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Boundary Influence On The Entropy Of A Lozi-Type Map

Yu-Chuan Chang and Jonq Juang

Abstract: Let T be a Henon-type map induced from a spatial discretization of a Reaction-Diffusion system. With the above-mentioned description of T, the following open problems were raised in [Afraimovich and Hsu, 2003]. Is it true that, in general, $h(T) = h_D(T) = h_N(T) = h_{\ell_{(1)},\ell_{(2)}}(T)$? Here $h(T)$ and $h_{\ell_{(1)},\ell_{(2)}}(T)$ (see Definitions **??** and **??**) are, respectively, the spatial entropy of the system T and the spatial entropy of T with respect to the lines $\ell_{(1)}$ and $\ell_{(2)}$, and $h_D(T)$ and $h_N(T)$ are spatial entropy with respect to the Dirichlet and Neuman boundary conditions. If it is not true, then which parameters of the lines $\ell_{(i)}$, $i = 1, 2$, are responsible for the value of $h(T)$? What kind of bifurcations occurs if the lines $\ell_{(i)}$ move? In this paper, we shed some light on these open problems with T being replaced by a Lozi-type map.

摘要: 若T是一個由反應-擴散系統之空間離散化誘導出的 Henon 型態映射。伴隨以上所提的T, 跟隨在後的開放性問題是 Afraimovich 和許教授在2003年出版的書中所提出來。一般來說,$h(T) = h_D(T) = h_N(T) = h_{\ell_{(1)},\ell_{(2)}}(T)$是否爲眞? 此處 $h(T)$與$h_{\ell_{(1)},\ell_{(2)}}(T)$(見定義1.1與1.2) 分別是系統$T$的空間熵及$T$對應於直線$\ell_{(1)}$、$\ell_{(2)}$、$h_D(T)$與$h_N(T)$的空間熵, 而$h_D(T)$與$h_N(T)$是對應於 Dirichlet 與 Neuman 邊界條件的空間熵。假設, 若不爲眞, 則直線$\ell_{(i)}$, $i = 1, 2$, 的那些參數對$h(T)$的値影響較大? 假使直線$\ell_{(i)}$移動, 將發生何種分岐點? 在此篇論文中, 我們將以 Lozi 型態映射取代T而提出了一些見解在這未解決的問題上。

Cellular Neural Networks : Defect Patterns And Stability

Jonq Juang, Chin-Lung Li and Shih-Chia Tseng

Abstract:Of concern is one-dimensional Cellular Neural Networks (CNNs) with a piecewise-linear output function for which the slope of the output outside linear zone is $r > 0$. We impose a symmetric coupling between the nearest neighbors. Two parameters a and β are used to describe the weights between the cell with itself and its nearest neighbors, respectively. We study patterns that exist as stable defect equilibria (see Definitions 1.1 and 1.2). In particular, we give an infinite-dimensional version of Gerschgorin's Theorem and derive a concept of δ-extendability to determine whether two local-defect patterns can be glued together. Using such tools, we give a region in (r, a, β)-space for which the corresponding defect patterns have non-zero spatial entropy, while the associated mosaic patterns have zero spatial entropy.

摘要: 在這篇論文中, 探討的是一維細胞類神經網路在其輸出函數是片段線性輸出函數, 且此函數在線性區域以外的斜率$r > 0$, 我們在最鄰近的細胞之間採用一組對稱的耦合, 用兩個參數來描述細胞本身與最鄰近細胞間各自的權數。在這些條件下我們研究存在穩定缺陷平衡的花樣 (參閱定義1.1和定義1.2)。特別地, 我們給予一個無窮維觀點的 Gerschgorin 定理並且導出一個δ-extendability 的概念來決定兩個局部花樣是否可以接合在一起。使用這些工具方法,

我們給定一個在(r, a, β)空間的區域, 其相對應的缺陷花樣擁有非零的空間熵而其相關聯的馬賽克花樣的空間熵卻爲零。更有甚者, 在那些區域所產生的花樣並不是有限型式的子替換。

Fourier Coefficients, Lyapunov Exponents, Invariant Measures and Chaos

Huan-Hsun Hsu and Jonq Juang

Abstract:A complex and unpredictable frequency spectrum of a signal has long been seen in physics and engineering as an indication of a chaotic signal. The first step to understand such phenomenon mathematically was taken up by Chen, Hsu, Huang and Roque-Sol. In particular, they look for possible connections between chaotic dynamical systems and the behavior of its Fourier coefficients. Among other things, they found variety of sufficient conditions on the Fourier coefficients of the n-th iterate f^n of an interval map f, for which the topological entropy of f is positive. In this thesis, we explore the relationship between the Fourier coefficients of an interval map and its Lyapunov exponent and invariant measure. Specifically, the relationships between those three quantities of two family of interval maps, piecewise linear maps admitting a Markov partition and quadratic family, are considered.

摘要: 長久以來, 在物理及工程上, 常利用對一個複雜且不可預測的信號作光譜分析來判斷此信號是否渾沌。首先將此現象做數學分析的是陳鞏老師等人。他們是希望尋求一種關於渾沌動態系統以及傅利葉係數之間的關係。陳鞏老師等人找到了許多關於一個系統做 n 次疊代之後的傅利葉係數, 可以使得這個系統的拓樸熵大於零的充分條件。在這篇論文當中, 我們創新出一個針對定義在一個區間的函數, 傅利葉係數, 黎阿普諾夫指數和不變測度的關係。尤其我們是針對一個定義在馬可夫分割上的片段線性函數以及二次函數來討論這三種特徵量。

Partial-State Coupling, Nonlinearities, and Synchronization

Jonq Juang, Chin-Lung Li, and Yu-Hao Liang

Abstract: Partial-state coupling plays a surprising role in determining if a coupled system can achieve synchronization. For instance, synchronization for the single z_i-component coupled Lorenz equations can not occur, while synchronization for other partial-state couplings can be realized. The purpose of this paper is to address these coupling issues in a general framework. Some sharp conditions are given on the nonlinearities of the subsystem and on the coupling scheme to ensure the synchronization. We apply our theorems to three examples: coupled Lorenz-equations [**?**]; coupled chaotic works [**?**]; coupled Duffing oscillators [**?**]. The results on the latter two examples appear to be new.

摘要: 局部狀態的耦合在耦合系統是否可以達到同步化, 扮演一個讓人驚訝的角色。舉個例子, 在Lorenz 方程式裡, 耦合在單一z_i分量並不能發生同步化, 但耦合在其他局部狀態下, 同步化可以被實現。這篇論文的主要目的是標記這些耦合的問題在一般廣義的架構下。一些明顯的條

件達到同步化, 已經被給定在非線性的子系統以及在耦合結構下。我們應用我們的定理在三個例子: 耦合 Lorenz 方程式; 耦合混沌工作; 耦合 Duffing 震盪子。在最後兩個例子的結果似乎是新的。
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Snap-Back Repellers And Chaotic Traveling Waves In One-Dimensional Cellular Neural Networks

Ya-Wen Chang, Jonq Juang and Chin-Lung Li

International Journal of Bifurcation and Chaos (IJBC), accepted.

