

Fig. 7 Effect of numerical compensation on the measured curves representing the molten steel level in CCP as measured by the optical distance meter with τ value at 0.1 s: (A) before compensation and (B) after compensation of transient. The curves obtained using the γ -ray level meter, are also shown for comparison. Timing at the powder application is indicated as *. The optical distance meter detected center of the target surface, while γ -ray detected mean level of the target surface.

distance due to a splash of the molten steel, appears in the optical method. The rapid increase of the recorded curve [parts (3)–(4) in Fig. 5] is caused by the bias of the powder.

Unexpected reflection of the optical beam arising from the application of the powder also leads to erroneous fluctuations in the measured level. On the other hand, the level recorded by the γ ray sensor increased gradually between parts (3) and (4) because the sensor is unaffected by the powder application. Both the optical method and the γ -ray method showed similar traces of the steel level, after the powder has been applied.

For further evaluation of the proposed method, the eddy current sensor was also employed. However, there is appreciable inconsistency on the recorded curves between these two methods as shown in Fig. 6. This is because the optical distance meter detected the powder surface, while the eddy current sensor did not sense the powder.

C. Response Time. It is necessary to find the appropriate response time to achieve a reliable monitoring of the steel level during CCP applications. Among the various τ values, $\tau = 0.1$ s was found to be the most suitable to apply a high-speed CCP, although the measurement accuracy slightly decreases comparing with that at a larger τ value. However, this response is so fast that the time dependent effects arising from the powder application and the fire-stirring are also detected as a strong spike noise as shown in Fig. 7. To remove such unwanted noises, the large and rapid level change that exceeds 10 mm/s was masked using numerical compensation [Fig. 7(B)]. However, the unwanted spike noise was reasonably suppressed without numerical compensation when τ value was set at 0.2 s (see Fig. 6).

V Conclusion

We have developed an optical distance meter that is suitable for monitoring the level of molten steel during CCP. Because the achievable measurement range and accuracy were 1000 mm and 1 mm, respectively, the meter is suitable for use during in-process CCP. The measurement range of the optical method is more than ten times those that could be attained using either the γ -ray and eddy current methods, so that continuous monitoring of the steel level can be done from the initiation of the casting process. This capability can be used to realize automatic start of casting and can contribute to increase of the casting quality.

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Robust Kalman Filter Synthesis for Uncertain Multiple Time-Delay Stochastic Systems

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The problem of robust Kalman filter synthesis is considered in this present study for discrete multiple time-delay stochastic systems with parametric and noise uncertainties. A discrete multiple time-delay uncertain stochastic system can be transformed into another uncertain stochastic system with no delay by properly defining state variables. Minimax theory and Bellman-Gronwall lemma are employed on the basis of the upper norm-bounds of parametric uncertainties and noise uncertainties. A robust criterion can consequently be derived which guarantees the asymptotic stability of the uncertain stochastic system. Designed procedures are finally elaborated upon with an illustrative example.

1 Introduction

The existence of delay, which is commonly encountered in various engineering systems, is often a source of instability. Moreover, multiple delays occur in most physical systems. For example, systems with computer control have delays, as it takes time for the computer to execute numerical operations. Besides, remote working, radar, electric networks, transport process, metal rolling systems, etc. all have delays. The output in these systems responds only to an input after some time interval. The problem of stability analysis of delay systems has consequently been a main concern of several previously published research efforts [1–9]. The introduction of time-delay factor generally complicates the analysis, and convenient methods to test stability have long been sought.

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The Kalman filter is verified to be the optimal estimator against noise with normal distribution by minimizing a wide class of reconstruction error performance index [10]. In many practical situations, however, noise covariances may not be accurately known, or else its distribution may not be normal. Numerous reports in regard to this subject have been published [10–14] from the minimax viewpoint by obtaining the saddlepoint solution for the least favorable distribution. The problem of robust Kalman filter synthesis for discrete multiple time-delay stochastic systems with parametric uncertainties and uncertain noise covariances is worthwhile studying since it has not appeared in previous literature. A discrete multiple time-delay uncertain stochastic system can be transformed into another uncertain stochastic system with no delay by properly defining new state variables. Minimax theory and Bellman-Gronwall lemma are employed for deriving a robust criterion which guarantees the asymptotic stability of the uncertain system under parametric uncertainties and uncertain noise covariances. This paper may be viewed as a generalization of Chen and Dong [10] to the discrete multiple time-delay systems.

