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Well-posedness of the Hydrodynamic Model for Semiconductors

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This paper concerns the well-posedness of the hydrodynamic model for semiconductor devices, a quasi-linear elliptic–parbolic–hyperbolic system. Boundary conditions for elliptic and parabolic equations are Dirichlet conditions while boundary conditions for the hyperbolic equations are assumed to be well-posed in L^2 sense. Maximally strictly dissipative boundary conditions for the hyperbolic equations satisfy the assumption of well-posedness in L^2 sense. The well-posedness of the model under the boundary conditions is demonstrated.

1. Introduction

This paper addresses the well-posedness of the hydrodynamic model for semiconductors. The model is derived from moments of the Boltzmann's equation, taken over group velocity space. When coupled with the charge conservation equation, it describes the behaviour of small semiconductor devices and accounts for special features such as hot electrons and velocity overshoots. The model consists of a set of non-linear conservation laws for particle number, momentum, and energy, coupled to Poisson's equation for the electric potential. It is a perturbation of the drift diffusion model [7]. We consider a ballistic diode problem which models the channel of a MOSFET, so the effect of holes in the model can be neglected. The model is [4, 11]

$$\partial_t \mathbf{n} + \nabla\cdot(\mathbf{n}V) = 0, \tag{1.1}$$

$$\partial_t V + (V\cdot\nabla)V + \frac{1}{\mathbf{mn}}\nabla(\mathbf{n}T) - \frac{\mathbf{q}}{\mathbf{m}}\nabla\Psi = -\frac{V}{\tau_{\mathrm{p}}}, \tag{1.2}$$

$$\partial_t T - \frac{1}{\mathbf{n}}\nabla\cdot(\mathbf{kn}\nabla T) + V\nabla T + \frac{2}{3}T\nabla\cdot V - \frac{2\mathbf{m}V^2}{3\tau_{\mathrm{p}}} + \frac{\mathbf{m}V^2}{3\tau_{\mathrm{w}}} + \frac{T - T_{\mathrm{s}}}{\tau_{\mathrm{w}}} = 0, \tag{1.3}$$

$$\Delta\Psi = \frac{\mathbf{q}}{\omega}(\mathbf{n} - Z(x)), \tag{1.4}$$

where $\mathbf{k}$ is a constant, and $t \in [0, t_M]$, $x \in \Omega$. The electron density is labeled $\mathbf{n}$, V is the average electron velocity, T the temperature in energy units (Boltzmann's constant

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has been set to 1), $\mathbf{m}$ the effective electron mass, $\mathbf{q}$ the electron charge, Ψ the electrostatic potential, T_s the surrounding temperature, $Z(x)$ the prescribed ion background density, ω the dielectric constant, and $\Omega \subset \mathbb{R}^3$ the bounded semiconductor domain. Electric field E is given by $E = -\nabla\Psi$. The collision terms of equations (1.2–1.3) are approximated in terms of momentum and energy relaxation times, respectively, [2,4]:

$$\tau_p = \beta_2 \frac{T_s}{T}, \quad \tau_w = \beta_2 \frac{T_s}{2T} + \beta_3 \frac{TT_s}{T + T_s} \tag{1.5}$$

for some constants β_2, β_3. The initial conditions of system (1.1–1.4) are

$$V(0,x) = V_0, \quad \mathbf{n}(0,x) = \mathbf{n}_0 > 0, \quad T(0,x) = T_0. \tag{1.6}$$

In addition, system (1.1–1.4) is supplemented by the following boundary conditions:

$$M(t,x,V,\mathbf{n},T)\begin{pmatrix} V \\ \rho \end{pmatrix}\Bigg|_{\partial\Omega'} = \mathscr{B}, \quad \Psi|_{\partial\Omega'} = \Psi_b, \quad T|_{\partial\Omega'} = T_b, \tag{1.7}$$

where $e^\rho = \mathbf{n}$ and the admissible form of M will be explained in section 3. The hydrodynamic model couples a hyperbolic system of two equations (equations (1.1–1.2)), a parabolic equation (equation (1.3)), and an elliptic equation (equation (1.4)).

For steady-state solutions of Euler–Poisson model, a simplified version of the hydrodynamic model, we refer the reader to the works [4, 7, 11, 13, 16], and references therein. The well-posedness of the transient model in the two-dimensional space was discussed in [13], where maximally strictly dissipative boundary conditions for hyperbolic equations were mainly concerned. For non-linear hyperbolic initial boundary value problems, we refer to [9, 10].

We consider the transient model in three-dimensional space. Boundary conditions for elliptic and parabolic equations are Dirichlet conditions while boundary conditions for the hyerbolic equations (1.1–1.2) are assumed to be well-posed in L^2 sense. Maximally strictly dissipative boundary conditions for the hyperbolic equations are shown to satisfy the assumption of well-posedness in L^2 sense. The well-posedness of the model under these boundary conditions is then demonstrated. To do this, we first establish an existence theorem for a linear hyperbolic system by extending the results of [8, 17]. Then the existence and the uniqueness of the solution of the transient hydrodynamic model are proved by employing contractive map and interpolation theorem [6, 12]. More precisely, we prove the following:

Theorem 1.1. *Under the assumptions* A1–8 (see section 3), *system* (1.1–1.7) *has a unique classical local-in-time solution.*

This paper is organized as follows: In section 2, pertinent notation is reviewed. In section 3, we reformulate the differential equations (1.1–1.7) and present the assumptions A1–8 of Theorem 1.1. In section 4, maximally strictly dissipative boundary conditions for hyperbolic system are shown to satisfy the assumption of well-posedness in L^2 sense. In section 5, we establish an existence theorem of a linear hyperbolic system for our application. The existence and the uniqueness of a local-in-time solution of (1.1–1.7) are proved in section 6.

2. Notation

Summation convention is used. c denotes various constants, which may not be the same. That τ is a tangential vector field in Ω means a vector field τ in Ω satisfies $\tau\cdot\vec{n}=0$ on $\partial\Omega$ (where $\vec{n}=(n_1,n_2,n_3)$ is the unit outward normal vector on $\partial\Omega$). $\Omega^t=[0,t_M]\times\Omega$, $\partial\Omega^t=[0,t_M]\times\partial\Omega$. For a system

$$\begin{aligned} \mathscr{L}U &:= A^0\partial_t U + A^j\partial_{x_j}U + BU = F && \text{in } \Omega^t,\\ \mathscr{M}(t,x)U &= g && \text{on } \partial\Omega^t,\\ U(0,x) &= f(x) && \text{in } \Omega, \end{aligned} \tag{2.1}$$

the partial derivative "$\partial_t^i U(0)$" of (2.1) is defined by formally taking $i-1$ time derivatives of the system, solving for $\partial_t^i U$, and evaluating at time $t=0$, e.g. "$U(0)$" $=f$, "$\partial_t U(0)$" $=(A^0(0))^{-1}[F(0)-A^j(0)\partial_{x_j}f-B(0)f]$, etc. That compatibility conditions for (2.1) hold up to $m-1$ means that, on $\partial\Omega$ for $0\leqslant p\leqslant m-1$,

$$\sum_{i=0}^{p}\binom{p}{i}\partial_t^i\mathscr{M}(0,x)\partial_t^{p-i}U(0,x)=\partial_t^p g(0,x).$$

Let $\mathscr{L}^*$ be, the formal adjoint of $\mathscr{L}$,

$$\mathscr{L}^* = -A^0\partial_t - A^j\partial_{x_j} + B^* - \partial_t A^0 - \partial_{x_j}A^j,$$

where B^* is the conjugate transpose of the matrix B. The kernel of $\mathscr{M}$ is the boundary subspace denoted by N. The adjoint boundary subspace is $N^*=(A_{\vec{n}}N)^{\perp}$, where $A_{\vec{n}}:=\sum A^j n_j$. Let $\mathscr{M}^*$, adjoint boundary operator, be a matrix-valued function on $\mathbb{R}\times\partial\Omega$ whose kernel is N^*.

For $p\geqslant 1$, $(H^p)'$ is the dual space of H^p. For a matrix A, $|A|$ denotes its operator norm. Suppose A is a positive definite matrix, we define the norm $\|U\|_{s,A}^2$ by

$$\|U\|_{s,A}^2=\sum_{|\gamma|\leqslant s}\int_\Omega(\partial^\gamma U, A\partial^\gamma U)\,\mathrm{d}x.$$

$C(H^s):=C(0,t_M,H^s(\Omega))$, $L^p(H^s):=L^p(0,t_M,H^s(\Omega))$. The space $X_s(0,t_M,\Omega)$ is

$$X_s(0,t_M,\Omega):=\bigcap_{i=0}^{s}C^i(0,t_M,H^{s-i}(\Omega))$$

with norm $|||U|||_{X_s}=\sup_t|U|_{s,0,\Omega}(t)$ where $|U|_{s,0,\Omega}(t)=(\sum_{i=0}^s\|\partial_t^i U\|_{H^{s-i}(\Omega)}^2(t))^{1/2}$. In addition, we introduce the norm

$$\|U\|_{0,[t_1,t_2]\times\Omega,\alpha}^2=\int_{t_1}^{t_2}\int_\Omega|U|^2\mathrm{e}^{-2\alpha t}\,\mathrm{d}x\,\mathrm{d}t,$$

$$\|U\|_{0,[t_1,t_2]\times\partial\Omega,\alpha}^2=\int_{t_1}^{t_2}\int_{\partial\Omega}|U|^2\mathrm{e}^{-2\alpha t}\,\mathrm{d}\sigma\,\mathrm{d}t.$$

3. Statement of the problem

We rewrite the system (1.1–1.4). Letting $e^\rho = \mathbf{n}$, (1.1–1.4) can be written as

$$\partial_t V + (V\cdot\nabla)V + \frac{T}{\mathbf{m}}\nabla\rho + \frac{V}{\tau_p} = \frac{\mathbf{q}}{\mathbf{m}}\nabla\Psi - \frac{1}{\mathbf{m}}\nabla T, \tag{3.1}$$

$$\partial_t\rho + V\nabla\rho + \nabla\cdot V = 0, \tag{3.2}$$

$$\partial_t T - \mathbf{k}\Delta T = \mathbf{k}\nabla\rho\nabla T - V\nabla T - \tfrac{2}{3}T\nabla\cdot V + \frac{2\mathbf{m}V^2}{3\tau_\mathrm{p}} - \frac{\mathbf{m}V^2}{3\tau_\mathrm{w}} - \frac{T-T_\mathrm{s}}{\tau_\mathrm{w}}, \tag{3.3}$$

$$\Delta\Psi = \frac{\mathbf{q}}{\omega}(e^\rho - Z(x)). \tag{3.4}$$

The new system (3.1–3.4) is equivalent to the old system (1.1–1.4).

