i] \\
&= \Pr[X_i = X_{i+1} \neq X_{i+2} \text{ and } ((X_i < X_{i+1} \text{ and } X_{i+1} > X_{i+2}) \text{ or} \\
&\quad (X_i > X_{i+1} \text{ and } X_{i+1} < X_{i+2}))] - E[\xi_i]E[\eta_i] \\
&= -E[\xi_i]E[\eta_i] \\
&= -\frac{q^2 + 2q^3 + q^4 - 4q^6}{(1+q)^2(1+q+q^2)^2},
\end{aligned}
\tag{16}
$$

and similarly, for $i \in \{1, \ldots, n-3\}$, the covariance of ξ_{i+1} and η_i is

$$
\begin{aligned}
&cov(\xi_{i+1}, \eta_i) \\
&= E[\xi_{i+1} \eta_i] - E[\xi_{i+1}]E[\eta_i] \\
&= \Pr[X_{i+1} = X_{i+2} \neq X_{i+3} \text{ and } ((X_i < X_{i+1} \text{ and } X_{i+1} > X_{i+2}) \text{ or} \\
&\quad (X_i > X_{i+1} \text{ and } X_{i+1} < X_{i+2}))] - E[\xi_{i+1}]E[\eta_i] \\
&= -E[\xi_{i+1}]E[\eta_i] \\
&= -\frac{q^2 + 2q^3 + q^4 - 4q^6}{(1+q)^2(1+q+q^2)^2}.
\end{aligned}
\tag{17}
$$

Then, by Lemma 1 and Eq. (16), for $i \in \{1, \ldots, n-3\}$, the covariance of ξ_i and η_{i+1} is

$$
\begin{aligned}
&cov(\xi_i, \eta_{i+1}) \\
&= E[\xi_i \eta_{i+1}] - E[\xi_i]E[\eta_{i+1}] \\
&= \Pr[X_i = X_{i+1} \neq X_{i+2} \text{ and } ((X_{i+1} < X_{i+2} \text{ and } X_{i+2} > X_{i+3}) \text{ or} \\
&\quad (X_{i+1} > X_{i+2} \text{ and } X_{i+2} < X_{i+3}))] - E[\xi_i]E[\eta_{i+1}] \\
&= \sum_{s \in \mathrm{N}} \Pr[X_i = X_{i+1} = s](\Pr[s < X_{i+2} \text{ and } X_{i+2} > X_{i+3}] + \\
&\quad \Pr[s > X_{i+2} \text{ and } X_{i+2} < X_{i+3}]) - E[\xi_i]E[\eta_{i+1}] \\
&= \sum_{s \in \mathrm{N}} (pq^{s-1})^2 \cdot \left(q^s - \frac{q^{2s}}{1+q} + \frac{q(1-q^{2s-2})}{1+q} \right) - E[\xi_i]E[\eta_{i+1}] \\
&= \frac{q - q^4 - q^5 + q^6 - 2q^7 + 2q^8}{(1+q)(1+q+q^2)(1+q+q^2+q^3)}.
\end{aligned}
\tag{18}
$$

Similarly, for $i \in \{1, \ldots, n-4\}$, the covariance of ξ_i and η_{i+2} is

$$
\begin{aligned}
&cov(\xi_i, \eta_{i+2}) \\
&= E[\xi_i \eta_{i+2}] - E[\xi_i]E[\eta_{i+2}] \\
&= \Pr[X_i = X_{i+1} \neq X_{i+2} \text{ and } ((X_{i+2} < X_{i+3} \text{ and } X_{i+3} > X_{i+4}) \text{ or} \\
&\quad (X_{i+2} > X_{i+3} \text{ and } X_{i+3} < X_{i+4}))] - E[\xi_i]E[\eta_{i+2}]
\end{aligned}
$$

$$= \sum_{k\in\mathrm{N}} \Pr[X_i = X_{i+1} = k] \sum_{s\in\mathrm{N}, s\neq k} \Pr[X_{i+2} = s]\cdot (\Pr[s < X_{i+3} \text{ and } X_{i+3} > X_{i+4}] +$$
$$\Pr[s > X_{i+3} \text{ and } X_{i+3} < X_{i+4}]) - E[\xi_i]E[\eta_{i+2}]$$
$$= \sum_{k\in\mathrm{N}} (pq^{k-1})^2 \sum_{s\in\mathrm{N}, s\neq k} pq^{s-1}\cdot \left(q^s - \frac{q^{2s}}{1+q} + \frac{q(1-q^{2s-2})}{1+q}\right) - E[\xi_i]E[\eta_{i+2}]$$
$$= \sum_{k\in\mathrm{N}} (pq^{k-1})^2 \cdot \left[\frac{q+q^2+2q^3}{(1+q)(1+q+q^2)} - pq^{k-1}\left(q^k - \frac{q^{2k}}{1+q} + \frac{q(1-q^{2k-2})}{1+q}\right)\right] - E[\xi_i]E[\eta_{i+2}]$$
$$= \frac{q^3+q^4-q^5-2q^6-2q^7+3q^9+2q^{10}-q^{11}-q^{12}}{(1+q)^2(1+q+q^2)^2(1+q+q^2+q^3)(1+q+q^2+q^3+q^4)}. \tag{19}$$