The organization of this paper is as follows: System description is presented in Section 2. The design procedure of a robust Kalman filter is proposed in Section 3. An example is provided in Section 4 for illustrating our main results. A conclusion is finally provided in Section 5.

2 System Description

A discrete uncertain stochastic system with multiple time delays, which are not exact integer multiples of the sampling interval, is depicted by the following difference equations:

$$x_p(k+1) = A_0 x_p(k) + \Delta A_0(k) x_p(k) + \sum_{i=1}^m A_i x_p(k-h_i) + \sum_{i=1}^m \Delta A_i(k) x_p(k-h_i) + v(k) \quad (2.1a)$$

$$y(k) = C_p x_p(k) + \Delta C_p(k) x_p(k) + e(k) \quad (2.1b)$$

where $x_p(k)$ is an $n \times 1$ state vector, $y(k)$ is a $p \times 1$ output vector, $v(k)$ is an $n \times 1$ random process vector, $e(k)$ is a $p \times 1$ random process vector; h_i (time delay) $i = 1, 2, \dots, m$ are positive real numbers and A_0, A_i, C_p ($\text{rank}(C_p) = p$) are constant matrices with appropriate dimensions. $\Delta A_0(k), \Delta A_i(k)$ and $\Delta C_p(k)$ denote linear time-varying parametric uncertainties with the following upper norm-bounds:

$$\|\Delta A_0(k)\| \leq \sigma \quad (2.2a)$$

$$\|\Delta A_i(k)\| \leq \eta_i, \quad i = 1, 2, \dots, m \quad (2.2b)$$

$$\|\Delta C_p(k)\| \leq \rho \quad (2.2c)$$

where σ, η_i , and ρ are given constants. The process noise $v(k)$ and measurement noise $e(k)$ are uncorrelated random sequences with zero mean. Moreover, they have no time correlation or are “white” noise, that is,

$$E\{v(i)v^T(j)\} = 0 \quad \text{if } i \neq j, \quad (2.3a)$$

$$E\{e(i)e^T(j)\} = 0 \quad \text{if } i \neq j, \quad (2.3b)$$

and have covariances or “noise levels” defined by

$$E\{v(k)v^T(k)\} \equiv R_1 \quad (2.3c)$$

$$E\{e(k)e^T(k)\} \equiv R_2 \quad (2.3d)$$

where R_1 and R_2 are symmetric, positive definite matrices and have the following norm-bounds:

$$\|R_1 - R_{10}\| \leq \epsilon_1 \quad (2.3e)$$

$$\|R_2 - R_{20}\| \leq \epsilon_2 \quad (2.3f)$$

in which ϵ_1, ϵ_2 are given positive constants, R_{10}, R_{20} denote the nominal parts of the actual covariances of the process noise $v(k)$ and measurement noise $e(k)$, respectively.

By defining an $n(m+1) \times 1$ new state vector

$$\bar{x}(k) = [x_p^T(k-h_m) \ x_p^T(k-h_{m-1}) \ \dots \ x_p^T(k-h_1) \ x_p^T(k)]^T, \quad (2.4)$$

the uncertain multiple time-delay system (2.1) can then be transformed into the following uncertain system with no delay:

$$\bar{x}(k+1) = \bar{A}\bar{x}(k) + \Delta\bar{A}(k)\bar{x}(k) + \bar{v}(k) \quad (2.5a)$$

$$y(k) = \bar{C}\bar{x}(k) + \Delta\bar{C}(k)\bar{x}(k) + e(k) \quad (2.5b)$$

in which

$$\bar{A} = \begin{bmatrix} 0 & I & 0 & \dots & 0 & 0 \\ 0 & 0 & I & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & I \\ A_m & A_{m-1} & A_{m-2} & \dots & A_1 & A_0 \end{bmatrix}_{n \times n}, \quad (2.6a)$$