To solve (3.1–3.4), we uncouple the system into a Poisson equation, a parabolic equation, and a linearized symmetric hyperbolic system. Next we show a fixed point of the uncoupled system exists in some weak spaces. Then by interpolation theorem, we show the fixed point is a classical solution of system (3.1–3.4).

System (3.1–3.4) is uncoupled as follows: Given Q, η, S, find Ψ by solving

$$\begin{aligned} &\Delta\Psi = \frac{\mathbf{q}}{\omega}(e^\eta - Z(x)),\\ &\Psi|_{\partial\Omega} = \Psi_\mathrm{b}. \end{aligned} \tag{3.5}$$

Then determine T by solving

$$\begin{aligned} &\partial_t T - \mathbf{k}\Delta T = \mathbf{k}\nabla\eta\nabla S - Q\nabla S - \tfrac{2}{3}S\nabla\cdot Q + \frac{2\mathbf{m}Q^2}{3\tau_\mathrm{p}(S)} - \frac{\mathbf{m}Q^2}{3\tau_\mathrm{w}(S)} - \frac{S-T_\mathrm{s}}{\tau_\mathrm{w}(S)},\\ &T|_{\partial\Omega'} = T_\mathrm{b},\\ &T(0,x) = T_0. \end{aligned} \tag{3.6}$$

Finally, we solve the following linearized system for V and ρ

$$\begin{aligned} &\partial_t V + (Q\cdot\nabla)V + \frac{T}{\mathbf{m}}\nabla\rho + \frac{V}{\tau_\mathrm{p}(T)} = \frac{\mathbf{q}}{\mathbf{m}}\nabla\Psi - \frac{1}{\mathbf{m}}\nabla T,\\ &\frac{T}{\mathbf{m}}\partial_t\rho + \frac{T}{\mathbf{m}}Q\nabla\rho + \frac{T}{\mathbf{m}}\nabla\cdot V = 0, \end{aligned} \tag{3.7}$$

$$M(t,x,Q,\eta,T)\begin{pmatrix} V\\ \rho\end{pmatrix}\Bigg|_{\partial\Omega'} = \mathscr{B}, \tag{3.8}$$

$$V(0,x) = V_0, \quad \rho(0,x) = \ln\mathbf{n}_0. \tag{3.9}$$

Obviously, a smooth fixed point of system (3.5–3.9) is a classic solution of the system (3.1–3.4) and vice versa. Therefore, we can proceed to work with system (3.5–3.9).

We note that if a smooth solution of the system (1.1–1.4) exists, the following compatibility conditions hold, for $p \geqslant 0$,

$$\sum_{i=0}^{p}\binom{p}{i}\partial_t^i M(0,x)\partial_t^{p-i}\begin{pmatrix} V\\ \rho\end{pmatrix}(0,x)\Bigg|_{\partial\Omega} = \partial_t^p\mathscr{B}(0,x), \tag{3.10}$$

$$\partial_t^p T(0,x)|_{\partial\Omega} = \partial_t^p T_\mathrm{b}(0,x), \tag{3.11}$$

where derivatives are obtained by differentiating (1.1–1.4) with respect to t and setting $t = 0$.

Define $(\bar{V}_i, \bar{\rho}_i, \bar{T}_i)$ as follows: When $i = 0$, it is the initial condition $(V_0, \ln \mathbf{n}_0, T_0)$; When $i = 1, 2$, it is obtained by formally taking $i - 1$ time derivatives of the system (1.1–1.3), solving for $(\partial_t^i V, \partial_t^i \rho, \partial_t^i T)$, and evaluating at time $t = 0$. Define K_{δ, t_M} by

$$K_{\delta,t_M} := \left\{ (Q,\eta,S) \;\middle|\; \begin{array}{l} (Q,\eta,S) \in X_3(\Omega^t) \cap H^3(\partial\Omega^t), \\ |||(Q,\eta,S) - (V_0, \ln \mathbf{n}_0, T_0)|||_{X_3(\Omega^t)} \\ \quad + \|(Q,\eta,S) - (V_0, \ln \mathbf{n}_0, T_0)\|_{H^3(\partial\Omega^t)} \leqslant \delta, \\ (\partial_t^i Q, \partial_t^i \eta, \partial_t^i S)(0,x) = (\bar{V}_i, \bar{\rho}_i, \bar{T}_i),\ i = 0,1,2. \end{array} \right\}. \tag{3.12}$$

Let us make the following assumptions

(A1) Ω is open, bounded, and smooth in $\mathbb{R}^3$,
(A2) $0 < \langle V_0, \vec{\mathbf{n}} \rangle^2|_{\partial\Omega}$ and $0 < |\langle V_0, \vec{\mathbf{n}} \rangle^2 - (T_0/\mathbf{m})|$ in $\partial\Omega$,
(A3) $0 < \mathbf{n}_0, T_0, T_b$ and $V_0, \mathbf{n}_0 \in H^4(\Omega)$,
(A4) $T_s \in C^3(\Omega^t)$, $Z(x) \in H^3(\Omega)$, $\Psi_b \in H^4(\partial\Omega^t)$, $\mathscr{B} \in H^3(\partial\Omega^t)$, $M \in H^3(\Omega^t \times \mathbb{R}^5)$,
(A5) $T_0 \in H^5(\Omega)$, $\bar{T}_1 \in H^4(\Omega)$, $\bar{T}_2 \in H^2(\Omega)$, and there is a $\tilde{T}$ satisfying $\tilde{T}|_{\partial\Omega^t} = T_b$, $\partial_t^i \tilde{T} \in L^2(H^{4-i})$, $i = 0,1,2,3$, $\partial_t^4 \tilde{T} \in L^2(H^{-1})$,
(A6) compatibility conditions (3.10–3.11) for system (1.1–1.4) hold up to 2.

Remark 1. *By* A3–5, *we know* $(\bar{V}_i, \bar{\rho}_i, \bar{T}_i) \in H^{3-i}(\Omega)$, $i = 0,1,2$. *By trace theorems* [3], *we see* K_{δ,t_M} *is not an empty set for some* δ.

Given $(Q,\eta,S) \in K_{\delta,t_M}$, *we are able to solve* (3.5–3.6) *to obtain* T. *Then by* (Q,η,T), *we want to solve* (3.7–3.9). *For convenience,* (3.7–3.9) *will be written as*

$$\begin{aligned} &\mathscr{L}W := A^0(T)\partial_t W + A^j(Q,T)\partial_{x_j} W + B(T)W = F && \text{in } \Omega^t, \\ &M(t,x,Q,\eta,T)W = \mathscr{B} && \text{on } \partial\Omega^t, \\ &W(0,x) = (V_0, \ln \mathbf{n}_0)^t && \text{in } \Omega, \end{aligned} \tag{3.13}$$

where

$$W := (V,\rho)^t,\ F := \left(\frac{\mathbf{q}}{\mathbf{m}} \nabla\Psi - \frac{1}{\mathbf{m}} \nabla T, 0\right)^t.$$

Let $\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ *be constants satisfying*

(1) $0 < \mathbf{a}_1 < A^0(T_0) < \mathbf{a}_0$ *in* Ω^t,
(2) $n_j A^j(V_0, T_0)$ *is non-singular and* $0 < \mathbf{a}_3 < |n_j A^j(V_0, T_0)| < \mathbf{a}_2$ *on* $\partial\Omega^t$.

By A2–3, *above four constants exist. Two more assumptions for* (3.13).

(A7) There are two constants M_1, M_2 such that, for any $(Q_i, \eta_i, S_i) \in K_{\delta,t_M}$, $i = 1,2,3$, the following hold on boundary $\partial\Omega^t$,

$$\left| (M(t,x,Q_1,\eta_1,S_1) - M(t,x,Q_2,\eta_2,S_2)) \begin{pmatrix} Q_3 \\ \eta_3 \end{pmatrix} \right| < (|V_0|M_1 + |\ln \mathbf{n}_0| M_2)(|Q_1 - Q_2| + |\eta_1 - \eta_2| + |S_1 - S_2|),$$

and

$$\frac{4\mathbf{a}_2}{\mathbf{a}_1 \mathbf{a}_3}(|V_0|M_1 + |\ln \mathbf{n}_0| M_2) < \min\{1, c(\Omega)\mathbf{k}\}.$$

(A8) There is a positive constant α_0 such that, for any $\alpha \geqslant \alpha_0$, $W \in H^1(\mathbb{R}\times\bar{\Omega})$, $(Q,\eta,T) \in K_{\delta,t_M}$

(a) the followings hold:

$$\mathbf{a}_1 \| W e^{-2\alpha t} \|^2_{L^2(\Omega)}(t_2) + \frac{\alpha \mathbf{a}_1}{2} \| W \|^2_{0,[t_1,t_2]\times\Omega,\alpha} + \frac{1}{2\mathbf{a}_2} \| W \|^2_{0,[t_1,t_2]\times\partial\Omega,\alpha}$$

$$< \frac{2}{\mathbf{a}_1\mathbf{a}_3} \| MW \|^2_{0,[t_1,t_2]\times\partial\Omega,\alpha} + \frac{2}{\alpha\mathbf{a}_1} \| \mathscr{L}W \|^2_{0,[t_1,t_2]\times\Omega,\alpha} + \mathbf{a}_0 \| W e^{-2\alpha t} \|^2_{L^2(\Omega)}(t_1), \tag{3.14}$$

$$\mathbf{a}_1 \| W e^{2\alpha t} \|^2_{L^2(\Omega)}(t_1) + \frac{\alpha \mathbf{a}_1}{2} \| W \|^2_{0,[t_1,t_2]\times\Omega,-\alpha} + \frac{1}{2\mathbf{a}_2} \| W \|^2_{0,[t_1,t_2]\times\partial\Omega,-\alpha}$$

$$< \frac{2}{\mathbf{a}_1\mathbf{a}_3} \| M^* W \|^2_{0,[t_1,t_2]\times\partial\Omega,-\alpha} + \frac{2}{\alpha\mathbf{a}_1} \| \mathscr{L}^* W \|^2_{0,[t_1,t_2]\times\Omega,-\alpha}$$

$$+ \mathbf{a}_0 \| W e^{2\alpha t} \|^2_{L^2(\Omega)}(t_2), \tag{3.15}$$

where $0 \leqslant t_1 < t_2 \leqslant t_M$, $\mathscr{L}, M$ are the operators in (3.13), and $\mathscr{L}^*, M^*$ are the adjoint operators of $\mathscr{L}, M$ (see section 2),

(b) there are B_k, M_k, and symmetric $A_k^0, A_k^j \subset C^\infty(\bar{\Omega}^t)$ satisfying

$$\begin{aligned} &\{A_k^0, A_k^j, B_k\} \to \{A^0(T), A^j(Q,T), B(T)\} && \text{in } X_3(\Omega^t), \\ &M_k \to M(t,x,Q,\eta,T) && \text{on } H^3(\partial\Omega^t), \end{aligned} \quad k\to\infty,$$

and inequalities (3.14–3.15) hold for the following

$$\begin{aligned} &\mathscr{L}_k W := A_k^0 \partial_t W + A_k^j \partial_{x_j} W + B_k W = F && \text{in } \Omega^t, \\ &M_k W = \mathscr{B} && \text{on } \partial\Omega^t, \\ &W(0,x) = (V_0, \ln \mathbf{n}_0)^t && \text{in } \Omega, \end{aligned} \tag{3.16}$$

where $\mathscr{L}, M, \mathscr{L}^*, M^*$ in (3.14–3.15) are replaced by $\mathscr{L}_k, M_k, \mathscr{L}_k^*, M_k^*$, respectively, ($\mathscr{L}_k^*, M_k^*$ are the adjoint operators of $\mathscr{L}_k, M_k$).