Abstract: In 1998, Chen et al., [6] found an error in Marotto's paper [24]. It was pointed out by them that the existence of an expanding fixed point $\mathbf{z}$ of a map $\mathbf{F}$ in $B_r(\mathbf{z})$, the ball of radius r with center at $\mathbf{z}$ which does not necessarily imply that $\mathbf{F}$ is expanding in $B_r(\mathbf{z})$. Subsequent efforts (see e.g., [6], [22]-[23].) in fixing the problems all have some discrepancies since they only give conditions for which $\mathbf{F}$ is expanding "locally". In this paper, we give sufficient conditions so that $\mathbf{F}$ is "globally" expanding. This, in turn, gives more satisfying definitions of a snap-back repeller. We then use those results to show the existence of chaotic backward traveling waves in a discrete time analogy of one-dimensional Cellular Neural Networks (CNNs). Some computer evidence of chaotic traveling waves is also given.

摘要: 在1998年,Chen 等人 [6]在 Marotto 的論文 [24]裡發現了一個錯誤。在 Chen 等人文章中指出, 一個 map$\mathbf{F}$如果存在一個擴展的固定點$\mathbf{z}$, 則$\mathbf{F}$在以半徑爲 r、圓心在$\mathbf{z}$所形成的球$B_r(\mathbf{z})$上, 並不必要的推得$\mathbf{F}$在$B_r(\mathbf{z})$是擴展的。後來接著在爲了修正這個問題 (詳見 例子, [6], [22]-[23].) 的結果都有些地方不一致, 因爲他們都僅僅給予$\mathbf{F}$是"局部"擴展的條件。在這篇論文裡, 我們給了一個充分的條件使得$\mathbf{F}$是全域擴展。接著我們給了snap-back repeller 更滿足的定義。然後, 我們使用這些結果去說明, 在比擬一維度離散時間的類神經網絡 (CNNs) 裡, 混沌的 backward traveling waves 的存在性。在這篇論文裡, 我們也給予了對於混沌的 backward traveling waves 一些電腦上的驗證。

Cellular Neural Networks : Mosaic Patterns, Bifurcation and Complexity

Jonq Juang, Chin-Lung Li and Ming-Huang Liu

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Abstract: We study a one-dimensional Cellular Neural Network with an output function which is non-flat at infinity. Spatial chaotic regions are completely characterized. Moreover, each of their exact corresponding entropy is obtained via the method of transition matrices. We also study the bifurcation phenomenon of mosaic patterns with bifurcation parameters z and β. Here z is a source (or bias) term and β is the interaction weight between the neighboring cells. In particular, we find that by injecting the source term, i.e. $z \neq 0$, a lot of new chaotic patterns emerge with a smaller interaction weight β. However, as β increases to a certain range, most of previously observed chaotic patterns disappear, while other new chaotic patterns emerge.

摘要: 我們主要探討一個細胞類神經網路模型的馬賽克花樣, 在這裡考慮的輸出函數在無窮遠的地方並不是平坦的。許多複雜的參數區域是可以被完整地描繪出來, 每一個參數區域的熵是可以藉由轉換矩陣的方法算出來; 我們也利用參數z和β 來討論一些馬賽克花樣的分歧現象, 在這裡z是一個偏壓項、β是和鄰近細胞的互動比重。特別地, 對於一個小的互動比重β, 我們發現當加入偏壓項之後, 許多新的複雜參數區域都會產生。然而當β增加到某一個範圍之後, 許多上述的複雜參數區域會消失, 但是又有一些新的複雜參數範圍會產生。

Eigenvalue Problems and their Application to The Wavelet Method of Chaotic Control

Jonq Juang and Chin-Lung Li

Journal of Mathematical Physics, 47, 072704 (2006) (16 pages)

Abstract:Controlling chaos via wavelet transform was recently proposed by Wei, Zhan and Lai ([11]). It was reported there that by modifying a tiny fraction of the wavelet subspace of a coupling matrix, the transverse stability of the synchronous manifold of a coupled chaotic system could be dramatically enhanced. The stability of chaotic synchronization is actually controlled by the second largest eigenvalue $\lambda_1(\alpha,\beta)$ of the (wavelet) transformed coupling matrix $C(\alpha,\beta)$ for each α and β. Here β is a mixed boundary constant and α is a scalar factor. In particular, $\beta = 1$(resp., 0) gives the nearest neighbor coupling with periodic(resp., Neumann) boundary conditions. The first rigorous work to understand the eigenvalues of $C(\alpha,1)$ was provided by Shieh, Wei, Wang and Lai ([9]). The purpose of this paper is two-fold. First, we apply a different approach to obtain the explicit formulas for the eigenvalues of $C(\alpha,1)$ and $C(\alpha,0)$. This, in turn, yields some new information

concerning $\lambda_1(\alpha, 1)$. Second, we shed some light on the question whether the wavelet method works for general coupling schemes. In particular, we show that the wavelet method is also good for the nearest neighbor coupling with Neumann boundary conditions.

摘要: 利用微波轉換控制混沌系統, 是由 Wei, Zhan and Lai ([11]) 在最近被提出。這篇報告中, 藉由修改一個耦合矩陣的微波子空間的極微小的分數, 對於一個耦合動態系統的同步流型橫截的穩定性, 可以被引人注目性地增大。對所有的α與β而言, 這種混沌同步化的穩定性, 實際上, 是被微波轉換的耦合矩陣$C(\alpha, \beta)$之第二大固有值$\lambda_1(\alpha, \beta)$所控制。在此,β是一個混合邊界常數, 並且 α是一個純量因子。特別地,$\beta = 1$(相對應於0) 給予了最鄰近耦合的週期 (相對應於 Neumann) 邊界條件。第一個嚴密的去瞭解$C(\alpha, 1)$的固有值工作是由 Shieh, Wei, Wang and Lai ([9]) 所提出。這一篇論文的目的被分爲兩大類, 第一, 我們應用了不同的逼近方式去獲得$C(\alpha, 1)$與$C(\alpha, 0)$明顯的固有值公式。在此, 對於考慮$\lambda_1(\alpha, 1)$依次產出了一些新的資訊。第二, 我們指出對於一般的耦合設計微波方式的工作。特別地, 我們顯示出微波方式對於最鄰近耦合的 Neumann 邊界條件也是好的。

Perturbed Block Circulant Matrices And Their Application to the Wavelet Method of Chaotic Control

Jing-Wei Chang, Jonq Juang and Chin-Lung Li

Journal of Mathematical Physics, accepted.

Abstract:Controlling chaos via wavelet transform was proposed by Wei, Zhan and Lai [Phys. Rev. Lett. 89, 284103 (2002)]. It was reported there that by modifying a tiny fraction of the wavelet subspace of a coupling matrix, the transverse stability of the synchronous manifold of a coupled chaotic system could be dramatically enhanced. The stability of chaotic synchronization is actually controlled by the second largest eigenvalue $\lambda_2(\alpha, \beta)$ of the (wavelet) transformed coupling matrix $C(\alpha, \beta)$ for each α and β. Here β is a mixed boundary constant and α is a scalar factor. In particular, $\beta = 1$(resp., 0) gives the nearest neighbor coupling with periodic(resp., Neumann) boundary conditions. In this paper, we obtain two main results. First, the reduced eigenvalue problem for $C(\alpha, 0)$ is completely solved. Some partial results for the reduced eigenvalue problem of $C(\alpha, \beta)$ are also obtained. Second, we are then able to understand behavior of $\lambda_2(\alpha, 0)$ and $\lambda_2(\alpha, 1)$ for any j and $n \in \mathbb{N}$. Our results complete and strengthen the work of Shieh et al. [J. Math. Phys. to appear] and Juang and Li [J. Math. Phys. to appear].