$\Delta\bar{A}(k)$

$$= \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ \Delta A_m(k) & \Delta A_{m-1}(k) & \dots & \Delta A_1(k) & \Delta A_0(k) \end{bmatrix}_{n \times n}, \quad (2.6b)$$

$$\bar{C} = [0 \ 0 \ \dots \ 0 \ C_p]_{p \times n}, \quad (2.6c)$$

$$\Delta\bar{C}(k) = [0 \ 0 \ \dots \ 0 \ \Delta C_p(k)]_{p \times n}, \quad (2.6d)$$

$$\bar{v}(k) = Gv(k), \quad (2.6e)$$

where

$$\bar{n} \equiv n(m+1) \text{ and } G = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I \end{bmatrix}_{n \times n}.$$

According to (2.3c) and (2.6e), the covariance of $\bar{v}(k)$ is given by

$$E\{\bar{v}(k)\bar{v}^T(k)\} = GR_1G^T \equiv \bar{R}_1 \quad (2.7)$$

and \bar{R}_1 has the following norm-bound:

$$\|\bar{R}_1 - \bar{R}_{10}\| \leq \bar{\epsilon}_1 \quad (2.8)$$

where $\bar{R}_{10} \equiv GR_{10}G^T$ and $\bar{\epsilon}_1 \equiv \epsilon_1 \|G\| \|G^T\|$.

Lemma 2.1: If $\text{rank}(C_p) = p$, the pair $\{\bar{A}, \bar{C}\}$ is observable.

Proof: From (2.6c), obtained here is $\text{rank}(\bar{C}) = \text{rank}(C_p)$. If $\text{rank}(C_p) = p$, the pair $\{\bar{A}, \bar{C}\}$ is then observable if and only if

$$\text{rank} \begin{bmatrix} \bar{C} \\ \bar{C}\bar{A} \\ \vdots \\ \bar{C}\bar{A}^{n-p} \end{bmatrix}_{(n-p+1)p \times n} \equiv \text{rank}(O_{n-p}) = \bar{n} [15]. \quad (2.9)$$

The matrix O_{n-p} is easily observed to have the following form:

$$O_{n-p} = \begin{bmatrix} 0 & 0 & \cdots & 0 & C_p \\ C_p A_m & C_p A_{m-1} & \cdots & C_p A_1 & C_p A_0 \\ C_p A_0 A_m & C_p A_m + C_p A_0 A_{m-1} & \cdots & C_p A_2 + C_p A_0 A_1 & C_p A_1 + C_p A_0^2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \times & \times & \cdots & \times & \times \end{bmatrix}_{(n-p+1)p \times n} \quad (2.10)$$

and O_{n-p} has \bar{n} linearly independent rows. Thus, $\text{rank}(O_{n-p}) = \bar{n}$. The proof is then complete.

3 Robust Kalman Filter

Prior to discussion of the design of robust Kalman filter for the uncertain system (2.5), let's consider first its nominal system (i.e., $\Delta \bar{A}(k) = 0$, $\Delta \bar{C}(k) = 0$ and $\bar{\epsilon}_1 = \epsilon_2 = 0$):

$$\bar{x}(k+1) = \bar{A}\bar{x}(k) + \bar{v}(k) \quad (3.1a)$$

$$y(k) = \bar{C}\bar{x}(k) + e(k). \quad (3.1b)$$

The optimal filter that estimates the state $\bar{x}(k)$ of system (3.1) is given by the Kalman filter:

$$\hat{x}(k+1) = \bar{A}\hat{x}(k) + F(k)[y(k) - \bar{C}\hat{x}(k)] \quad (3.2a)$$

$$F(k) = \bar{A}P(k)\bar{C}^T[R_2 + \bar{C}P(k)\bar{C}^T]^{-1} \quad (3.2b)$$

where Kalman filter gain $F(k)$ is chosen for minimizing the performance index $J = \|\bar{x} - \hat{x}\|^{2,3}$ and $P(k)$ is the solution of the following equation:

$$P(k+1) = \bar{A}P(k)\bar{A}^T + \bar{R}_1 - \bar{A}P(k)\bar{C}^T[R_2 + \bar{C}P(k)\bar{C}^T]^{-1}\bar{C}P(k)\bar{A}^T. \quad (3.2c)$$

Objective here lies in formulating a robust Kalman filter for a given observable system (3.1) such that the filter still asymptotically tracks the actual states in the presence of parametric uncertainties and uncertain noise covariances.