Actually, A8 assume that boundary condition $M(t,x,Q,\eta,T)$ for the hyperbolic system (3.13) is well-posed in L^2 sense. In section 6 we will show that if A1–8 hold, then a fixed point of system (3.5–3.9) exists uniquely in a short-time period, which implies Theorem 1.1.

4. Example

We give one example for M of (1.7) such that assumptions A1–8 hold. Multiplying (3.2) by $T/\mathbf{m}$ and combining it with equation (3.1) along with initial and boundary conditions, we obtain

$$\begin{aligned} &A^0(T)\partial_t W + A^j(V,T)\partial_{x_j} W + B(T)W = F && \text{in } \Omega^t, \\ &M(t,x,V,\rho,T)W = \mathscr{B} && \text{on } \partial\Omega^t, \\ &W(0,x) = (V_0, \ln \mathbf{n}_0)^t && \text{in } \Omega, \end{aligned} \tag{4.1}$$

where $W := (V, \rho)^t$. Assume that the matrix $A_{\vec{n}}(=A^j n_j)$ of (4.1) on $\partial\Omega^t$ has negative eigenvalues λ_μ^-, $\mu = 1 \ldots \ell$, and positive eigenvalues λ_ν^+, $\nu = \ell + 1 \ldots 4$, and, corresponding to them, orthonormal eigenvectors r_μ^-, r_ν^+, $\mu = 1 \ldots \ell$, $\nu = \ell + 1 \ldots 4$. The boundary condition M is taken as

$$MW = \sum_{\mu,\nu=1}^{\ell} (r_\nu^- \cdot W) a_{\mu,\nu} r_\mu^- + \sum_{\mu=1}^{\ell} \sum_{\nu=\ell+1}^{4} (r_\nu^+ \cdot W) b_{\mu,\nu} r_\mu^- \tag{4.2}$$

where $a_{\mu,\nu}, b_{\mu,\nu}$ are smooth functions and $(a_{\mu,\nu})$ is an invertible matrix in $\partial\Omega^t$. If $b_{\mu,\nu} = 0$ for all μ, ν, then M is maximally strictly dissipative boundary condition [17].

Remark 2. *The eigenvalues of $A_{\vec{n}}$* for (4.1) *are* $\lambda_1 = \lambda_2 = V_{\vec{n}}(:= V \cdot \vec{\mathbf{n}})$,

$$\lambda_3 = \frac{(1 + T/\mathbf{m})V_{\vec{n}} + \sqrt{[V_{\vec{n}}^2(1 + T/\mathbf{m})^2 - 4T/\mathbf{m}(V_{\vec{n}}^2 - T/\mathbf{m})]}}{2},$$

$$\lambda_4 = \frac{(1 + T/\mathbf{m})V_{\vec{n}} - \sqrt{[V_{\vec{n}}^2(1 + T/\mathbf{m})^2 - 4T/\mathbf{m}(V_{\vec{n}}^2 - T/\mathbf{m})]}}{2}.$$

Let $\tau_1 := (\tau_{11}, \tau_{12}, \tau_{13})$, $\tau_2 := (\tau_{21}, \tau_{22}, \tau_{23})$ be two orthonormal tangential vectors in Ω, and ζ be defined by

$$\zeta = \frac{(1 - T/\mathbf{m})V_{\vec{n}} + \sqrt{[V_{\vec{n}}^2(1 + T/\mathbf{m})^2 - 4T/\mathbf{m}(V_{\vec{n}}^2 - T/\mathbf{m})}}{2}.$$

The corresponding eigenvectors of $\lambda_1, \lambda_2, \lambda_4$ are $(\tau_{11}, \tau_{12}, \tau_{13}, 0)$, $(\tau_{21}, \tau_{22}, \tau_{23}, 0)$, and $(n_1, n_2, n_3, -\mathbf{m}\zeta/T)$, respectively.

Therefore on inflow boundary $\{(t,x) \in \partial\Omega^t | V_{\vec{n}}(0,x) < 0\}$, (1) if $V_{\vec{n}}^2(0,x) < T_0/\mathbf{m}$, we specify three boundary conditions (because $\lambda_1, \lambda_2, \lambda_4$ are negative and λ_3 positive), (2) if $T_0/\mathbf{m} < V_{\vec{n}}^2(0,x)$, M is the identity matrix (since all of the eigenvalues are negative). The former corresponds to subsonic case, and the latter to supersonic case. However an outflow boundary $\{(t,x) \in \partial\Omega^t | V_{\vec{n}}(0,x) > 0\}$, (3) if $V_{\vec{n}}^2(0,x) < T_0/\mathbf{m}$, we need one boundary condition (because only λ_4 is negative), (4) if $T_0/\mathbf{m} < V_{\vec{n}}^2(0,x)$, M is the zero matrix (because no boundary condition is needed). Similarly, (3) and (4) correspond to subsonic and supersonic cases, respectively.

For example, $MW = \mathscr{B}$ of (1.7) can be: in case (1)

$$MW = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} & 0 \\ \tau_{21} & \tau_{22} & \tau_{23} & 0 \\ n_1 & n_2 & n_3 & -\mathbf{m}\zeta/T \end{pmatrix} \begin{pmatrix} W_1 \\ W_2 \\ W_3 \\ \rho \end{pmatrix} = \mathscr{B},$$

which means $(W_{\tau_1}, W_{\tau_2}, W_{\vec{n}} - \rho\mathbf{m}\zeta/T) = \mathscr{B}^t$ on $\partial\Omega$, where $W_{\tau_i} := W \cdot \tau_i$, $i = 1, 2$, in case (2) $(W, \rho) = \mathscr{B}^t$ on $\partial\Omega$, in case (3) $M = (n_1, n_2, n_3, -\mathbf{m}\zeta/T)$, i.e. $W_{\vec{n}} - \rho\mathbf{m}\zeta/T = \mathscr{B}$ on $\partial\Omega$, and in case (4) $M = 0$.

If A1–6 and (4.2) hold, and if $|\rho|(0,x)|_{\partial\Omega}$, $\{b_{\mu,\nu}\}$ of (4.2) are sufficiently small, then they imply assumptions A1–8 hold. A7 holds obviously if $|\rho|(0,x)|_{\partial\Omega}$ is small. The following lemma implies A8.

Lemma 4.1. *Consider* (1.1–1.7). *Besides assumptions* A1–6, *if the following holds*: M *of* (1.7) *is constructed as* (4.2), $\{a_{\mu,v}\}$ *and* $\{b_{\mu,v}\}$ *are in* $H^3(\partial\Omega^t)$, *and they satisfy*

$$\max_{\partial\Omega^t} |a_{\mu,v}^{-1}| < \mathbf{b}_1^{-1}, \quad \max_{\partial\Omega^t} |b_{\mu,v}| < \mathbf{b}_0, \quad \forall \mu, v,$$

$$4(1/\mathbf{a}_2 + 1/\mathbf{a}_3)\mathbf{b}_1^{-2}\mathbf{b}_0^2 < 1/\mathbf{a}_2,$$

for some constants $\mathbf{b}_1, \mathbf{b}_0$, *then* A8 *holds for any* δ *as long as* t_M *is small enough.*

Proof. (3.14) is proved by energy L^2 estimate. By means of a local co-ordinate change and a partition of unity, it is sufficient to prove (3.14) in a half space. The estimate in a half-plane can be obtained as follows: Multiplying (3.13) by $W\,e^{-2\alpha t}$, integrating the resulting equation over Ω^t, and employing assumptions of boundary conditions of this lemma.

(3.15) can be proved in a similar way as (3.14). A8(b) holds by using Friedrichs' mollifiers and arguing as the proof for (3.14). ∎

5. Existence results for linear problems

In this section, we present existence results for a parabolic equation (Lemma 5.1) and for a linear symmetric hyperbolic system (Lemma 5.4). These are used to show the existence of a fixed point of system (3.5–3.9) in section 6. Lemma 5.1 is a standard result. Lemma 5.4 is established by Lemmas 5.2, 5.3. Lemma 5.2 gives an a priori estimate for a linear symmetric hyperbolic system (5.3) by energy method. Lemma 5.3 is to present approximate systems of (5.3) so that the solutions of these approximate systems exist. Lemma 5.4 is to show that the solutions of the approximate systems converge (by a priori estimate obtained in Lemma 5.2). Furthermore, we show the limit is a solution of (5.3). Domain Ω will be assumed in $\mathbb{R}^d$.