摘要: 關心的是, 對於β任意固定, 擾動區塊循環矩陣$C(\alpha, \beta)$種類的特徵曲線問題。這裡$\alpha > 0$是 (微波) 純量積因子且$\beta \in \mathbb{R}$表示混合邊界常數。$C(\alpha, \beta)$是一個區塊循環

矩陣只有在$\beta = 1$。對於每個α,$C(\alpha,1)$的特徵值包含它的區塊矩陣作線性組合後的特徵值這件事是已經被知道的。這樣的結果被稱作對於$C(\alpha,1)$的降低特徵值問題。在這篇論文裡，我們得到二個主要結果。首先，對於$C(\alpha,0)$的降低特徵值問題被完全解決了。$C(\alpha,\beta)$的降低特徵值問題也得到一些部分結果。第二，對於$C(\alpha,0)$和$C(\alpha,1)$利用微波方法控制混沌, 扮演必要角色的第二大特徵曲線問題將被討論。

Perturbed Block Circulant Matrices And Their Application to the Wavelet Method of Chaotic Control

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Abstract: Controlling chaos via wavelet transform was proposed by Wei, Zhan and Lai [Phys. Rev. Lett. 89, 284103 (2002)]. It was reported there that by modifying a tiny fraction of the wavelet subspace of a coupling matrix, the transverse stability of the synchronous manifold of a coupled chaotic system could be dramatically enhanced. The stability of chaotic synchronization is actually controlled by the second largest eigenvalue $\lambda_2(\alpha, \beta)$ of the (wavelet) transformed coupling matrix $C(\alpha, \beta)$ for each α and β. Here β is a mixed boundary constant and α is a scalar factor. In particular, $\beta = 1$(resp., 0) gives the nearest neighbor coupling with periodic(resp., Neumann) boundary conditions. In this paper, we obtain two main results. First, the reduced eigenvalue problem for $C(\alpha, 0)$ is completely solved. Some partial results for the reduced eigenvalue problem of $C(\alpha, \beta)$ are also obtained. Second, we are then able to understand behavior of $\lambda_2(\alpha, 0)$ and $\lambda_2(\alpha, 1)$ for any wavelet dimension $j \in \mathbb{N}$ and block dimension $n \in \mathbb{N}$. Our results complete and strengthen the work of Shieh et al. [J. Math. Phys. 47, 082701 (2006)] and Juang and Li [J. Math. Phys. 47, 072704 (2006)].

1. Introduction

Of concern here is the eigencurve problem for a class of "perturbed" block circulant matrices.

$$C(\alpha, \beta)\mathbf{b} = \lambda(\alpha, \beta)\mathbf{b}. \tag{1.1a}$$

Here $C(\alpha, \beta)$ is an $n \times n$ block matrix of the following form.

$$C(\alpha,\beta)=\begin{pmatrix} C_1(\alpha,\beta) & C_2(\alpha,1) & 0 & \cdots & 0 & C_2^T(\alpha,\beta) \\ C_2^T(\alpha,1) & C_1(\alpha,1) & C_2(\alpha,1) & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & C_2^T(\alpha,1) & C_1(\alpha,1) & C_2(\alpha,1) \\ C_2(\alpha,\beta) & 0 & \cdots & 0 & C_2^T(\alpha,1) & \hat{I}C_1(\alpha,\beta)\hat{I} \end{pmatrix}_{n\times n.} \tag{1.1b}$$

Here

$$C_1(\alpha,\beta)=\begin{pmatrix} -1-\beta & 1 & 0 & \cdots & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & -2 & 1 \\ 0 & \cdots & \cdots & 0 & 1 & -2 \end{pmatrix}_{2^j\times 2^j} - \frac{\alpha(1+\beta)}{2^{2j}}ee^T$$

$$=: A_1(\beta,2^j)-\frac{\alpha(1+\beta)}{2^{2j}}ee^T, \tag{1.1c}$$

where $e=(1,1,...,1)^T$, j is a positive integer, $\alpha>0$ is a (wavelet) scalar factor and $\beta\in\mathbb{R}$ represents a mixed boundary constant. Moreover,

$$C_2(\alpha,\beta)=\begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & & & 0 \\ \beta & 0 & \cdots & 0 \end{pmatrix}+\frac{\alpha\beta}{2^{2j}}ee^T$$

$$=: A_2(\beta,2^j)+\frac{\alpha\beta}{2^{2j}}ee^T, \tag{1.1d}$$

$$\hat{I}=\begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ \vdots & & \cdot & \cdot & \cdot & \vdots \\ \vdots & \cdot & \cdot & \cdot & & \vdots \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & \cdots & 0 & 0 \end{pmatrix}. \tag{1.1e}$$

The dimension of $C(\alpha,\beta)$ is $n2^j\times n2^j$. From here on, we shall call n and j the block and the wavelet dimensions of $C(\alpha,\beta)$, respectively. $C(\alpha,\beta)$ is a block circulant matrix (see e.g., [1]) only if $\beta=1$. It is well-known, see e.g., Theorem 5.6.4 of [1], that for each α the eigenvalues of $C(\alpha, 1)$ consists of eigenvalues of a certain linear combinations of its block matrices. Such results are called the reduced eigenvalue problem for $C(\alpha, 1)$.

This problem arises in the wavelet method for a chaotic control ([7]). It is found there that the modification of a tiny fraction of wavelet subspaces of a coupling matrix could lead to a dramatic change in chaos synchronizing properties. We begin with describing their work. Let there be N nodes (oscillators). Assume $\mathbf{u}_i$ is the m-dimensional vector of dynamical variables of the ith node. Let the isolated (uncoupling) dynamics be $\dot{\mathbf{u}}_i = f(\mathbf{u}_i)$ for each node. Used in the coupling, $h : \mathbb{R}^m \to \mathbb{R}^m$ is an arbitrary function of each node's variables. Thus, the dynamics of the ith node is

$$\dot{\mathbf{u}}_i = f(\mathbf{u}_i) + \epsilon \sum_{j=1}^{N} a_{ij} h(\mathbf{u}_j), i = 1, 2, ..., N, \tag{1.2a}$$

where ϵ is a coupling strength. The sum $\displaystyle\sum_{j=1}^{N} a_{ij} = 0$. Let $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_N)^T$, $F(\mathbf{u}) = (f(\mathbf{u}_1), f(\mathbf{u}_2), ..., f(\mathbf{u}_N))^T$, $H(\mathbf{u}) = (h(\mathbf{u}_1), h(\mathbf{u}_2), ..., h(\mathbf{u}_N))^T$, and $A = (a_{ij})$. We may write (1.1a) as

$$\dot{\mathbf{u}} = F(\mathbf{u}) + \epsilon A \times H(\mathbf{u}). \tag{1.2b}$$

Here $\times$ is the direct product of two matrices B and C defined as follows. Let $B = (b_{ij})_{k_1 \times k_2}$ be a $k_1 \times k_2$ matrix and $C = (C_{ij})_{k_2 \times k_3}$ be a $k_2 \times k_3$ block matrix. Then

$$B \times C = (\sum_{l=1}^{k_2} b_{il} C_{lj})_{k_1 \times k_3}.$$

Many coupling schemes are covered by Equation(1.2b). For example, if the Lorenz system is used and the coupling is through its three components x, y, and z, then the function h is just the matrix