The approach for the design of a robust Kalman filter is divided into two steps. In the first step, we only consider the system (3.1) under uncertain noise covariances, i.e.,

$$\bar{x}(k+1) = \bar{A}\bar{x}(k) + \bar{v}(k) \quad (3.3a)$$

$$y(k) = \bar{C}\bar{x}(k) + e(k) \quad (3.3b)$$

with

$$\bar{R}_1 \in S_1 = \{\|\bar{R}_1 - \bar{R}_{10}\| \leq \bar{\epsilon}_1, \bar{R}_1 > 0\} \quad (3.4a)$$

$$R_2 \in S_2 = \{\|R_2 - R_{20}\| \leq \epsilon_2, R_2 > 0\}. \quad (3.4b)$$

The design of a robust Kalman filter for the system (3.3) can therefore be viewed as a saddlepoint problem which deals with the uncertain (but bounded) noise covariances problem. By means of minimax theory [13] and following the same procedure as that in Chen and Dong [10], the following lemma is obtained as:

Lemma 3.1 [10, 13]: The robust Kalman filter under noise uncertainties is the Kalman filter in (3.2) with the worst noise covariances, $\bar{R}_{10} + \bar{\epsilon}_1 I$ and $R_{20} + \epsilon_2 I$; i.e.,

$$\hat{x}(k+1) = \bar{A}\hat{x}(k) + \bar{F}(k)[y(k) - \bar{C}\hat{x}(k)] \quad (3.5a)$$

$$\bar{F}(k) = \bar{A}P(k)\bar{C}^T[(R_{20} + \epsilon_2 I) + \bar{C}P(k)\bar{C}^T]^{-1} \quad (3.5b)$$

$$P(k+1) = \bar{A}P(k)\bar{A}^T + (\bar{R}_{10} + \bar{\epsilon}_1 I) - \bar{A}P(k)\bar{C}^T [(R_{20} + \epsilon_2 I) + \bar{C}P(k)\bar{C}^T]^{-1}\bar{C}P(k)\bar{A}^T. \quad (3.5c)$$

³ $\|f\| = (\sum_{k=0}^{\infty} E[f^T(k)f(k)])^{1/2}$ in which $E(\cdot)$ denotes the expected value of (\cdot) .

Next, the Kalman filter in (3.5) may still not be robust if the system (3.1) is perturbed not only by noise uncertainties but also by parametric uncertainties (i.e. the uncertain system (2.5) is considered). Consequently, it is necessary that more restrictions must be imposed to let the Kalman filter in (3.5) be robust under parametric uncertainties.

Define the state reconstruction error $\bar{x}(k) \equiv \bar{x}(k) - \hat{x}(k)$ and subtract (3.5a) from (2.5a). Consequently obtained here is

$$\bar{x}(k+1) = \bar{x}(k+1) - \bar{x}(k+1) = (\bar{A} - \bar{F}\bar{C})(\bar{x}(k) - \hat{x}(k)) + (\Delta \bar{A}(k) - \bar{F}\Delta \bar{C}(k))\bar{x}(k) + \bar{v}(k) - \bar{F}e(k). \quad (3.6)$$

Yielded through a combination of (3.5a) with (3.6) is

$$x(k+1) = Ax(k) + \Delta A(k)x(k) + Hn(k) \quad (3.7)$$

where

$$x(k) = \begin{bmatrix} \bar{x}(k) \\ \hat{x}(k) \end{bmatrix}, \quad (3.8a)$$

$$A = \begin{bmatrix} \bar{A} & 0 \\ 0 & \bar{A} - \bar{F}\bar{C} \end{bmatrix}, \quad (3.8b)$$

$$\Delta A(k) = \begin{bmatrix} \Delta \bar{A}(k) & 0 \\ \Delta \bar{A}(k) - \bar{F}\Delta \bar{C}(k) & 0 \end{bmatrix}, \quad (3.8c)$$

$$H = \begin{bmatrix} I & 0 \\ I & -\bar{F} \end{bmatrix}, \quad (3.8d)$$

$$n(k) = \begin{bmatrix} \bar{v}(k) \\ e(k) \end{bmatrix}. \quad (3.8e)$$

A robust criterion is derived in the following for guaranteeing the asymptotic stability of the system (3.7). Prior to examination of robust stability, Bellman-Gronwall lemma, which is useful in the proof of the next theorem, is given below.