Lemma 5.1. *Consider the following system*

$$\begin{aligned} &\partial_t U - \mathbf{a}\Delta U = F \quad && \text{in } \Omega^t, \\ &U = g && \text{on } \partial\Omega^t, \\ &U(0,x) = f && \text{in } \Omega, \end{aligned} \tag{5.1}$$

where $\mathbf{a}$ *is a constant. If* $F \in H^2(\Omega^t)$, $\partial_t^3 F \in L^2(H^{-1})$ *and if there exists a function* $\tilde{g}$ *satisfying*

$$\tilde{g}|_{\partial\Omega^t} = g, \quad \partial_t^i \tilde{g} \in L^2(H^{4-i}),\ i = 0,1,2,3, \quad \partial_t^4 \tilde{g} \in L^2(H^{-1}),$$

$$\partial_t^i(U - \tilde{g})(0,x) \in H_0^1(\Omega),\ i = 0,1,2, \quad \partial_t^3(U - \tilde{g})(0,x) \in L^2(\Omega),$$

then there is a $U \in X_3(\Omega^t)$ *such that* $\partial_t^i U \in L^2(H^{4-i})$, $i = 0,1,2,3$, *and*

$$|||U|||_{X_3}^2 + \sum_{i=0}^{3} \|\partial_t^i U\|_{L^2(H^{4-i})}^2 \leqslant c\left[\sum_{i=0}^{3} \|\partial_t^i \tilde{g}\|_{L^2(H^{4-i})}^2 + \|\partial_t^4 \tilde{g}\|_{L^2(H^{-1})}^2 + \|F\|_{H^2(\Omega^t)}^2 \right.$$
$$\left. + \|\partial_t^3 F\|_{L^2(H^{-1})}^2 + \sum_{i=2}^{3} \|\partial_t^i U(0,x)\|_{H^{3-i}(\Omega)}^2 \right]. \tag{5.2}$$

Proof. This lemma can be proved by employing the results in [14, 15]. ∎

Lemma 5.2. *Consider the system*

$$\begin{aligned}
&\mathscr{L}U := A^0\partial_t U + A^j\partial_{x_j}U + BU = F && \text{in } \Omega^t,\\
&\mathscr{M}U = g && \text{on } \partial\Omega^t,\\
&U(0,x) = f(x) && \text{in } \Omega.
\end{aligned} \tag{5.3}$$

If the following conditions hold:

1. Ω *is open, bounded, and smooth in* $\mathbb{R}^{\mathbf{d}}$,
2. $A^0, A^j, B \in X_s(\Omega^t)$ *are* $\mathbf{j}\times\mathbf{j}$ *matrices,* $\mathscr{M} \in H^s(\partial\Omega^t)$, *and* $s \geqslant [\mathbf{d}/2] + 2$,
3. A^0, A^j *are symmetric and* $0 < \mathbf{a}_1 < A^0 < \mathbf{a}_0$ *in* Ω^t,
4. $A_{\bar{\mathbf{n}}} := \sum A^j n_j$ *is nonsingular on* $\partial\Omega^t$ *and* $0 < \mathbf{a}_3 < |A_{\bar{\mathbf{n}}}^{-1}| < \mathbf{a}_2$ *on* $\partial\Omega^t$,
5. $F \in H^m(\Omega^t), f \in H^m(\Omega)$, $g \in H^m(\partial\Omega^t)$, $1 \leqslant m \leqslant s$,
6. *for any* $U \in H^1(\mathbb{R}\times\bar{\Omega})$, $0 \leqslant t_1 < t_2 \leqslant t_M$, *and* $\mathbf{a}_1\alpha \geqslant c|||L|||_{X_s(\Omega^t)}$, (3.14) *holds for* (5.3), *i.e.*

$$\mathbf{a}_1\|U\mathrm{e}^{-2\alpha t}\|^2_{L^2(\Omega)}(t_2) + \frac{\alpha\mathbf{a}_1}{2}\|U\|^2_{0,[t_1,t_2]\times\Omega,\alpha} + \frac{1}{2\mathbf{a}_2}\|U\|^2_{0,[t_1,t_2]\times\partial\Omega,\alpha}$$

$$< \frac{2}{\mathbf{a}_2\mathbf{a}_3}\|\mathscr{M}U\|^2_{0,[t_1,t_2]\times\partial\Omega,\alpha} + \frac{2}{\alpha\mathbf{a}_1}\|\mathscr{L}U\|^2_{0,[t_1,t_2]\times\Omega,\alpha} + \mathbf{a}_0\|U\mathrm{e}^{-2\alpha t}\|^2_{L^2(\Omega)}(t_1), \tag{5.4}$$

then there exists a polynomial P such that, if $\alpha\mathbf{a}_1 \geqslant P(|||L|||_{X_s(\Omega^t)}, \|\mathscr{M}\|_{H^s(\partial\Omega^t)}, \mathbf{a}_2)$, *the following holds*

$$|||U|||^2_{X_m(\Omega^t)} + \|U\|^2_{H^m(\partial\Omega^t)}$$

$$\leqslant \frac{\mathrm{e}^{2\alpha t_M}}{\mathbf{a}_1} P(|||L|||_{X_s(\Omega^t)}, \|\mathscr{M}\|_{H^s(\partial\Omega^t)}, \mathbf{a}_2)[\|F\|^2_{H^m(\Omega^t)} + \|g\|^2_{H^m(\partial\Omega^t)} + |U|^2_{m,0,\Omega}(0)], \tag{5.5}$$

where $L = \{A^0, A^j, B\}$.

Proof. By means of a local co-ordinate change and a partition of unity, it is sufficient to prove (5.5) in a half-space. The estimate in a half plane will be obtained by differentiating the equation $(5.3)_1$ to estimate tangential derivatives and then, using $\det A_{\bar{\mathbf{n}}} \neq 0$ to solve for the normal variables in terms of the tangential ones.

For convenience, we assume (5.4) holds in a half plane. (5.5) is proved by induction. Proof of this lemma is close to that of Lemma 3.2 in [17], so similar arguments will not be repeated here. The main difference between the two proofs is the estimate for $\|\mathscr{M}\partial^m_{t,x'}U - \partial^m_{t,x'}(\mathscr{M}U)\|^2_{0,\partial\Omega^t,\alpha}$.

Consider the following in a half space $\{(t,x_1,\dots,x_{\mathbf{d}})|x_1 \geqslant 0\}$

$$\begin{aligned}
&\mathscr{L}\partial^\gamma_{t,x'}U = \partial^\gamma_{t,x'}\mathscr{L}U + (\mathscr{L}\partial^\gamma_{t,x'}U - \partial^\gamma_{t,x'}\mathscr{L}U),\\
&\mathscr{M}\partial^\gamma_{t,x'}U|_{x_1=0} = \partial^\gamma_{t,x'}(\mathscr{M}U) + (\mathscr{M}\partial^\gamma_{t,x'}U - \partial^\gamma_{t,x'}(\mathscr{M}U))|_{x_1=0},\\
&\partial^\gamma_{t,x'}U(0,x) = \partial^\gamma_{t,x'}U(0,x),
\end{aligned} \tag{5.6}$$

where $\partial^{\gamma}_{t,x'}$ is the γ derivative with respect to $t, x' = (x_2, \ldots, x_d)$ and $|\gamma| = 0, 1, \ldots, s$. By (5.4), (5.6)

$$\mathbf{a}_1 \| \partial^1_{t,x'} U \|^2_{0,\Omega} e^{-2\alpha t} + \frac{\alpha \mathbf{a}_1}{2} \| \partial^1_{t,x'} U \|^2_{0,\Omega',\alpha} + \frac{1}{2\mathbf{a}_2} \| \partial^1_{t,x'} U \|^2_{0,\partial\Omega',\alpha}$$
$$\leqslant \frac{2}{\alpha \mathbf{a}_1} \| \partial^1_{t,x'} F \|^2_{0,\Omega',\alpha} + \frac{2}{\alpha \mathbf{a}_1} \| \mathscr{L} \partial^1_{t,x'} U - \partial^1_{t,x'} \mathscr{L} U \|^2_{0,\Omega',\alpha}$$
$$+ \frac{2}{\mathbf{a}_2 \mathbf{a}_3} \| \partial^1_{t,x'} g \|^2_{0,\partial\Omega',\alpha} + \frac{2}{\mathbf{a}_2 \mathbf{a}_3} \| \mathscr{M} \partial^1_{t,x'} U - \partial^1_{t,x'} (\mathscr{M} U) \|^2_{0,\partial\Omega',\alpha}$$
$$+ \mathbf{a}_0 \| \partial^1_{t,x'} U \|^2_{0,\Omega}(0). \tag{5.7}$$

By Hölder inequality, embedding theorem, using $\det A_{\mathbf{n}} \neq 0$ to solve for the normal variables in terms of the tangential ones, and an argument similar to (3.20–3.28) of [17], one can show that, for $0 \leqslant t \leqslant t_M$,

$$\mathbf{a}_1 | U |^2_{1,0,\Omega} e^{-2\alpha t} + \frac{\alpha \mathbf{a}_1}{4} \| U \|^2_{1,\Omega',\alpha} + \frac{1}{\mathbf{a}_2} \| U \|^2_{1,\partial\Omega',\alpha}$$
$$\leqslant \frac{e^{2\alpha t_M}}{\mathbf{a}_1} P(|||L|||_{X_s}, \| \mathscr{M} \|_{H^s(\partial\Omega')}, \mathbf{a}_2) [\| F \|^2_{H^1(\Omega')} + \| g \|^2_{H^1(\partial\Omega')} + | U |^2_{1,0,\Omega}(0)]. \tag{5.8}$$

Next we assume the following inequality holds:

$$\mathbf{a}_1 | U |^2_{i,0,\Omega} e^{-2\alpha t} + \frac{\alpha \mathbf{a}_1}{4} \| U \|^2_{i,\Omega',\alpha} + \frac{1}{\mathbf{a}_2} \| U \|^2_{i,\partial\Omega',\alpha}$$
$$\leqslant \frac{e^{2\alpha t_M}}{\mathbf{a}_1} P(|||L|||_{X_s}, \| \mathscr{M} \|_{H^s(\partial\Omega')}, \mathbf{a}_2) [\| F \|^2_{H^i(\Omega')} + \| g \|^2_{H^i(\partial\Omega')} + | U |^2_{i,0,\Omega}(0)], \tag{5.9}$$

where $i = 0, 1, \ldots, m-1$.