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.3}$$

The choice of A will provide the connectivity of nodes. For instance, the nearest neighbor coupling with periodic, Neumann boundary conditions and mixed boundary conditions are, respectively, given as $A = A_1(1, N) + A_2(1, N) + A_2^T(1, N) =: A_P$, $A = A_1(0, N) + A_2(1, N)\hat{I} =: A_N$ and $A = A_1(\beta, N) + A_2(\beta, N) + A_2^T(\beta, N) + (1 - \beta)A_2(1, N)\hat{I} =: A_M$, where those $A_i' s$, $i = 1, 2$, are defined in (1.1c,d).
Mathematical speaking ([5]), the second largest eigenvalue λ_2 of A is dominant in controlling the stability of chaotic synchronization, and the critical strength ϵ_c for synchronization can be determined in term of λ_2,

$$\epsilon_c = \frac{L_{max}}{-\lambda_2}. \tag{1.4}$$

The eigenvalues of $A = A_P$ are given by $\lambda_i = -4\sin^2 \frac{\pi(i-1)}{N}$, i=1,2,...,N. In general, a larger number of nodes gives a smaller nonzero eigenvalue λ_2 in magnitude and, hence, a larger ϵ_c. In controlling a given system, it is desirable to reduce the critical coupling strength ϵ_c. The wavelet method in [7] will, in essence, transform A into $C(\alpha, \beta)$. Consequently, it is of great interest to study the second eigencurve of $C(\alpha, \beta)$ for each β. By the second largest eigencurve $\lambda_2(\alpha, \beta)$ of $C(\alpha, \beta)$ for fixed β, we mean that for given $\alpha > 0$, $\lambda_2(\alpha, \beta)$ is the second largest eigenvalue of $C(\alpha, \beta)$. We remark that 0 is the largest eigenvalue of $C(\alpha, \beta)$ for any $\alpha > 0$ and $\beta \in \mathbb{R}$. This is to say for fixed β, $\lambda_2(\alpha, \beta) = 0$ is the first eigencurve of $C(\alpha, \beta)$. A numerical simulation [7] of a coupled system of $N = 512$ Lorenz oscillators shows that with $h = I_3$ and $A = A_P$, the critical coupling strength ϵ_c decreases linearly with respect to the increase of α up to a critical value α_c. The smallest ϵ_c is about 6, which is about 10^3 times smaller than the original critical coupling strength, indicating the efficiency of the proposed approach.

The mathematical verification of such phenomena is first achieved by Shieh, Wei, Wang and Lai [6]. Specifically, they solved the second eigencurve problem of $C(\alpha, 1)$ with n being a multiple of 4 and j being any positive integer. Subsequently, in [4], the second eigencurve problem for $C(\alpha, 0)$ and $C(\alpha, 1)$ with n being any positive integer and $j = 1$ are solved without touching on the reduced eigenvalue problem. In this paper, we obtain two main results. First, the reduced eigenvalue problem for $C(\alpha, 0)$ is completely solved. Some partial results for the reduced eigenvalue problem of $C(\alpha, \beta)$ are also obtained. Second, we are then able to understand behavior of $\lambda_2(\alpha, 0)$ and $\lambda_2(\alpha, 1)$ for any j and $n \in \mathbb{N}$.

2. Reduced Eigenvalue problems

Writing the eigenvalue problem $C(\alpha, \beta)\mathbf{b} = \lambda\mathbf{b}$, where $\mathbf{b} = (\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_n)^T$ and $\mathbf{b}_i \in \mathbb{C}^{2^j}$, in block component form, we get

$$C_2^T(\alpha, 1)\mathbf{b}_{i-1} + C_1(\alpha, 1)\mathbf{b}_i + C_2(\alpha, 1)\mathbf{b}_{i+1} = \lambda\mathbf{b}_i, \quad 1 \leq i \leq n. \tag{2.1a}$$

Mixed boundary conditions would yield that

$$C_2^T(\alpha, 1)\mathbf{b}_0 + C_1(\alpha, 1)\mathbf{b}_1 + C_2(\alpha, 1)\mathbf{b}_2 = \lambda\mathbf{b}_1 = C_1(\alpha, \beta)\mathbf{b}_1 + C_2(\alpha, 1)\mathbf{b}_2 + C_2^T(\alpha, \beta)\mathbf{b}_n,$$

and

$$\begin{aligned} C_2^T(\alpha, 1)\mathbf{b}_{n-1} + C_1(\alpha, 1)\mathbf{b}_n + C_2(\alpha, 1)\mathbf{b}_{n+1} &= \lambda\mathbf{b}_n \\ &= C_2(\alpha, \beta)\mathbf{b}_1 + C_2^T(\alpha, 1)\mathbf{b}_{n-1} + \hat{I}C_1(\alpha, \beta)\hat{I}\mathbf{b}_n, \end{aligned}$$

or, equivalently,

$$C_2^T(\alpha,1)\mathbf{b}_0 = (C_1(\alpha,\beta) - C_1(\alpha,1))\mathbf{b}_1 + C_2^T(\alpha,\beta)\mathbf{b}_n$$

$$= [\begin{pmatrix} 1-\beta & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} + \frac{\alpha(1-\beta)}{2^{2j}} ee^T]\mathbf{b}_1 + [\begin{pmatrix} 0 & \cdots & 0 & \beta \\ 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix} + \frac{\alpha\beta}{2^{2j}} ee^T]\mathbf{b}_n$$

$$= (1-\beta)C_2^T(\alpha,1)\hat{I}\mathbf{b}_1 + \beta C_2(\alpha,1)\mathbf{b}_n, \tag{2.1b}$$

and

$$C_2(\alpha,1)\mathbf{b}_{n+1} = (\hat{I}C_1(\alpha,\beta)\hat{I} - C_1(\alpha,1))\mathbf{b}_n + C_2(\alpha,\beta)\mathbf{b}_1$$

$$= (1-\beta)C_2^T(\alpha,1)\hat{I}\mathbf{b}_n + \beta C_2(\alpha,1)\mathbf{b}_1. \tag{2.1c}$$

To study the block difference equation (2.1), we set

$$\mathbf{b}_j = \delta^j \boldsymbol{v}, \tag{2.2}$$

where $\boldsymbol{v} \in \mathbb{C}^{2^j}$ and $\delta \in \mathbb{C}$.
Substituting (2.2) into (2.1a), we have

$$[C_2^T(\alpha,1) + \delta(C_1(\alpha,1) - \lambda I) + \delta^2 C_2(\alpha,1)]\boldsymbol{v} = 0. \tag{2.3}$$

To have a nontrivial solution $\boldsymbol{v}$ satisfying (2.3), we need to have

$$det[C_2^T(\alpha,1) + \delta(C_1(\alpha,1) - \lambda I) + \delta^2 C_2(\alpha,1)] = 0. \tag{2.4}$$

Definition 2.1. Equation (2.4) is to be called the characteristic equation of the block difference equation (2.1a). Let $\delta_k = \delta_k(\lambda) \neq 0$ and $\boldsymbol{v}_k = \boldsymbol{v}_k(\lambda) \neq 0$ be complex numbers and vectors, respectively, satisfying (2.3). Here $k = 1, 2, ..., m$ and $m \leq 2^j$. Assume that there exists a $\lambda \in \mathbb{C}$, such that $\mathbf{b}_j = \Sigma_{k=1}^m c_k \delta_k^j(\lambda)\boldsymbol{v}_k(\lambda)$, j=0,1,...,n+1, satisfy equation (2.1b,c), where $c_k \in \mathbb{C}$. If, in addition, $\mathbf{b}_j$, $j = 1, 2, ..., n$, are not all zero vectors, then such $\delta_k(\lambda)$ is called a characteristic value of equation (2.1) or (1.1a) with respect to λ and $\boldsymbol{v}_k(\lambda)$ its corresponding characteristic vector.