Lemma 3.2 [16]: Let Z_+ denote the set of nonnegative integers: $\{0, 1, 2, \dots\}$ and u_k, f_k, h_k be real-valued sequence on Z_+ . Let

$$h_k \geq 0, \quad \forall k \in Z_+$$

U.t.c., if

$$u_k \leq f_k + \sum_{i=0}^{k-1} h_i u_i, \quad k = 0, 1, 2, \dots \quad (3.9)$$

$$u_k \leq f_k + \sum_{k=0}^{k-1} \left[\prod_{i<j<k} (1 + h_j) h_i f_i \right], \quad k = 0, 1, 2, \dots \quad (3.10)$$

where $\prod_{i<j<k} (1 + h_j)$ is set equal to 1 when $i = k - 1$.

Note that

(a) If for some constant $h_M, h_i \leq h_M, \forall i$, then

$$u_k \leq f_k + h_M \sum_{i=0}^{k-1} (1 + h_M)^{k-i-1} f_i. \quad (3.11)$$

(b) If for some constant $f_M, f_i \leq f_M, \forall i$ then

$$u_k \leq f_M \prod_{0 \leq i < k} (1 + h_i). \quad (3.12)$$

Theorem 1: If the matrix A is diagonalizable and Hurwitz (i.e., all the eigenvalues of A are inside the unit circle) such that the state transition matrix A^k satisfies the inequality:

$$\|A^k\| \leq M r^{k^4}, \quad k = 0, 1, 2, \dots \quad (3.13)$$

in which $M \geq 1$ and $0 < r < 1$ and if the following inequality holds:

$$|r(1 + h)| < 1 \quad (3.14)$$

where

$$h = \frac{M}{r} \left[2 \left(\sigma + \sum_{i=1}^m \eta_i \right) + \rho \| \bar{F} \| \right] \quad (3.15)$$

then the Kalman filter in (3.5) is a robust filter under parametric and noise uncertainties, that is, the system (3.7) is asymptotically stable.

Proof: See Appendix.

4 Example

The following uncertain stochastic time-delay system is considered as:

$$\begin{aligned} x_p(k+1) &= \begin{bmatrix} 0 & -0.1 \\ -0.2 & -0.1 \end{bmatrix} x_p(k) \\ &+ \begin{bmatrix} 0 & 0.2 \\ 0.2 & 0.01 \end{bmatrix} x_p(k-1.5) + \Delta A_0(k) x_p(k) \\ &+ \Delta A_1(k) x_p(k-1.5) + v(k) \end{aligned} \quad (4.1a)$$

$$y(k) = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} x_p(k) + \Delta C_p(k) x_p(k) + e(k) \quad (4.1b)$$

with

$$\| \Delta A_0(k) \| \leq 0.02^5 \quad (4.2a)$$

$$\| \Delta A_1(k) \| \leq 0.01 \quad (4.2b)$$

$$\| \Delta C_p(k) \| \leq 0.03 \quad (4.2c)$$

and

$$E\{v(k)\} = E\{e(k)\} = 0 \quad (4.2d)$$

$$E\{v(k)v^T(k)\} \equiv R_1 \quad (4.2e)$$

$$E\{e(k)e^T(k)\} \equiv R_2 \quad (4.2f)$$

where

$$R_1 \in S_1 = \left\{ R_{10} - \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} \leq R_1 \leq R_{10} + \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} \right\} \quad (4.2g)$$

⁴ If A is a $q \times q$ diagonalizable and Hurwitz matrix, one obtains the following result:

$$\|A^k\| = \|Q \hat{A}^k Q^{-1}\| \leq \|Q\| \|\hat{A}^k\| \|Q^{-1}\| \leq M r^k$$

where $\hat{A} = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_q)$ and $M = \|Q\| \|Q^{-1}\|$ (the condition number of A). The i th column of Q is an eigenvector corresponding to λ_i . Moreover, r is chosen as the distance from the origin to the eigenvalue (of the matrix A) nearest the unit circle.