We want to show (5.9) also hold for $i = m$, $m \leqslant s$. By (5.4), (5.6) again

$$\mathbf{a}_1 \| \partial^m_{t,x'} U \|^2_{0,\Omega} e^{-2\alpha t} + \frac{\alpha \mathbf{a}_1}{2} \| \partial^m_{t,x'} U \|^2_{0,\Omega',\alpha} + \frac{1}{2\mathbf{a}_2} \| \partial^m_{t,x'} U \|^2_{0,\partial\Omega',\alpha}$$
$$\leqslant \frac{2}{\alpha \mathbf{a}_1} \| \partial^m_{t,x'} F \|^2_{0,\Omega',\alpha} + \frac{2}{\alpha \mathbf{a}_1} \| \mathscr{L} \partial^m_{t,x'} U - \partial^m_{t,x'} \mathscr{L} U \|^2_{0,\Omega',\alpha}$$
$$+ \frac{2}{\mathbf{a}_2 \mathbf{a}_3} \| \partial^m_{t,x'} g \|^2_{0,\partial\Omega',\alpha} + \frac{2}{\mathbf{a}_2 \mathbf{a}_3} \| \mathscr{M} \partial^m_{t,x'} U - \partial^m_{t,x'} (\mathscr{M} U) \|^2_{0,\partial\Omega',\alpha}$$
$$+ \mathbf{a}_0 \| \partial^m_{t,x'} U \|^2_{0,\Omega}(0). \tag{5.10}$$

By Hölder inequality, (5.9), and arguing as (3.31–3.38) of [17], one can show that

$$\| \mathscr{L} \partial^m_{t,x'} U - \partial^m_{t,x'} \mathscr{L} U \|^2_{0,\Omega',\alpha}$$
$$\leqslant P(|||L|||_{X_s}, \mathbf{a}_2) (\| F \|^2_{m-1,\Omega',\alpha} + \| \partial^m_{t,x'} U \|^2_{0,\Omega',\alpha} + \| U \|^2_{m-1,\Omega',\alpha}).$$

The term $\| \mathscr{M}\partial^m_{t,x'}U - \partial^m_{t,x'}(\mathscr{M}U)\|^2_{0,\partial\Omega',\alpha}$ is estimated as follows:

$$\| \mathscr{M}\partial^m_{t,x'}U - \partial^m_{t,x'}\mathscr{M}U\|^2_{0,\partial\Omega',\alpha} \leqslant c\sum_{\mu=0}^{m-1}\int_{\partial\Omega'}|\partial^{m-\mu}_{t,x'}\mathscr{M}|^2|\partial^{\mu}_{t,x'}U|^2 e^{-2\alpha t}\,dx'\,dt$$

$$\leqslant c\sum_{\mu=0}^{m-1}\|\partial^{m-\mu}_{t,x'}\mathscr{M}\|^2_{L^{2p}(\partial\Omega')}\|\partial^{\mu}_{t,x'}U e^{-\alpha t}\|^2_{L^{2q}(\partial\Omega')}, \quad (5.11)$$

where for (i) $m = s$, $\mu = 0$ case we pick $p = 1, q = \infty$ in (5.11), and for (ii) $m < s$, or for (iii) $m = s$, $\mu > 0$ case we pick $p, q \geqslant 1$ in (5.11) in such a way that

$$1 = \frac{1}{p} + \frac{1}{q}, \qquad \frac{1}{p} \geqslant \frac{\mathbf{d} - 2(s - m + \mu)}{\mathbf{d} - 1}, \qquad \frac{1}{q} \geqslant \frac{\mathbf{d} - 2(m - \mu - 1/2)}{\mathbf{d} - 1}.$$

For case (i), inequality (5.11) implies that

$$\| \mathscr{M}\partial^m_{t,x'}U - \partial^m_{t,x'}\mathscr{M}U\|^2_{0,\partial\Omega',\alpha} \leqslant \max_{\Omega'}|U|^2\,\|\partial^m_{t,x'}\mathscr{M}\|^2_{0,\partial\Omega',\alpha}$$

$$\leqslant c\,e^{2\alpha t}\|U\|^2_{s-1,\Omega',\alpha}\|\partial^m_{t,x'}\mathscr{M}\|^2_{0,\partial\Omega',\alpha}. \quad (5.12)$$

For cases (ii), (iii), inequality (5.11) implies that

$$\| \mathscr{M}\partial^m_{t,x'}U - \partial^m_{t,x'}\mathscr{M}U\|^2_{0,\partial\Omega',\alpha} \leqslant c\sum_{\mu=0}^{m-1}\|\partial^{m-\mu}_{t,x'}\mathscr{M}\|^2_{H^{s-m+\mu}(\partial\Omega')}\|\partial^{\mu}_{t,x'}U e^{-\alpha t}\|^2_{H^{m-\mu-1/2}(\partial\Omega')}$$

$$\leqslant c\|\mathscr{M}\|^2_{H^s(\partial\Omega')}\sum_{\mu=0}^{m}\int_0^{t_M}\|\partial^{\mu}_{t,x'}U\|^2_{m-\mu,2,\Omega}e^{-2\alpha t}\,dt. \quad (5.13)$$

By using $\det A_{\bar{\mathbf{n}}} \neq 0$ to estimate $\|\partial^m_{t,x_1,x'}U\|^2_{0,\Omega}e^{-2\alpha t}$ as (3.39–3.42) in [17] and combining (5.10–5.13), we complete the proof. ∎

Remark 3. *Consider system* (5.3). *Besides the assumptions of* Lemma 5.2, *we assume*:

1. *compatibility conditions for* (5.3) *hold up to* $m - 1$, $2 \leqslant m \leqslant s$,
2. *there are* $B_k, \mathscr{M}_k$, *and symmetric* $\{A^0_k, A^j_k\} \subset C^\infty(\bar{\Omega}')$ *such that, as* $k \to \infty$,

$$\{A^0_k, A^j_k, B_k\} \to \{A^0, A^j, B\} \quad \text{in } X_s(\Omega'),$$
$$\mathscr{M}_k \to \mathscr{M} \qquad \text{on } H^s(\partial\Omega').$$

By $\{A^0_k, A^j_k, B_k, \mathscr{M}_k\}$ *above we consider the following*:

$$\begin{aligned} &\mathscr{L}_k U := A^0_k\partial_t U + A^j_k\partial_{x_j}U + B_k U = F && \text{in } \Omega^t,\\ &\mathscr{M}_k U = g && \text{on } \partial\Omega^t,\\ &U(0,x) = f(x) && \text{in } \Omega. \end{aligned} \quad (5.14)$$

It is clear that if k is large, assumptions 2–4 *of* Lemma 5.2 *hold in* (5.14), *where* $\{A^0, A^j, B, \mathscr{M}, A_{\bar{\mathbf{n}}}\}$ *are replaced by* $\{A^0_k, A^j_k, B_k, \mathscr{M}_k, A_{k,\bar{\mathbf{n}}}(:=A^j_k n_j)\}$. *Let* $\mathscr{L}^*_k, \mathscr{M}^*_k$ *be the adjoint operators of* $\mathscr{L}_k, \mathscr{M}_k$ *of* (5.14) (see section 2). *We also assume*:

3. *for* $U \in H^1(\mathbb{R}\times\bar{\Omega})$, $0 \leqslant t_1 < t_2 \leqslant t_M$, *and* $\mathbf{a}_1\alpha \geqslant c|||A^0, A_j, B|||_{X_s(\Omega^t)}$, (5.4) *and the following hold for* (5.14):

$$\mathbf{a}_1 \| U e^{2\alpha t} \|^2_{L^2(\Omega)}(t_1) + \frac{\alpha \mathbf{a}_1}{2} \| U \|^2_{0,[t_1,t_2]\times\Omega,-\alpha} + \frac{1}{2\mathbf{a}_2} \| U \|^2_{0,[t_1,t_2]\times\partial\Omega,-\alpha}$$
$$< \frac{2}{\mathbf{a}_1\mathbf{a}_3} \| \mathscr{M}_k^* U \|^2_{0,[t_1,t_2]\times\partial\Omega,-\alpha} + \frac{2}{\alpha\mathbf{a}_1} \| \mathscr{L}_k^* U \|^2_{0,[t_1,t_2]\times\Omega,-\alpha}$$
$$+ \mathbf{a}_0 \| U e^{2\alpha t} \|^2_{L^2(\Omega)}(t_2). \tag{5.15}$$

Assumption 3 *above is similar to* A8(b) *but for* (5.14).

Lemma 5.3. *Under assumptions of* Lemma 5.2 *and* Remark 3, *there exist a sequence of functions* $\{F_k, f_k, g_k\}$ *such that* $F_k \in H^{m+2}(\Omega^t)$, $f_k \in H^{m+2}(\Omega)$, $g_k \in H^{m+2}(\partial\Omega^t)$,

$$\begin{aligned} F_k &\to F \quad \text{in } H^m(\Omega^t) \\ f_k &\to f \quad \text{in } H^m(\Omega) \qquad \text{as } k \to \infty, \\ g_k &\to g \quad \text{on } H^m(\partial\Omega^t) \end{aligned}$$

and, if we consider the following (*replacing* F, f, g *of* (5.14) *by* F_k, f_k, g_k)

$$\begin{aligned} \mathscr{L}_k U_k = A_k^0 \partial_t U_k + A_k^j \partial_{x_j} U_k + B_k U_k &= F_k \quad \text{in } \Omega^t, \\ \mathscr{M}_k U_k = g_k & \quad \text{on } \partial\Omega^t \\ U_k(0,x) = f_k(x) & \quad \text{in } \Omega, \end{aligned} \tag{5.16}$$

then (5.4), (5.15), *and compatibility conditions up to m also hold in* (5.16).

Remark 4. *First let us give definition* (*for* (5.3)) *for* $\mathbf{G}_i, \mathbf{f}_p$:

$$\mathbf{G}_i := -(\partial_t^i[(A^0)^{-1}A^j])\partial_j - \partial_t^i[(A^0)^{-1}B], \quad i \geqslant 0,$$
$$\mathbf{f}_0 := U(0,x) = f,$$
$$\mathbf{f}_p := \sum_{i=0}^{p-1} \binom{p-1}{i} \mathbf{G}_i(0,x)\mathbf{f}_{p-1-i} + (\partial_t^{p-1}[(A^0)^{-1}F])(0,x), \quad p \geqslant 1. \tag{5.17}$$

By definition, $\mathbf{f}_p$ *corresponds to* "$\partial_t^p U(0,x)$". *We define* $\mathbf{B}_p, \mathbf{E}_p$ *as*

$$\mathbf{B}_0 f := \mathbf{f}_0 = f \quad \text{and} \quad \mathbf{B}_p f + \mathbf{E}_p (A^0)^{-1} F := \mathbf{f}_p, \quad p \geqslant 1,$$

where

$$\mathbf{B}_p f := \mathbf{G}_0^p(0,x) f + \text{term of the form } \mathbf{G}_0^\gamma \mathbf{G}_{i_1} \ldots \mathbf{G}_{i_q}(0,x) f \tag{5.18}$$

with $\gamma + i_1 + \cdots + i_q \leqslant p - 1$ *and*

$$\mathbf{E}_p(A^0)^{-1}F := \text{sum of the terms of the form } \mathbf{G}_0^\gamma \mathbf{G}_{i_1} \ldots \mathbf{G}_{i_q}(0)\partial_t^\mu[(A^0)^{-1}F](0) \tag{5.19}$$

with $\gamma + i_1 + \cdots + i_q + \mu \leqslant p - 1$.

Proof. (1) That (5.4), (5.15) hold in (5.16) is because (5.4), (5.15) hold in (5.14).