Remark 2.1. Clearly, for each α and β, λ in the Definition of 2.1 is an eigenvalue of $C(\alpha,\beta)$.

Should no ambiguity arises, we will write $C_2^T(\alpha,1) = C_2^T$, $C_1(\alpha,1) = C_1$ and $C_2(\alpha,1) = C_2$. Likewise, we will write $A_2(\beta,2^j) = A_2(\beta)$ and $A_1(\beta,2^j) = A_1(\beta)$.

Proposition 2.1. *Let $\rho(\lambda) = \{\delta_i(\lambda) : \delta_i(\lambda)$ is a root of equation (2.4)$\}$, and let $\overline{\rho}(\lambda) = \{\frac{1}{\delta_i(\lambda)} : \delta_i(\lambda)$ is a root of equation (2.4)$\}$. Then $\rho(\lambda) = \overline{\rho}(\lambda)$. Let δ_i and δ_k be in $\rho(\lambda)$. We further assume that δ_i and $\boldsymbol{v}_i = (v_{i1}, \cdots, v_{i2^j})^T$ satisfy (2.3). Suppose $\delta_i \cdot \delta_k = 1$. Then δ_k and $\boldsymbol{v}_k = (v_{i2^j}, v_{i2^j-1}, \cdots, v_{i2}, v_{i1})^T =: \boldsymbol{v}_i^s$ also satisfy (2.3). Conversely, if $\delta_i \cdot \delta_k \neq 1$, then $\boldsymbol{v}_k \neq \boldsymbol{v}_i^s$.*

Proof. To proof $\rho(\lambda) = \overline{\rho}(\lambda)$, we see that

$$det[C_2^T + \delta(C_1 - \lambda I) + \delta^2 C_2] = \delta^2 det[\frac{1}{\delta^2} C_2^T + \frac{1}{\delta}(C_1 - \lambda I) + C_2]$$
$$= \delta^2 det[\frac{1}{\delta^2} C_2^T + \frac{1}{\delta}(C_1 - \lambda I) + C_2]^T = \delta^2 det[C_2^T + \frac{1}{\delta}(C_1 - \lambda I) + \frac{1}{\delta^2} C_2].$$

Thus, if δ is a root of equation (2.4), then so is $\frac{1}{\delta}$. To see the last assertion of the proposition, we write equation (2.3) with $\delta = \delta_i$ and $\boldsymbol{v} = \boldsymbol{v}_i$ in component form.

$$\sum_{m=1}^{2^j} [(C_2^T)_{lm} v_{im} + \delta_i (\bar{C}_1)_{lm} v_{im} + \delta_i^2 (C_2)_{lm} v_{im}] = 0, l = 1, 2, ..., 2^j. \tag{2.5}$$

Here $\bar{C}_1 = C_1 - \lambda I$. Now the right hand side of (2.5) becomes

$$(\frac{1}{\delta_k})^2 \{\sum_{m=1}^{2^j} [(C_2)_{l(2^j+1-m)} v_{i(2^j+1-m)} + \delta_k (\bar{C}_1)_{l(2^j+1-m)} v_{i(2^j+1-m)}$$
$$+ \delta_k^2 (C_2^T)_{l(2^j+1-m)} v_{i(2^j+1-m)}]\}$$
$$= (\frac{1}{\delta_k})^2 \{\sum_{m=1}^{2^j} [(C_2^T)_{(2^j+1-l)m} v_{i(2^j+1-m)} + \delta_k (\bar{C}_1)_{(2^j+1-l)m} v_{i(2^j+1-m)}$$
$$+ \delta_k^2 (C_2)_{(2^j+1-l)m)} v_{i(2^j+1-m)}]\}, l = 1, 2, ..., 2^j. \tag{2.6}$$

We have used the fact that

$$(A)_{(2^j+1-l)m} = (A^T)_{l(2^j+1-m)}, \tag{2.7}$$

where $A = C_2^T$ or $\bar{C}_1$ or C_2 to justify the equality in (2.6). However, (2.7) follows from (1.1c) and (1.1d). Letting $v_{i(2^j+1-m)} = v_{km}$, we have that the pair $(\delta_k, \boldsymbol{v}_k)$ satisfies (2.3). Suppose $\boldsymbol{v}_k = \boldsymbol{v}_i^s$, we see, similarly, that the pair $(\frac{1}{\delta_i}, \boldsymbol{v}_k)$ also satisfy (2.3). Thus $\frac{1}{\delta_i} = \delta_k$.

□

Remark 2.2. Equation (2.4) is a palindromic equation. That is for each λ, δ and δ^{-1} are both the roots of (2.4). However, eigenvalue problem discussed here is not a palindromic eigenvalue problem [3].

Definition 2.2. We shall call $\boldsymbol{v}^s$ and $-\boldsymbol{v}^s$, the symmetric vector and antisymmetric vector of $\boldsymbol{v}$, respectively. A vector $\boldsymbol{v}$ is symmetric (resp., antisymmetric) if $\boldsymbol{v} = \boldsymbol{v}^s$ (resp., $\boldsymbol{v} = -\boldsymbol{v}^s$).

Theorem 2.1. *Let $\delta_k = e^{\frac{\pi k}{n} i}$, k is an integer and $i = \sqrt{-1}$, then δ_{2k}, k=0,1,...,n-1, are characteristic values of equation (2.1) with $\beta = 1$. For each α, if $\lambda \in \mathbb{C}$ satisfies*

$$det[C_2^T + \delta_{2k}(C_1 - \lambda I) + \delta_{2k}^2 C_2] = 0,$$

for some $k \in \mathbb{Z}$, $0 \le k \le n-1$, then λ is an eigenvalue of $C(\alpha, 1)$.

Proof. Let λ be as assumed. Then there exists a $\boldsymbol{v} \in \mathbb{C}^{2^j}$, $\boldsymbol{v} \neq \mathbf{0}$ such that

$$[C_2^T + \delta_{2k}(C_1 - \lambda I) + \delta_{2k}^2 C_2]\boldsymbol{v} = \mathbf{0}.$$

Let $\mathbf{b}_j = \delta_{2k}^j \boldsymbol{v}$, $0 \le j \le n+1$. Then such $\mathbf{b}_j's$ satisfy (2.1a), (2.1b), and (2.1c). We just proved the assertion of the theorem. □

Corollary 2.1. *Set*

$$\Gamma_k = C_1 + \delta_{2n-k} C_2^T + \delta_k C_2. \tag{2.8}$$

Then the eigenvalues of $C(\alpha, 1)$, for each α, consists of eigenvalues of Γ_k, $k = 0, 2, 4, ..., 2(n-1)$. That is $\rho(C(\alpha, 1)) = \bigcup_{k=0}^{n-1} \rho(\Gamma_{2k})$. Here $\rho(A)$ = the spectrum of the matrix A.

Remark 2.3. $C(\alpha, 1)$ is a block circulant matrix. The assertion of Corollary 2.1 is not new (see e.g., Theorem 5.6.4 of [1]). Here we merely gave a different proof.

To study the eigenvalue of $C(\alpha, 0)$ for each α, we begin with considering the eigenvalues and eigenvectors of $C_2^T + C_1 + C_2$ and $C_2^T - C_1 + C_2$.