⁵ 2-norm case is considered in this example.

$$R_2 \in S_2 = \left\{ R_{20} - \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix} \leq R_2 \leq R_{20} + \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix} \right\} \quad (4.2h)$$

and

$$R_{10} = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.1 \end{bmatrix}, R_{20} = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix}.$$

By defining a new state vector

$$\bar{x}(k) = \begin{bmatrix} x_p(k-1.5) \\ x_p(k) \end{bmatrix}, \quad (4.3)$$

the uncertain stochastic time-delay system (4.1) is then transformed into the following uncertain system with no delay:

$$\bar{x}(k+1) = \bar{A} \bar{x}(k) + \Delta \bar{A}(k) \bar{x}(k) + \bar{v}(k) \quad (4.4a)$$

$$y(k) = \bar{C} \bar{x}(k) + \Delta \bar{C}(k) \bar{x}(k) + e(k) \quad (4.4b)$$

where

$$\bar{A} = \begin{bmatrix} 0 & I \\ A_1 & A_0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0.2 & 0 & -0.1 \\ 0.2 & 0.01 & -0.2 & -0.1 \end{bmatrix}, \quad (4.5a)$$

$$\Delta \bar{A}(k) = \begin{bmatrix} 0 & 0 \\ \Delta A_1(k) & \Delta A_0(k) \end{bmatrix}, \quad (4.5b)$$

$$\bar{v}(k) = Gv(k), \quad (4.5c)$$

$$G = \begin{bmatrix} 0 \\ I \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad (4.5d)$$

$$\bar{C} = \begin{bmatrix} 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0.1 \end{bmatrix}, \quad (4.5e)$$

$$\Delta \bar{C}(k) = [0 \quad \Delta C_p(k)]. \quad (4.5f)$$

Additionally, the pair $\{\bar{A}, \bar{C}\}$ is definitely observable from the fact of Lemma 2.1.

The Kalman filter according to (3.5) is described as follows:

$$\hat{x}(k+1) = \bar{A} \hat{x}(k) + \bar{F}[y(k) - \bar{C} \hat{x}(k)] \quad (4.6a)$$

in which

$$\bar{F} = \begin{bmatrix} -0.0015 & -0.0208 \\ -0.0068 & -0.0095 \\ 0.1018 & 0.0015 \\ 0.0015 & 0.0495 \end{bmatrix}. \quad (4.6b)$$

The condition number $M = 5.3438$ and $r = 0.5692$ are obtained here through substituting \bar{F} into (3.8b) and then applying the inequality (3.13). Yielded here through substituting $\|\bar{F}\|$ and the upper norm-bounds (4.2a)–(4.2c) into (3.15) is

$$\begin{aligned} h &= \frac{M}{r} [2(\sigma + \eta_1) + \rho \|\bar{F}\|] \\ &= \frac{5.3438}{0.5692} [2(0.02 + 0.01) + 0.03 \times 0.1021] = 0.5921 \end{aligned}$$

and then

$$|r(1 + h)| = 0.5692 \times (1 + 0.5921) = 0.9062 < 1.$$

The robust stability condition (3.14) is consequently satisfied. Namely, the Kalman filter in (4.6) is a robust filter in the

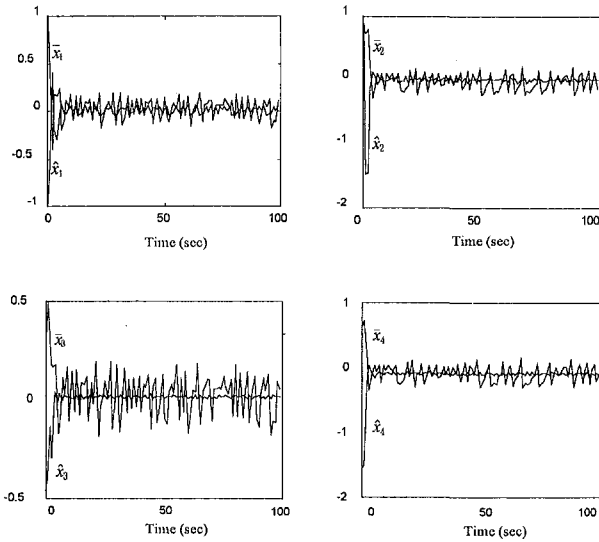


Fig. 1 Simulation of robust Kalman filter under parametrical and noise uncertainties in the example with true state $\bar{x} = [\bar{x}_1 \bar{x}_2 \bar{x}_3 \bar{x}_4]^T$ with initial condition $[1 \ 0.9 \ 0.35 \ 0.75]^T$ and state estimate $\hat{x} = [\hat{x}_1 \hat{x}_2 \hat{x}_3 \hat{x}_4]^T$ with initial condition $[-0.95 \ 0.85 \ -0.45 \ -1.45]^T$