By assumption 2 of Remark 3, there are smooth functions $A_k^0, A_k^j, B_k, \mathscr{M}_k$ which converge to $A^0, A^j, B, \mathscr{M}$. Consider system (5.14) for each k. Let us use $\mathbf{f}_{k,p}, \mathbf{B}_{k,p}, \mathbf{E}_{k,p}$ to denote the $\mathbf{f}_p, \mathbf{B}_p, \mathbf{E}_p$ of (5.14) (see (5.17–5.19)). By the assumption of compatibility conditions for (5.3), on boundary $\partial\Omega$, for $0 \leqslant p \leqslant m-1$,

$$\sum_{\mu=0}^{p} \binom{p}{\mu} \partial_t^\mu \mathcal{M}(0,x)\mathbf{f}_{p-\mu} = \partial_t^p g(0,x),$$

where $\mathbf{f}_{p-\mu}$ is the $(p-\mu)$th derivative, "$\partial_t^{p-\mu}U(0,x)$", of (5.3). Since

$$\sum_{\mu=0}^{p} \binom{p}{\mu} \partial_t^\mu \mathcal{M}_k(0,x)\mathbf{f}_{k,p-\mu} - \partial_t^p g(0,x)$$
$$= \sum_{\mu=0}^{p} \binom{p}{\mu} \partial_t^\mu(\mathcal{M}_k - \mathcal{M})(0,x)\mathbf{f}_{k,p-\mu} + \sum_{\mu=0}^{p} \binom{p}{\mu} \partial_t^\mu \mathcal{M}(0,x)(\mathbf{f}_{k,p-\mu} - \mathbf{f}_{p-\mu}),$$

we may choose k such that

$$\left\| \sum_{\mu=0}^{p} \binom{p}{\mu} \partial_t^\mu \mathcal{M}_k \mathbf{f}_{k,p-\mu} - \partial_t^p g \right\|_{H^{m-p-1/2}(\partial\Omega)} (0) < \frac{1}{2k}, \quad 0 \leqslant p \leqslant m-1. \tag{5.20}$$

Let $q \geqslant m+2$ be an integer. For fixed k, we choose sequences $F_k^i \in H^{m+q}(\Omega^t)$, $\phi_k^i \in H^{m+q}(\Omega)$, and $g_k^i \in H^{m+q}(\partial\Omega^t)$ such that

$$\begin{aligned} &F_k^i \to F \quad \text{in } H^m(\Omega^t) \\ &\phi_k^i \to f \quad \text{in } H^m(\Omega) \qquad \text{as } i \to \infty. \\ &g_k^i \to g \quad \text{on } H^m(\partial\Omega^t) \end{aligned} \tag{5.21}$$

Then we want to find $i(k)$ and $h_k^{i(k)}$ such that, for $0 \leqslant p \leqslant m$,

$$h_k^{i(k)} \in H^{m+2}(\Omega), \quad \| h_k^{i(k)} \|_{H^m(\Omega)} \leqslant \frac{1}{2k}, \tag{5.22}$$

$$\sum_{\mu=0}^{p} \binom{p}{\mu} \partial_t^\mu \mathcal{M}_k(\mathbf{B}_{k,p-\mu}(-h_k^{i(k)} + \phi_k^{i(k)}) + \mathbf{E}_{k,p-\mu}(A_k^0)^{-1} F_k^{i(k)}) - \partial_t^p g_k^{i(k)} = 0, \tag{5.23}$$

for $(t,x) \in \{0\} \times \partial\Omega$. Suppose $\{h_k^{i(k)}\}$ is available, we define

$$f_k^{i(k)} = -h_k^{i(k)} + \phi_k^{i(k)}.$$

Then we see, when $i(k)$ is large,

$$\| F_k^{i(k)} - F \|_{H^m(\Omega^t)} < c/k, \quad \| f_k^{i(k)} - f \|_{H^m(\Omega)} < c/k, \quad \| g_k^{i(k)} - g \|_{H^m(\partial\Omega^t)} < c/k,$$

and system (5.16) satisfies compatibility conditions up to m. So Lemma 5.3 holds true. Next step is to search for $h_k^{i(k)}$.

(2) Equation (5.23) can be written as, for $0 \leqslant p \leqslant m$, $(t,x) \in \{0\} \times \partial\Omega$,

$$\sum_{\mu=0}^{p} \binom{p}{\mu} \partial_t^\mu \mathcal{M}_k \mathbf{B}_{k,p-\mu} h_k^i$$
$$= \sum_{\mu=0}^{p} \binom{p}{\mu} \partial_t^\mu \mathcal{M}_k(\mathbf{B}_{k,p-\mu}\phi_k^i + \mathbf{E}_{k,p-\mu}(A_k^0)^{-1} F_k^i) - \partial_t^p g_k^i. \tag{5.24}$$

Let $\mathbf{R}_k$ be the inverse of $\mathscr{M}_k$ when it is restricted to the orthogonal complement of the kernel of $\mathscr{M}_k$. Note $\partial_t^p g_k^i(0,x)$ is in the image of $\mathscr{M}_k$. For $0 \leqslant p \leqslant m$ on $(t,x) \in \{0\} \times \partial\Omega$, to solve (5.24) is sufficient to solve

$$\mathbf{B}_{k,p} h_k^i + \mathbf{R}_k \sum_{\mu=1}^{p} \binom{p}{\mu} \partial_t^\mu \mathscr{M}_k \mathbf{B}_{k,p-\mu} h_k^i$$
$$= \mathbf{R}_k \sum_{\mu=0}^{p} \binom{p}{\mu} \partial_t^\mu \mathscr{M}_k (\mathbf{B}_{k,p-\mu} \phi_k^i + \mathbf{E}_{k,p-\mu}(A_k^0)^{-1} F_k^i) - \mathbf{R}_k \partial_t^p g_k^i := a_{p,i,k}. \tag{5.25}$$

The left-hand side of equation (5.25) can be written as

$$\mathbf{B}_{k,p} h_k^i + \mathbf{R}_k \sum_{\mu=1}^{p} \binom{p}{\mu} \partial_t^\mu \mathscr{M}_k \mathbf{B}_{k,p-\mu} h_k^i = \mathbf{D}_{k,\mathbf{\bar{n}}}^p \partial_{\mathbf{\bar{n}}}^p h_k^i + \sum_{\gamma=0}^{p-1} \mathbf{C}_{k,p,p-\gamma} \partial_{\mathbf{\bar{n}}}^\gamma h_k^i,$$

where $\mathbf{D}_{k,\mathbf{\bar{n}}} := -(A_k^0)^{-1} A_k^j n_j$, $\partial_{\mathbf{\bar{n}}} h_k^i$ denotes the normal derivative of h_k^i on $\partial\Omega$, and $\mathbf{C}_{k,p,p-\gamma}$ an operator of order $(p-\gamma)$ which only contains derivatives tangential to $\partial\Omega$. So equation (5.25) becomes

$$b_{p,i,k} = (\mathbf{D}_{k,\mathbf{\bar{n}}}^p)^{-1} \left(a_{p,i,k} - \sum_{\gamma=0}^{p-1} \mathbf{C}_{k,p,p-\gamma} b_{\gamma,i,k} \right), \tag{5.26}$$

where $\partial_{\mathbf{\bar{n}}}^p h_k^i := b_{p,i,k}$. Since $a_{p,i,k} \in H^{m+q-p-1/2}(\partial\Omega)$, it follows that

$$b_{p,i,k} \in H^{m+q-p-1/2}(\partial\Omega).$$

Also by (5.20–5.21) and (5.25–5.26), there is a large $i(k)$ such that

$$\| b_{p,i(k),k} \|_{H^{m-p-1/2}(\partial\Omega)} < c/k, \quad 0 \leqslant p \leqslant m-1.$$

Furthermore, by a trace theorem [10], there is a $v_k \in H^{m+q}(\Omega)$ such that

$$\partial_{\mathbf{\bar{n}}}^p v_k = b_{p,i(k),k}, \quad 0 \leqslant p \leqslant m-1, \quad \| v_k \|_{H^m(\Omega)} \leqslant c/k.$$

Finally, we write $h_k^{i(k)} = v_k + w_k$, where w_k is required to satisfy $w_k \in H^q(\Omega)$, and,

$$\| w_k \|_{H^m(\Omega)} \leqslant c/k; \quad \partial_{\mathbf{\bar{n}}}^m w_k|_{\partial\Omega} = b_{m,i(k),k} - \partial_{\mathbf{\bar{n}}}^m v_k; \quad \partial_{\mathbf{\bar{n}}}^p w_k|_{\partial\Omega} = 0, \ 0 \leqslant p \leqslant m-1.$$

To find such a w_k, we refer the reader to the proof of Lemma 3.3 of [8]. Therefore the desired $h_k^{i(k)}$ can be obtained. ■

Lemma 5.4. *Under the assumptions of* Lemma 5.3, *system* (5.3) *has a unique solution* $U \in X_m(\Omega') \cap H^m(\partial\Omega')$ *and there is a polynomial* P *such that*

$$|||U|||_{X_m(\Omega')}^2 + \| U \|_{H^m(\partial\Omega')}^2$$
$$\leqslant \frac{e^{2\alpha t_M}}{\mathbf{a}_1} P(|||L|||_{X_s(\Omega')}, \| \mathscr{M} \|_{H^s(\partial\Omega')}, \mathbf{a}_2) [\| F \|_{H^m(\Omega')}^2 + \| g \|_{H^m(\partial\Omega')}^2 + \| f \|_{H^m(\Omega)}^2], \tag{5.27}$$

where $L = \{A^0, A^j, B\}$ *and* α *is a constant with* $\mathbf{a}_1 \alpha \geqslant P(|||L|||_{X_s(\Omega')}, \| \mathscr{M} \|_{H^s(\partial\Omega')}, \mathbf{a}_2)$.