Proposition 2.2. *Let $T_1(C)$ (resp., $T_2(C)$) be the set of linearly independent eigenvectors of the matrix C that are symmetric (resp., antisymmetric). Then $|T_1(C_2^T + C_1 + C_2)| = |T_2(C_2^T + C_1 + C_2)| = |T_1(C_2^T - C_1 + C_2)| = |T_2(C_2^T - C_1 + C_2)| = 2^{j-1}$. Here $|A|$ denote the cardinality of the set A.*

Proof. We will only illustrate the case for $C_2^T - C_1 + C_2 =: C$. We first observe that $|T_1(C)|$ is less than or equal to 2^{j-1}. So is $|T_2(C)|$. We also remark the cardinality of the set of all linearly independent eigenvectors of C is 2^j. If $0 < |T_1(C)| < 2^{j-1}$, there must exist an eigenvector $\boldsymbol{v}$ for which $\boldsymbol{v} \neq \boldsymbol{v}^s$, $\boldsymbol{v} \neq -\boldsymbol{v}^s$ and $\boldsymbol{v} \notin span\{T_1(C), T_2(C)\}$, the span of the vectors in $T_1(C)$ and $T_2(C)$. It then follows from Proposition 2.1 that $\boldsymbol{v} + \boldsymbol{v}^s$, a symmetric vector, is in the $span\{T_1(C)\}$. Moreover, $\boldsymbol{v} - \boldsymbol{v}^s$ is in $span\{T_2(C)\}$. Hence $\boldsymbol{v} \in span\{T_1(C), T_2(C)\}$, a contradiction. Hence, $|T_1(C)| = 2^{j-1}$. Similarly, we conclude that $|T_2(C)| = 2^{j-1}$. □

Theorem 2.2. *Let $\delta_k = e^{\frac{\pi k}{n} i}$, k is an integer, $i = \sqrt{-1}$. For each α, if $\lambda \in \mathbb{C}$ satisfies*

$$det[C_2^T + \delta_k(C_1 - \lambda I) + \delta_k^2 C_2] = 0,$$

for some $k \in \mathbb{Z}$, $1 \leq k \leq n-1$, then λ is an eigenvalue of $C(\alpha, 0)$. Let λ be the eigenvalue of $C_2^T + C_1 + C_2$ (resp., $-C_2^T + C_1 - C_2$) for which its associated eigenvector $\boldsymbol{v}$ satisfies $\hat{I}\boldsymbol{v} = \boldsymbol{v}$ (resp., $\hat{I}\boldsymbol{v} = -\boldsymbol{v}$), then λ is also an eigenvalue of $C(\alpha, 0)$.

Proof. For any $1 \leq k \leq n-1$, let δ_k be as assumed. Let λ_k and ν_k be a number and a nonzero vector, respectively, satisfying

$$[C_2^T + \delta_k(C_1 - \lambda_k I) + \delta_k^2 C_2]\boldsymbol{v}_k = \mathbf{0}. \tag{2.9}$$

Using Proposition 2.1, we see that λ_k satisfies

$$det[C_2^T + \delta_{2n-k}(C_1 - \lambda_k I) + \delta_{2n-k}^2 C_2] = 0. \tag{2.10}$$

Let $\boldsymbol{v}_{2n-k}$ be a nonzero vector satisfying $[C_2^T + \delta_{2n-k}(C_1 - \lambda_k I) + \delta_{2n-k}^2 C_2]\boldsymbol{v}_{2n-k} = \mathbf{0}$. Letting

$$\mathbf{b}_i = \delta_k^i \boldsymbol{v}_k + \delta_k \delta_{2n-k}^i \boldsymbol{v}_{2n-k}, i = 0, 1, ..., n+1,$$

we conclude, via (2.9) and (2.10), that $\mathbf{b}_i$ satisfy (2.1a) with $\lambda = \lambda_k$. Moreover,

$$\hat{I}\mathbf{b}_1 = \delta_k \hat{I}\boldsymbol{v}_k + \hat{I}\boldsymbol{v}_{2n-k} = \delta_k \boldsymbol{v}_{2n-k} + \boldsymbol{v}_k = \mathbf{b}_0.$$

We have used Proposition 2.1 to justify the second equality above. Similarly, $\mathbf{b}_{n+1} = \hat{I}\mathbf{b}_n$. To see $\lambda = \lambda_k$, $1 \leq k \leq n-1$, is indeed an eigenvalue of $C(\alpha, 0)$ for each α, it remains to show that $\mathbf{b}_i \neq \mathbf{0}$ for some i. Using Proposition 2.1, we have that there exists an m, $1 \leq m \leq 2^j$ such that $v_{km} = v_{(2n-k)(2^j-m+1)} \neq 0$. We first show that $\mathbf{b}_0 \neq \mathbf{0}$. Let m be the index for which $v_{km} \neq 0$. Suppose $\mathbf{b}_0 = \mathbf{0}$. Then

$$v_{km} + \delta_k v_{(2n-k)m} = 0$$

and

$$v_{k(2^j-m+1)} + \delta_k v_{(2n-k)(2^j-m+1)} = v_{(2n-k)m} + \delta_k v_{km} = 0.$$

And so, $v_{km} = \delta_k^2 v_{km}$, a contradiction. Let λ and $\boldsymbol{v}$ be as assumed in the last assertion of theorem. Letting $\mathbf{b}_i = \boldsymbol{v}$ (resp., $\mathbf{b}_i = (-1)^i \boldsymbol{v}$), we conclude that λ is an eigenvalue of $C(\alpha, 0)$ with corresponding eigenvector $(\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n)^T$. Thus, λ_k is an eigenvalue of $C(\alpha, 0)$ for each α. □

Corollary 2.2. *Let* $\delta_k = e^{\frac{\pi k}{n}i}$*, k is an integer,* $i = \sqrt{-1}$*. Then, for each α,* $\rho(C(\alpha,0)) = \bigcup_{k=1}^{n-1} \rho(\Gamma_k) \bigcup \rho^S(\Gamma_0) \bigcup \rho^{AS}(\Gamma_n)$*, where $\rho^S(A)$ (resp., $\rho^{AS}(A)$) the set of eigenvalues of A for which their corresponding eigenvectors are symmetric (resp., antisymmetric).*

We next consider the eigenvalues of $C(\alpha,\beta)$.

Theorem 2.3. *Let* $\delta_k = e^{\frac{\pi k}{n}i}$*, k is an integer,* $i = \sqrt{-1}$*. Then, for each α,*

$$\rho(C(\alpha,\beta)) \supset \begin{cases} \displaystyle\bigcup_{k=1}^{[\frac{n}{2}]} \rho(\Gamma_{2k}) \bigcup \rho^S(\Gamma_0), & n \ \textit{is odd}, \\ \displaystyle\bigcup_{k=1}^{\frac{n}{2}-1} \rho(\Gamma_{2k}) \bigcup \rho^S(\Gamma_0) \bigcup \rho^{AS}(\Gamma_n), & n \ \textit{is even}. \end{cases}$$

Here $[\frac{n}{2}]$ is the greatest integer that is less than or equal to $\frac{n}{2}$.