presence of parametric and noise uncertainties. The result of simulation with

$$\Delta A_0(k) = \begin{bmatrix} 0.02 \sin(k) & 0 \\ 0 & 0.01 \exp(-k) \end{bmatrix},$$

$$\Delta A_1(k) = \begin{bmatrix} 0.005 \cos(k) & 0 \\ 0 & 0.01 \sin(k) \end{bmatrix}$$

and

$$\Delta C_p(k) = [0 \ 0.03 \sin(k)]$$

is shown in the Fig. 1.

5 Conclusion

A robust Kalman filter is introduced in this present study by properly defining new state variables for discrete multiple time-delay stochastic systems with parametric and noise uncertainties. Minimax theory and Bellman-Gronwall lemma are employed on the basis of upper norm-bounds of parametric uncertainties and noise uncertainties. A robust criterion is consequently derived which guarantees the asymptotic stability of the uncertain stochastic system. Design procedures are finally elaborated upon with an illustrative example.

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APPENDIX

The solution $x(k)$ to Eq. (3.7) is expressed as

$$x(k) = A^k x(0) + \sum_{j=0}^{k-1} A^{k-j-1} \Delta A(j) x(j) + \sum_{j=0}^{k-1} A^{k-j-1} H n(j). \quad (A.1)$$

Taking norms on both sides of (A.1), we obtain

$$\|x(k)\| = \|A^k x(0) + \sum_{j=0}^{k-1} A^{k-j-1} \Delta A(j) x(j) + \sum_{j=0}^{k-1} A^{k-j-1} H n(j)\| \leq \|A^k\| \|x(0)\| + \sum_{j=0}^{k-1} \|A^{k-j-1}\| \|\Delta A(j)\| \|x(j)\| + \sum_{j=0}^{k-1} \|A^{k-j-1}\| \|H\| \|n(j)\|. \quad (A.2)$$

Similarly, taking norms on both sides of (3.8c)-(3.8e), we have the following inequalities:

$$\|\Delta A(k)\| \leq \|\Delta \bar{A}(k)\| + \|\Delta \bar{A}(k) - \bar{F} \Delta \bar{C}(k)\| \leq 2 \|\Delta \bar{A}(k)\| + \|\bar{F}\| \|\Delta \bar{C}(k)\| \leq 2 \sum_{i=0}^m \|\Delta A_i(k)\| + \|\bar{F}\| \|\Delta C_p(k)\| \leq 2(\sigma + \sum_{i=1}^m \eta_i) + \rho \|\bar{F}\| \quad (A.3)$$

$$\|H\| \leq 2 + \|\bar{F}\| \quad (A.4)$$

$$\|n(k)\| \leq \|\bar{v}(k)\| + \|e(k)\| \leq \|v(k)\| + \|e(k)\| \leq [\text{tr}(R_{10} + \epsilon_1 I)]^{1/2} + [\text{tr}(R_{20} + \epsilon_2 I)]^{1/2}. \quad (A.5)$$

Substituting (3.13) and the inequalities (A.3)–(A.5) into (A.2) yields

$$\begin{aligned} \|x(k)\| \leq & Mr^k \|x(0)\| + \sum_{j=0}^{k-1} Mr^{k-j-1} [2(\sigma + \sum_{i=1}^m \eta_i) \\ & + \rho \|\bar{F}\|] \|x(j)\| + \sum_{j=0}^{k-1} Mr^{k-j-1} (2 + \|\bar{F}\|) \\ & \times \{ [\text{tr}(R_{10} + \epsilon_1 I)]^{1/2} + [\text{tr}(R_{20} + \epsilon_2 I)]^{1/2} \}. \end{aligned} \quad (\text{A.6})$$