Proof. (1) By Lemma 5.3, there exist B_k and symmetric matrices $A_k^0, A_k^j \in C^\infty(\Omega^t)$, $\mathscr{M}_k \in C^\infty(\partial\Omega^t)$, $F_k \in H^{m+2}(\Omega^t)$, $f_k \in H^{m+2}(\Omega)$, and $g_k \in H^{m+2}(\partial\Omega^t)$ such that

$$\begin{aligned}
&\{A_k^0, A_k^j, B_k\} \to \{A^0, A^j, B\} && \text{in } X_s(\Omega^t) \\
&\mathscr{M}_k \to \mathscr{M} && \text{on } H^s(\partial\Omega^t) \\
&F_k \to F && \text{in } H^m(\Omega^t) \quad \text{as } k \to \infty. \\
&f_k \to f && \text{in } H^m(\Omega) \\
&g_k \to g && \text{on } H^m(\partial\Omega^t)
\end{aligned}$$

Furthermore (5.4), (5.15), and the compatibility conditions up to m hold for system (5.16) for each k. Consider the system (5.16). By Proposition 5.1 of [8], one can show that system (5.16) has a solution $U_k \in X_{m+1}(\Omega^t)$. By Lemma 5.2, $\{U_k\}$ is bounded, i.e.,

$$|||U_k|||_{X_m} + \| U_k \|_{H^m(\partial\Omega^t)} \leqslant c, \quad \forall k. \tag{5.28}$$

Let us consider the following:

$$\begin{aligned}
&\mathscr{L}(U_k - U_l) = F_k - F_l + (\mathscr{L} - \mathscr{L}_k)U_k - (\mathscr{L} - \mathscr{L}_l)U_l && \text{in } \Omega^t, \\
&\mathscr{M}(U_k - U_l) = (\mathscr{M} - \mathscr{M}_k)U_k + (\mathscr{M}_l - \mathscr{M})U_l + g_k - g_l && \text{on } \partial\Omega^t, \\
&(U_k - U_l)(0,x) = f_k - f_l && \text{in } \Omega.
\end{aligned} \tag{5.29}$$

Inequality (5.4) of Lemma 5.2 applied to $\{U_k - U_l\}$ shows that $\{U_k\}$ converges in $C(L^2)$. Let U be the limit of $\{U_k\}$, then $U \in C(L^2)$. By (5.28), $\| U \|_{H^m(\partial\Omega^t)} \leqslant c$.

By (5.28) and the Sobolev's interpolation theorem [1, 6], we have for arbitrary $v > 0$,

$$\| U_k - U_l \|_{H^{m-v}(\Omega)}(t) \leqslant c \| U_k - U_l \|_{L^2(\Omega)}^{v/m}(t) \| U_k - U_l \|_{H^m(\Omega)}^{1-v/m}(t),$$

so

$$U_k \to U \quad \text{in } C(H^{m-v}) \quad \text{as } k \to \infty. \tag{5.30}$$

By (5.3) and (5.16), we see that

$$U_k \to U \in \bigcap_{i=0}^{m-1} C^i(H^{m-i-v}) \quad \text{as } k \to \infty,$$

$$\partial_t^i U \in L^\infty(H^{m-i}), \quad i = 0 \ldots m.$$

By (5.5), we see U satisfies the estimate (5.27). Equation (5.27) implies solution of (5.3) is unique. So the proof is complete as long as we know $\partial_t^i U \in C(H^{m-i})$, $i = 0 \ldots m$.

(2) That $\partial_t^i U \in C(H^{m-i})$, $i = 0 \ldots m$, is proved by three steps:

Step 1: As $t \to t_0$, $\partial_t^i U(t,\cdot) \to \partial_t^i U(t_0,\cdot)$ weakly in $H^{m-i}(\Omega)$ for all $i \leqslant m$.

Step 2: $\partial_{t,x}^m U(t,\cdot)$ is continuous at $t = 0$ in $L^2(\Omega)$.

Step 3: $U \in X_m(\Omega^t)$.

Proof of Step 1. If $q < m$ and $\psi_1 \in (H^q(\Omega))'$, by (5.30), $\psi_1(U_k(t,\cdot)) \to \psi_1(U(t,\cdot))$ uniformly in t as $k \to \infty$. Now if $\psi \in (H^m)'$, there is a $\psi_1 \in (H^q)'$, for some $q < m$, such that $\| \psi - \psi_1 \|_{(H^m)'} \leqslant 1/2^k$ for any k. By (5.28), for all t,

$$\begin{aligned}
&|\psi(U_k(t,\cdot)) - \psi(U(t,\cdot))| \\
&\quad \leqslant |\psi(U_k) - \psi_1(U_k)| + |\psi_1(U_k) - \psi_1(U)| + |\psi_1(U) - \psi(U)| \\
&\quad \leqslant c/2^{k-1} + |\psi_1(U_k) - \psi_1(U)|.
\end{aligned}$$

This implies that $\psi(U_k) \to \psi(U)$ uniformly in t. Because $\psi(U_k)$ is continuous in t, $\psi(U)$ is also continuous in t for all $\psi \in (H^m(\Omega))'$; that is, as $t \to t_0$, $U(t,\cdot) \to U(t_0,\cdot)$ weakly in $H^m(\Omega)$. Similarly, one can show that if $t \to t_0$ and $1 \leqslant i < m$, then $\partial_t^i U(t) \to \partial_t^i U(t_0)$ weakly in $H^{m-i}(\Omega)$ by (5.30). Then, by (5.3), we see that $\partial_t^m U(t,\cdot) \to \partial_t^m U(t_0,\cdot)$ weakly in $L^2(\Omega)$ as $t \to t_0$.

Proof of Step 2. To prove this step, we will show the convergence result in an equivalent norm $\|\cdot\|_{0,A^0(0)}$. By differentiating (5.16) with respect to tangential vectors or time $\partial^\gamma_{\tau,t}$, we have

$$\begin{aligned} \mathscr{L}_k \partial^\gamma_{\tau,t} U_k &= \partial^\gamma_{\tau,t} \mathscr{L}_k U_k + (\mathscr{L}_k \partial^\gamma_{\tau,t} U_k - \partial^\gamma_{\tau,t} \mathscr{L}_k U_k) && \text{in } \Omega^t,\\ \mathscr{M}_k \partial^\gamma_{\tau,t} U_k &= \partial^\gamma_{\tau,t} (\mathscr{M}_k U_k) + (\mathscr{M}_k \partial^\gamma_{\tau,t} U_k - \partial^\gamma_{\tau,t} (\mathscr{M}_k U_k)) && \text{on } \partial\Omega^t,\\ \partial^\gamma_{\tau,t} U_k(0,x) &= \partial^\gamma_{\tau,t} U_k(0,x) && \text{in } \Omega. \end{aligned} \tag{5.31}$$

Multiplying above by $\partial^\gamma_{\tau,t} U_k$, and integrating the resulting equation over Ω, we derive, for $|\gamma| \leqslant m$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \partial^\gamma_{\tau,t} U_k A_k^0 \partial^\gamma_{\tau,t} U_k \,\mathrm{d}x = \int_{\partial\Omega} \partial^\gamma_{\tau,t} U_k A_{k,\bar{n}} \partial^\gamma_{\tau,t} U_k \,\mathrm{d}\sigma + \sum_{|\zeta| \leqslant |\gamma|} \int_\Omega G(\partial^\zeta D_k)\,\mathrm{d}x, \tag{5.32}$$

where $G(\partial^\zeta D_k)$ is a function depending on $\partial^\zeta D_k$, $D_k = \{A_k^0, A_k^j, B_k, U_k, F_k\}$, and $A_{k,\bar{n}} := \sum A_k^j n_j$. We integrate (5.32) over $(0,\xi)$, then

$$\begin{aligned} \int_\Omega \partial^\gamma_{\tau,t} U_k A_k^0 \partial^\gamma_{\tau,t} U_k(\xi,x)\,\mathrm{d}x &\leqslant \int_\Omega \partial^\gamma_{\tau,t} U_k A_k^0 \partial^\gamma_{\tau,t} U_k(0,x)\,\mathrm{d}x \\ &\quad + \int_0^\xi \int_{\partial\Omega} |\partial^\gamma_{\tau,t} U_k A_{k,\bar{n}} \partial^\gamma_{\tau,t} U_k| \,\mathrm{d}\sigma\,\mathrm{d}t + \int_0^\xi R(z)\,\mathrm{d}z, \end{aligned} \tag{5.33}$$

where $R \in L^1(0,t_M)$ is a positive function (since $\|\partial^\gamma_{x,t}(A_k^0, A_k^j, B_k, U_k)\|_{L^2(\Omega)}$, $|\gamma| \leqslant m$, are uniformly bounded in t). Since $U_k(0,\cdot)$ converges to $U(0,\cdot)$ in $H^m(\Omega)$, it is easy to see

$$\lim_{k\to\infty} \int_\Omega \partial^\gamma_{\tau,t} U_k A_k^0 \partial^\gamma_{\tau,t} U_k(0,x)\,\mathrm{d}x = \|\partial^\gamma_{\tau,t} U(0)\|^2_{0,A^0(0)}. \tag{5.34}$$

Now, since $U_k(t,\cdot)$ converges weakly to $U(t,\cdot)$ in $H^m(\Omega)$ for each t, $U_k(t,\cdot)$ converges weakly to $U(t,\cdot)$ in $H^m(\Omega)$ in an equivalent norm for each t. So for all $|\gamma| \leqslant m$,

$$\|\partial^\gamma_{\tau,t} U(\xi)\|^2_{0,A^0(\xi)} \leqslant \varliminf_{k\to\infty} \|\partial^\gamma_{\tau,t} U_k(\xi)\|^2_{0,A^0(\xi)}. \tag{5.35}$$

Now since $A_k^0 \to A^0$ in $X_s(\Omega^t)$, then

$$\lim_{k\to\infty} \|U_k(\xi)\|^2_{m,A^0(\xi)} = \lim_{k\to\infty} \|U_k(\xi)\|^2_{m,A_k^0(\xi)}. \tag{5.36}$$

By (5.33–5.36), letting $k \to \infty$, we obtain, for $|\gamma| \leqslant m$,

$$\begin{aligned} \|\partial^\gamma_{\tau,t} U(\xi)\|^2_{0,A^0(\xi)} &\leqslant \|\partial^\gamma_{\tau,t} U(0)\|^2_{0,A^0(0)} \\ &\quad + \varlimsup_{k\to\infty} \int_0^\xi \int_{\partial\Omega} |\partial^\gamma_{\tau,t} U_k A_{k,\bar{n}} \partial^\gamma_{\tau,t} U_k| \,\mathrm{d}\sigma\,\mathrm{d}t + \int_0^\xi R(z)\,\mathrm{d}z. \end{aligned} \tag{5.37}$$

Let Ω^ε be the intersection of Ω and a ε-neighborhood of $\partial\Omega$, and let ψ^ε be a half bell function satisfying $\psi^\varepsilon(x) = 1$ for $x \in \partial\Omega$ and with support in $\bar{\Omega}^\varepsilon$. Multiplying (5.31) by ψ^ε, then applying (5.4) to the resulting equation, one can show that, for any ε,

$$\overline{\lim_{k\to\infty}} \int_0^\xi \int_{\partial\Omega} |e^{-2\alpha t}\psi^\varepsilon \partial^\gamma_{\tau,t} U_k|^2 \, d\sigma \, dt$$

$$\leqslant c\left(\|\psi^\varepsilon \partial^\gamma_{\tau,t} U\|_{L^2(\Omega)}(0) + \int_0^\xi \int_{\partial\Omega} |e^{-2\alpha t}\partial^\gamma_{\tau,t} g|^2 \, d\sigma \, dt + \int_0^\xi R(z)\, dz \right), \tag{5.38}$$

for some constant c. To obtain the result (5.38), we need to control the boundary term $\| \mathcal{M}_k \partial^\gamma_{\tau,t} U_k - \partial^\gamma_{\tau,t}(\mathcal{M}_k U_k)\|^2_{0,\partial\Omega^t,\alpha}$. It can be estimated by the same technique as (5.12–5.13). (5.38) implies that

$$\overline{\lim_{\xi\to 0^+}}\ \overline{\lim_{k\to\infty}} \int_0^\xi \int_{\partial\Omega} |\partial^\gamma_{\tau,t} U_k A_{k,\bar{n}} \partial^\gamma_{\tau,t} U_k| \, d\sigma \, dt \leqslant c \|\psi^\varepsilon \partial^\gamma_{\tau,t} U\|_{L^2(\Omega)}(0). \tag{5.39}$$