Proof. We illustrate only the case that n is even. Assume that k is such that $1 \le k \le \frac{n}{2} - 1$. Let $\mathbf{b}_i = \delta_{2k}^i \boldsymbol{v}_{2k} + \delta_{2k}\delta_{2n-2k}^i \boldsymbol{v}_{2n-2k}$, we see clearly that such $\mathbf{b}_i$, $i = 0, 1, n, n+1$, satisfy both Neumann and periodic boundary conditions, respectively. And so

$$\mathbf{b}_0 = (1-\beta)\mathbf{b}_0 + \beta\mathbf{b}_0 = (1-\beta)\hat{I}\mathbf{b}_1 + \beta\mathbf{b}_n,$$

and

$$\mathbf{b}_{n+1} = (1-\beta)\mathbf{b}_{n+1} + \beta\mathbf{b}_{n+1} = (1-\beta)\hat{I}\mathbf{b}_n + \beta\mathbf{b}_1.$$

Here, δ_{2k}, $1 \le k \le \frac{n}{2} - 1$, are characteristic values of equation of (2.1). Thus, if $\lambda \in \rho(\Gamma_{2k})$, then λ is an eigenvalue of $C(\alpha,\beta)$. The assertions for Γ_0 and Γ_n can be done similarly. □

Remark 2.4. If n is an even number, for each α and β, half of the eigenvalues of $C(\alpha,\beta)$ are independent of the choice of β. The other characteristic values of (2.1) seem to depend on β. It is of interest to find them.

3. The Second Eigencurve of $C(\alpha,0)$ and $C(\alpha,1)$

We begin with considering the eigencurves of Γ_k, as given in (2.8). Clearly,

$$\Gamma_k = \begin{pmatrix} -2 & 1 & 0 & \cdots & \cdots & \delta_{2n-k} \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & -2 & 1 \\ \delta_k & \cdots & \cdots & 0 & 1 & -2 \end{pmatrix}_{m\times m} - \frac{\alpha(2 - 2\cos\frac{\pi k}{n})}{m} ee^T$$

$$=: D_1(k) - \alpha(k)ee^T, \tag{3.1}$$

where $m = 2^j$. We next find a unitary matrix to diagonalize $D_1(k)$.

Remark 3.1. Let $(\lambda(k), \boldsymbol{v}(k))$ be the eigenpair of $D_1(k)$. If $e^T\boldsymbol{v}(k) = 0$, then $\lambda(k)$ is also an eigenvalue of Γ_k.

Proposition 3.1. *Let*

$$\theta_{l,k} = \frac{2l\pi}{m} + \frac{k\pi}{nm}, l = 0, 1, ..., m-1, \tag{3.2a}$$

$$\boldsymbol{p}_l(k) = \left(e^{i\theta_{l,k}}, e^{i2\theta_{l,k}}, \cdots, e^{im\theta_{l,k}}\right)^T \tag{3.2b}$$

and

$$P(k) = \left(\ \tfrac{\boldsymbol{p}_0(k)}{\sqrt{m}}, \cdots, \tfrac{\boldsymbol{p}_{m-1}(k)}{\sqrt{m}} \ \right). \tag{3.2c}$$

(i) Then $P(k)$ is a unitary matrix and $P^H(k)D_1(k)P(k) = Diag(\lambda_{0,k} \cdots \lambda_{m-1,k})$, where P^H is the conjugate transpose of P, and

$$\lambda_{l,k} = 2\cos\theta_{l,k} - 2, l = 0, 1, .., m-1. \tag{3.2d}$$

(ii) Moreover, for $0 \le k \le 2n$, the eigenvalues of $D_1(k)$ are distinct if and only if $k \ne 0, n$ or $2n$.

Proof. Let $\mathbf{b} = (b_1, ..., b_m)^T$. Writing the eigenvalue problem $D_1(k)\mathbf{b} = \lambda\mathbf{b}$ in component form, we get

$$b_{j-1} - (2+\lambda)b_j + b_{j+1} = 0, j = 2, 3, ..., m-1, \tag{3.3a}$$

$$-(2+\lambda)b_1 + b_2 + \delta_{2n-k}b_m = 0, \tag{3.3b}$$

$$\delta_k b_1 + b_{m-1} - (2+\lambda)b_m = 0. \tag{3.3c}$$

Set $b_j = \delta^j$, where δ satisfies the characteristic equation $1 - (2+\lambda)\delta + \delta^2 = 0$ of the system $D_1(k)\mathbf{b} = \lambda\mathbf{b}$. Then the boundary conditions (3.3b) and (3.3c) are reduced to

$$\delta^m = \delta_k. \tag{3.4}$$

Thus, the solutions $e^{i\theta_{l,k}}$, $l = 0, 1, ..., m-1$, of (3.4) are the candidates for the characteristic values of (3.3). Substituting $e^{i\theta_{l,k}}$ into (3.3a) and solving for λ, we see that $\lambda = \lambda_{l,k}$ are the candidates for the eigenvalues of $D_1(k)$. Clearly, $(\lambda, \mathbf{b}) = (\lambda_{l,k}, \boldsymbol{p}_l(k))$ satisfies $D_1(k)\mathbf{b} = \lambda\mathbf{b}$ and $\mathbf{b} = \boldsymbol{p}_l(k) \ne 0$. Thus, $\lambda = \lambda_{l,k}$ are, indeed, the eigenvalues of $D_1(k)$. To complete the proof of the proposition, it suffices to show that $P(k)$ is unitary. To this end, we need to compute $\boldsymbol{p}_l^H(k) \cdot \boldsymbol{p}_{l'}(k)$. Clearly, $\boldsymbol{p}_l^H(k) \cdot \boldsymbol{p}_l(k) = m$. Now, let $l \ne l'$, we have that

$$\boldsymbol{p}_l^H(k) \cdot \boldsymbol{p}_{l'}(k) = \sum_{j=1}^{m} e^{ij(\theta_{l,k} - \theta_{l',k})} = \sum_{j=1}^{m} e^{ij(\frac{2(l-l')}{m}\pi)} = \frac{r(1-r^m)}{1-r} = 0,$$

where $r = e^{i(\frac{2(l-l')}{m}\pi)}$. Hence, $P(k)$ is unitary. The last assertion of the proposition is obvious. $\square$

To prove the main results in this section, we also need the following proposition. Some of assertions of the proposition are from Theorem 8.6.2 of [2].

Proposition 3.2. *Suppose $D = diag(d_1, ..., d_m) \in \mathbb{R}^{m\times m}$ and that the diagonal entries satisfy $d_1 > \cdots > d_m$. Let $\gamma \neq 0$ and $\boldsymbol{z} = (z_1, ..., z_m)^T \in \mathbb{R}^n$. Assume that $(\lambda_i(\gamma), \boldsymbol{v}_i(\gamma))$ are the eigenpairs of $D + \gamma \boldsymbol{z}\boldsymbol{z}^T$ with $\lambda_1(\gamma) \geq \lambda(\gamma) \geq ... \geq \lambda_m(\gamma)$. (i) Let $A = \{k : 1 \leq k \leq m, z_k = 0\}$, $A^c = \{1, ..., m\} - A$. If $k \in A$, then $d_k = \lambda_k$. (ii) Assume $\alpha > 0$. Then the following interlacing relations hold $\lambda_1(\gamma) \geq d_1 \geq \lambda_2(\gamma) \geq d_2 \geq ... \geq \lambda_m(\gamma) \geq d_m$. Moreover, the strict inequality holds for these indexes $i \in A^c$. (iii) Let $i \in A^c$, $\lambda_i(\gamma)$ are strictly increasing in γ and $\lim_{\alpha\to\infty} \lambda_i(\gamma) = \bar{\lambda}_i$ for all i, where $\bar{\lambda}_i$ are the roots of $g(\lambda) = \sum_{k\in A^c} \frac{z_i^2}{d_k - \lambda}$ with $\bar{\lambda}_i \in (d_i, d_{i-1})$. In case that $1 \in A^c$, $d_0 = \infty$.*