Multiplying both sides of (A.6) by r^{-k} leads to

$$\begin{aligned} \|x(k)\| r^{-k} \leq & M \|x(0)\| + \sum_{j=0}^{k-1} Mr^{-j-1} [2(\sigma + \sum_{i=1}^m \eta_i) \\ & + \rho \|\bar{F}\|] \|x(j)\| + \sum_{j=0}^{k-1} Mr^{-j-1} (2 + \|\bar{F}\|) \\ & \{ [\text{tr}(R_{10} + \epsilon_1 I)]^{1/2} + [\text{tr}(R_{20} + \epsilon_2 I)]^{1/2} \}. \end{aligned} \quad (\text{A.7})$$

The inequality (A.7) can be changed to

$$\begin{aligned} \|x(k)\| r^{-k} \leq & M \|x(0)\| + M \frac{1 - r^{-k}}{r - 1} (2 + \|\bar{F}\|) \\ & \times \{ [\text{tr}(R_{10} + \epsilon_1 I)]^{1/2} + [\text{tr}(R_{20} + \epsilon_2 I)]^{1/2} \} \\ & + \sum_{j=0}^{k-1} Mr^{-1} [2(\sigma + \sum_{i=1}^m \eta_i) + \rho \|\bar{F}\|] \|x(j)\| r^{-j}. \end{aligned} \quad (\text{A.8})$$

Applying Lemma 3.2 to (A.8), we obtain the following inequality:

$$\begin{aligned} \|x(k)\| r^{-k} \leq & M \|x(0)\| + M \frac{1 - r^{-k}}{r - 1} (2 + \|\bar{F}\|) \\ & \times \{ [\text{tr}(R_{10} + \epsilon_1 I)]^{1/2} + [\text{tr}(R_{20} + \epsilon_2 I)]^{1/2} \} \\ & + M \|x(0)\| [(1 + h)^k - 1] + hM(2 + \|\bar{F}\|) \end{aligned}$$

$$\begin{aligned} & \times \{ [\text{tr}(R_{10} + \epsilon_1 I)]^{1/2} + [\text{tr}(R_{20} + \epsilon_2 I)]^{1/2} \} \\ & \times \left\{ \frac{(1 + h)^k - 1}{h(r - 1)} - \frac{r(1 + h)^k - r^{1-k}}{(r - 1)[r(1 + h) - 1]} \right\} \end{aligned} \quad (\text{A.9})$$

in which

$$h \equiv Mr^{-1} [2(\sigma + \sum_{i=1}^m \eta_i) + \rho \|\bar{F}\|].$$

Multiplying r^k to both sides of (A.9), we get the following result:

$$\begin{aligned} \|x(k)\| \leq & Mr^k \|x(0)\| + M \frac{r^k - 1}{r - 1} (2 + \|\bar{F}\|) \\ & \times \{ [\text{tr}(R_{10} + \epsilon_1 I)]^{1/2} + [\text{tr}(R_{20} + \epsilon_2 I)]^{1/2} \} \\ & + M \|x(0)\| [r(1 + h)]^k - Mr^k \|x(0)\| + M(2 + \|\bar{F}\|) \\ & \times \{ [\text{tr}(R_{10} + \epsilon_1 I)]^{1/2} + [\text{tr}(R_{20} + \epsilon_2 I)]^{1/2} \} \\ & \times \frac{[r(1 + h)]^k - r^k}{r - 1} - M(2 + \|\bar{F}\|) \{ [\text{tr}(R_{10} + \epsilon_1 I)]^{1/2} \\ & + [\text{tr}(R_{20} + \epsilon_2 I)]^{1/2} \} \frac{h \{ [r(1 + h)]^k r - r \}}{(r - 1)[r(1 + h) - 1]}. \end{aligned} \quad (\text{A.10})$$

As k approaches infinity, $\|x(k)\|$ will approach to a certain value

$$\begin{aligned} & \frac{M(2 + \|\bar{F}\|)}{1 - r} \left\{ 1 + \frac{M[2(\sigma + \sum_{i=1}^m \eta_i) + \rho \|\bar{F}\|]}{r + M[2(\sigma + \sum_{i=1}^m \eta_i) + \rho \|\bar{F}\|] - 1} \right\} \\ & \times \{ [\text{tr}(R_{10} + \epsilon_1 I)]^{1/2} + [\text{tr}(R_{20} + \epsilon_2 I)]^{1/2} \}, \end{aligned} \quad (\text{A.11})$$

because $0 < r < 1$ and $|r(1 + h)| < 1$. Thus, the system (3.7) is robustly stable.