The right-hand side goes to 0 as ε approaches 0. Since the left-hand side is independent of ε, it equals 0. By the continuity of A^0, we get, for $|\gamma| \leqslant m$,

$$\overline{\lim_{\xi\to 0^+}} \int_\Omega \partial^\gamma_{\tau,t} U A^0 \partial^\gamma_{\tau,t} U(\xi, x)\, dx = \overline{\lim_{\xi\to 0^+}} \int_\Omega \partial^\gamma_{\tau,t} U(\xi,x) A^0(0,x) \partial^\gamma_{\tau,t} U(\xi,x)\, dx. \tag{5.40}$$

By (5.37–5.40), we have, for $|\gamma| \leqslant m$,

$$\overline{\lim_{\xi\to 0^+}} \int_\Omega \partial^\gamma_{\tau,t} U(\xi,x) A^0(0,x) \partial^\gamma_{\tau,t} U(\xi,x)\, dx \leqslant \|\partial^\gamma_{\tau,t} U(0)\|^2_{0,A^0(0)}. \tag{5.41}$$

Combining the result of *Step* 1, Theorem 3.8 in [1], and (5.41), we see $\partial^\gamma_{\tau,t} U(t,\cdot)$ is continuous at $t = 0$ in $L^2(\Omega)$ for all $|\gamma| = m$.

Next we show that $\partial^m_{t,x} U(t,\cdot)$ is also continuous at $t = 0$ in $L^2(\Omega)$. By partition of unity, we can reduce the problem to the boundary $\partial\Omega$. By using $\det A_{\bar{n}} \neq 0$ on $\partial\Omega^t$, we can solve for the normal variables in terms of the tangential ones and time. Because $\partial^\gamma_{\tau,t} U(t,\cdot)$ is continuous at $t = 0$ for $|\gamma| = m$, $\partial^m_{t,x} U(t,\cdot)$ is also continuous at $t = 0$ in L^2 around the boundary $\partial\Omega$. Therefore, we conclude that $\partial^m_{t,x} U(t,\cdot)$ is continuous at $t = 0$ in $L^2(\Omega)$.

Proof of Step 3. By the same argument as that in Step 2, one can prove strong right continuity at any point $t \in [0, t_M)$. Note that equation (5.3) and the argument of *Step* 2 are reversible in time, so the proof of strong right continuity in $[0, t_M)$ implies strong left continuity on $(0, t_M]$. Therefore, we conclude that $U \in X_m(\Omega^t)$. ■

6. Existence of a unique local-in-time solution

In this section we prove the existence of a unique local-in-time solution of the system (1.1–1.7) under assumptions A1–8 of section 3. That is equivalent to showing the existence of a fixed point of the system (3.5–3.9). By A3–5, we know K_{δ,t_M} (see (3.12)) is not an empty set for some δ. Given $(Q,\eta,S) \in K_{\delta,t_M}$, we are able to solve (3.5–3.9) to obtain (V,ρ,T) by A1–8 of section 3 and Lemmas 5.1, 5.4. Set

$\Pi(Q,\eta,S)=(V,\rho,T)$. The existence of a unique local-in-time solution of the system (1.1–1.7) is proved as follows: First we show that, as t_M is small enough, there exists a δ such that Π is a map from K_{δ,t_M} to itself (Lemma 6.1). Next we prove that Π is a contractive map in some weak spaces. So we obtain a fixed point in these weak spaces. Then we show the fixed point is a smooth solution of the system (3.5–3.9) (Lemma 6.2). The existence and uniqueness of classical solution of system (1.1–1.7) follows the result of Lemma 6.2.

Lemma 6.1. *Under assumptions* A1–8 *of* section 3, *as t_M is small enough, there exists δ such that* (3.5–3.9) *is uniquely solvable and the solution of* (3.5–3.9) *is in K_{δ,t_M} for all* $(Q,\eta,S)\in K_{\delta,t_M}$.

Proof. Given $(Q,\eta,S)\in K_{\delta,t_M}$, we solve the system (3.5–3.9) to obtain (V,ρ,T) by A1–8 of section 3 and Lemmas 5.1, 5.4. By Lemma 5.1, the solution of (3.6) satisfies

$$T\in X_3(\Omega^t),\qquad \partial_t^i T\in L^2(H^{4-i}),\quad i=0,1,2,3,$$
$$\partial_t^i T(0,x)=\bar{T}_i,\quad i=0,1,2.$$

By (5.2), $|||T-T_0|||_{X_3}+\|T-T_0\|_{H^3(\partial\Omega^t)}\leqslant\delta_1$. If t_M is small enough, δ_1 depends on $T_0,\bar{T}_1,\bar{T}_2,T_s,\mathbf{n}_0,V_0$ but is independent of δ.

By Lemma 5.4 for $\mathbf{d}=m=s=3$, we see that the solution (3.7–3.9) satisfies

$$(V,\rho)\in X_3(\Omega^t)\cap H^3(\partial\Omega^t),\qquad (\partial_t^i V,\partial_t^i\rho)(0,x)=(\bar{V}_i,\bar{\rho}_i),\quad i=0,1,2.$$

By (5.27), $|||(V,\rho)-(V_0,\ln\mathbf{n}_0)|||_{X_3}^2+\|(V,\rho)-(V_0,\ln\mathbf{n}_0)\|_{H^3(\partial\Omega^t)}^2\leqslant\delta_2$. If t_M is small enough, we see that δ_2 is dependent on $\mathbf{n}_0,V_0,T_0,\Psi_b$ but independent of δ. If we take $\delta_0:=\max\{\delta,\delta_1,\delta_2\}$, repeat the above procedure, and let t_M smaller, we see $(V,\rho,T)\in K_{\delta_0,t_M}$ for all $(Q,\eta,S)\in K_{\delta_0,t_M}$. That is, Π is a map from K_{δ_0,t_M} to itself. ∎

Lemma 6.2. *Under assumptions* A1–8 *of* section 3, *system* (3.5–3.9) *has a unique fixed point if t_M is small enough. Moreover, the solution satisfies*

$$(\partial_t^i V,\partial_t^i\rho,\partial_t^i T)\in L^\infty(H^{3-i}),\quad \partial_t^i T\in L^2(H^{4-i}),\quad \partial_t^i\Psi\in L^\infty(H^{5-i}),\tag{6.1}$$

for $i=0,1,2,3$.

Proof. By Lemma 6.1, $\Pi(Q,\eta,S)=(V,\rho,T)$ is a map from K_{δ,t_M} to K_{δ,t_M} for some δ. Given (Q_1,η_1,S_1) and (Q_2,η_2,S_2), by solving (3.5–3.9) we get two solutions (V_1,ρ_1,T_1,Ψ_1) and (V_2,ρ_2,T_2,Ψ_2). Subtracting one solution from the other, multiplying the difference of (3.6) by $(T_1-T_2)e^{-2\alpha t}$, integrating the resulting equations over Ω^t, using (5.4) for the difference of system (3.7–3.9), we see, if α is large and t_M is small enough, Π is a contractive map in $V,\rho\in C(L^2)\cap L^2(\partial\Omega^t)$, $T\in L^2(\Omega^t)$. That is,

$$\begin{aligned}&\|T_1-T_2\|_{0,\Omega^t,\alpha}^2+\|(V_1-V_2,\rho_1-\rho_2)e^{-2\alpha t}\|_{L^2(\Omega)}^2+\|(V_1-V_2,\rho_1-\rho_2)\|_{0,\partial\Omega^t,\alpha}^2\\&\quad<c\|S_1-S_2\|_{0,\Omega^t,\alpha}^2+\|(Q_1-Q_2,\eta_1-\eta_2)e^{-2\alpha t}\|_{L^2(\Omega)}^2\\&\qquad+\|(Q_1-Q_2,\eta_1-\eta_2)\|_{0,\partial\Omega^t,\alpha}^2,\end{aligned}$$

for some constant $c<1$.

By the above, if we define $\Pi(V_{k-1},\rho_{k-1},T_{k-1})=(V_k,\rho_k,T_k)$, then sequence $\{V_k,\rho_k,T_k\}$ converges to the unique fixed point $\{V,\rho,T\}$ where $V,\rho\in C(L^2)\cap L^2(\partial\Omega^t)$, $T\in L^2(\Omega^t)$. Since $\{V_k,\rho_k,T_k\}\subset K_{\delta,t_M}$, $(V,\rho,T)\in L^\infty(H^3)\cap H^3(\partial\Omega^t)$. By Lemma 5.1, we know that $\partial_t^i T_k\in L^2(H^{4-i})$, $i=0,1,2,3$, so

$\partial_t^i T \in L^2(H^{4-i})$, $i = 0, 1, 2, 3$. By Sobolev's interpolation theorem [1, 6], we have for arbitrary $\nu > 0$ (set $W_k := (V_k, \rho_k)$)

$$\| W_k - W_i \|_{H^{3-\nu}(\Omega)}(t) \leqslant c \| W_k - W_i \|_{L^2(\Omega)}^{\nu/3}(t) \| W_k - W_i \|_{H^3(\Omega)}^{1-\nu/3}(t),$$

which implies that $V, \rho \in C(H^{3-\nu})$. By equation (3.7), we see $\partial_t^i V, \partial_t^i \rho \in \bigcap_{i=1}^2 C(H^{3-\nu-i})$. Therefore $\{V, \rho, T\}$ is the unique classical solution of system (3.5–3.9). Moreover, by (3.1–3.4), we see the fixed point satisfies, for $i = 0, 1, 2, 3$,

$$(\partial_t^i V, \partial_t^i \rho, \partial_t^i T) \in L^\infty(H^{3-i}), \quad \partial_t^i T \in L^2(H^{4-i}), \quad \partial_t^i \Psi \in L^\infty(H^{5-i}). \qquad \blacksquare$$

Theorem 1.1 is a direct result of Lemma 6.2.

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