Finally, for $i \in \{1, \ldots, n-4\}$, the covariance of ξ_{i+2} and η_i is

$$cov(\xi_{i+2}, \eta_i)$$
$$= E[\xi_{i+2}\eta_i] - E[\xi_{i+2}]E[\eta_i]$$
$$= \Pr[X_{i+2} = X_{i+3} \neq X_{i+4} \text{ and } ((X_i < X_{i+1} \text{ and } X_{i+1} > X_{i+2}) \text{ or}$$
$$(X_i > X_{i+1} \text{ and } X_{i+1} < X_{i+2}))] - E[\xi_{i+2}]E[\eta_i]$$
$$= \sum_{k\in\mathrm{N}} \Pr[X_{i+1} = k]\Pr[X_i < k]\sum_{s=1}^{k-1} \Pr[X_{i+2} = s]\Pr[X_{i+3} = s]\Pr[X_{i+4} \neq s]$$
$$+ \sum_{k\in\mathrm{N}} \Pr[X_{i+1} = k]\Pr[X_i < k]\sum_{s=k+1}^{\infty} \Pr[X_{i+2} = s]\Pr[X_{i+3} = s]\Pr[X_{i+4} \neq s]$$
$$- E[\xi_{i+2}]E[\eta_i]$$
$$= \sum_{k\in\mathrm{N}} pq^{k-1}\cdot(1-q^{k-1})\sum_{s=1}^{k-1}(pq^{s-1})^2(1-pq^{s-1}) +$$
$$\sum_{k\in\mathrm{N}} pq^{k-1}\cdot q^k \sum_{s=k+1}^{\infty}(pq^{s-1})^2(1-pq^{s-1}) - E[\xi_{i+2}]E[\eta_i]$$
$$= \sum_{k\in\mathrm{N}} pq^{k-1}(1-q^{k-1})\left[\frac{p(1-q^{2k-2})}{1+q} - \frac{p^2(1-q^{3k-3})}{1+q+q^2}\right] +$$
$$\sum_{k\in\mathrm{N}} pq^{k-1}q^k\left[\frac{pq^{2k}}{1+q} - \frac{p^2q^{3k}}{1+q+q^2}\right] - E[\xi_{i+2}]E[\eta_i]$$
$$= \frac{-q^5-q^6-q^7+q^8+4q^9+2q^{10}-2q^{11}-2q^{12}}{(1+q)^2(1+q+q^2)^2(1+q+q^2+q^3)(1+q+q^2+q^3+q^4)}. \tag{20}$$

Now, substituting Theorems 1 and 2 along with Eqs. (14) and (16) into Eq. (13), we can obtain the variance of AR_3, and substituting Theorems 1 and 2 along with Eqs. (14)-(20) into Eq. (13), we can obtain the variance of AR_n for $n \geq 4$. This completes the proof of Theorem 3.

5. THE DISTRIBUTION OF THE NUMBER OF ALTERNATING RUNS

In this section, we prove Theorem 4. Note that

$$AR_n = MAR_n + T_n = 1 + \varsigma_1 + \sum_{i=1}^{n-2} (\xi_i + \eta_i).$$

We observe that the sequence of random variables ς_1, ξ_1, η_1, ξ_2, η_2, … is 5-dependent, where a sequence of random variables $\{Y_i\}$ is called m-dependent, if the two sets (Y_1, Y_2, …, Y_r) and (Y_{r+m+1}, Y_{r+m+2}, …, Y_n) are independent for any r. Hence, we can apply the central limit theorem for dependent random variables [20] to obtain the distribution of AR_n. The following lemma is immediately from Theorem 1 in [20].

Lemma 2 Let Y_1, Y_2, … be an m-dependent sequence of random variables such that

1. $E[|Y_i - E[Y_i]|^3] < \infty$ for any i and
2. $\lim_{\ell\to\infty} \ell^{-1} \sum_{k=1}^{\ell} A_{i+k} = A$ exists, uniformly for all i, where

$$A_i = Var[Y_{i+m}] + 2\sum_{j=1}^{m} cov(Y_{i+m-j},\ Y_{i+m}).$$

Then as $n \to \infty$, the random variables $n^{-1/2}\sum_{i=1}^{n}(Y_i - E[Y_i])$ has a limiting normal distribution with mean 0 and variance A.

For the sequence of random variables ς_1, ξ_1, η_1, ξ_2, η_2, …, we have that for any $i \geq 1$,

$$\begin{aligned} A_{2i-1} &= Var[\xi_{i+2}] + 2\{cov(\eta_{i+1},\ \xi_{i+2}) + cov(\xi_{i+1},\ \xi_{i+2}) + cov(\eta_i,\ \xi_{i+2}) + cov(\xi_i,\ \xi_{i+2})\} = A_1, \\ A_{2i} &= Var[\eta_{i+2}] + 2\{cov(\xi_{i+2},\ \eta_{i+2}) + cov(\eta_{i+1},\ \eta_{i+2}) + cov(\xi_{i+1},\ \eta_{i+2}) + cov(\eta_i,\ \eta_{i+2}) \\ &\quad + cov(\xi_i,\ \eta_{i+2})\} = A_2. \end{aligned}$$

Then, by Eqs. (3), (5), (6), (10)-(12), (16)-(20), we obtain that for any i,

$$\begin{aligned} &\lim_{\ell\to\infty} \frac{1}{\ell}\sum_{k=1}^{\ell} A_{i+k} = \frac{1}{2}(A_1 + A_2) \\ &= \frac{1}{2}\cdot\frac{4q + 4q^2 + 6q^4 + 6q^5 + 2q^6 + 10q^7 + 12q^8 + 6q^9 + 8q^{10} + 6q^{11}}{(1+q)(1+q+q^2)^2(1+q+q^2+q^3)(1+q+q^2+q^3+q^4)}. \end{aligned}$$

Hence, according to Lemma 2 and taking the limit $q \to 1$, we complete the proof of Theorem 4.

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