Proof. The proof of interlacing relations in (ii) and the assertion in (i) can be found in Theorem 8.6.2 of [2]. We only prove the remaining assertions of the proposition. Rearranging $\mathbf{z}$ so that $\mathbf{z}^T = (0, 0, ..., 0, z_{i_1}, ..., z_{i_k}) =: (0, ..., 0, \bar{\boldsymbol{z}}^T)$, where $i_1 < i_2 < ... < i_k$ and $i_j \in A^c$, $j = 1, ..., k$. The diagonal matrix D is rearranged accordingly. Let $D = diag(D_1, D_2)$, where $D_2 = diag(d_{i_1}, ..., d_{i_k})$. Following Theorem 8.6.2 of [2], we see that $\lambda_{i_j}(\gamma)$ are the roots of the scalar equation $f_\gamma(\lambda)$, where

$$f_\gamma(\lambda_{i_j}(\gamma)) = 1 + \gamma \sum_{j=1}^{k} \frac{z_j^2}{d_{i_j} - \lambda_{i_j}(\gamma)} = 0. \tag{3.5}$$

Differentiate the equation above with respect to γ, we get

$$\sum_{j=1}^{k} \frac{z_{i_j}^2}{d_{i_j} - \lambda_{i_j}(\gamma)} + (\gamma \sum_{j=1}^{k} \frac{z_{i_j}^2}{(d_{i_j} - \lambda_{i_k}(\gamma))^2}) \frac{d\lambda_{i_j}(\gamma)}{d\gamma} = 0.$$

Thus,

$$\frac{d\lambda_{i_j}(\gamma)}{d\gamma} = \frac{1}{\gamma^2} \sum_{j=1}^{k} \frac{z_{i_j}^2}{(d_{i_j} - \lambda_{i_j}(\gamma))^2} > 0.$$

Clearly, for each i_j, the limit of $\lambda_{i_j}(\gamma)$ as $\gamma \to \infty$ exists, say $\bar{\lambda}_{i_j}$. Since, for $d_{i_j} < \lambda < d_{i_j-1}$,

$$\sum_{j=1}^{k} \frac{z_{i_j}^2}{d_{i_j} - \lambda_{i_j}(\gamma)} = \frac{1}{\gamma}.$$

Taking the limit as $\alpha \to \infty$ on both side of the equation above, we get

$$\sum_{j=1}^{k} \frac{z_{i_j}^2}{d_{i_j} - \bar{\lambda}_{i_j}} = 0 \tag{3.6}$$

as desired. $\square$

We are now in the position to state the following theorems.

Theorem 3.1. *let n and $m = 2^j$ be given positive integers. For each k, $k = 1, 2, \cdots, n-1$, and α, we denote by $\lambda_{l,k}(\alpha)$, $l = 0, 1, \cdots, 2^j - 1$, the eigenvalues of Γ_k. For $k = 1, 2, \cdots, n-1$, we let $(\lambda_{l,k}, u_{l,k})$, $l = 0, 1, \cdots, 2^j - 1$, be the eigenpairs of $D_1(k)$, as defined in (3.1). Then the following hold true.*
(i) $\lambda_{l,k}(\alpha)$ is strictly decreasing in α, $l = 0, 1, \cdots, 2^j - 1$ and $k = 1, 2, \cdots, n-1$.
(ii)There exist $\lambda^*_{l,k}$ such that $\lim_{\alpha \to \infty} \lambda_{l,k}(\alpha) = \lambda^*_{l,k}$. Moreover, $g_k(\lambda^*_{l,k}) = 0$, where

$$g_k(\lambda) = \sum_{l=1}^{m} \frac{1}{(\lambda_{l-1,k})(\lambda_{l-1,k} + \lambda)}. \tag{3.7}$$

Proof. The first assertion of the theorem follows from proposition 3.2-(iii). Let k be as assumed. Set, for $l = 0, 1, ..., m-1$,

$$z_{l+1} = \boldsymbol{p}_l^H(k)e = \sum_{j=1}^{m} e^{ij\theta_{l,k}} = \frac{e^{-\theta_{l,k}}(1 - e^{-im\theta_{l,k}})}{1 - e^{-\theta_{l,k}}} = \frac{e^{-\theta_{l,k}}(1 - e^{-ik\frac{\pi}{n}})}{1 - e^{-\theta_{l,k}}}.$$

Then

$$\bar{\boldsymbol{z}}_{l+1}\boldsymbol{z}_{l+1} = \frac{2 - 2\cos m\theta_{l,k}}{2 - 2\cos\theta_{l,k}} = \frac{2\cos\frac{k\pi}{n} - 2}{\lambda_{l,k}} \neq 0. \tag{3.8}$$

Let $P(k)$ be as given in (3.2c). Then

$$-P^H(k) \cdot \Gamma_k \cdot P(k) = Diag(-\lambda_{0,k}, ..., -\lambda_{m-1,k}) + \alpha(k)P_l^H(k)e(P_l^H(k)e)^H.$$

Note that if k is as assumed, it follows from Proposition 3.1-(ii) that $\lambda_{l,k}$, $l = 0, ..., m-1$, are distinct. Thus, we are in the position to apply Proposition 3.2. Specifically, by noting $A^c = \phi$, we see that $\lambda^*_{0,k}$ satisfies $g(\lambda) = 0$, where

$$g(\lambda) = \sum_{l=1}^{m} \frac{1}{(\lambda_{l-1,k})(\lambda_{l-1,k} + \lambda)}.$$

We have used (3.2d), (3.6) and (3.8) to find $g(\lambda)$. $\square$

We next give an upper bound for $\lambda^*_{0,k}$, $k = 1, 2, \cdots, n-1$.

Theorem 3.2. *The following inequalities hold true.*

$$\lambda^*_{0,k} < \lambda_{0,n}, \qquad k = 1, 2, \cdots, , n-1. \tag{3.9}$$

Proof. To complete the proof of (3.9), it suffices to show that $g_k(-\lambda_{0,n}) < 0$. Now,

$$\begin{aligned} g_k(-\lambda_{0,n}) &= \sum_{l=1}^{m} \frac{1}{[2cos(\frac{2(l-1)\pi}{m} + \frac{k\pi}{nm}) - 2][2cos(\frac{2(l-1)\pi}{m} + \frac{k\pi}{nm}) - 2cos\frac{\pi}{m}]} \\ &=: h(m, n, k) = h(2^j, n, k). \end{aligned} \tag{3.10}$$

We shall prove that $h(2^j,n,k)<0$ by the induction on j. For $j=1$, $h(2,n,k)=\frac{1}{2}\left[\frac{1}{cos^2(\frac{k\pi}{2n})-1}\right]<0$, $k=1,2,\cdots,,n-1$. Assume $h(2^j,n,k)<0$. Here, $n\in\mathbb{N}$ and $k=1,2,\cdots,,n-1$. We first note that

$$cos\left(\frac{2(2^j+i-1)\pi}{2^{j+1}}+\frac{k\pi}{2^{j+1}n}\right)=-cos\left(\frac{2(i-1)\pi}{2^{j+1}}+\frac{k\pi}{2^{j+1}n}\right)$$
$$=:-cos\theta_{i-1,k,j+1},\quad i=1,2,\cdots,2^j. \tag{3.11}$$

Moreover, upon using (3.11), we get that

$$\frac{1}{(cos\theta_{i-1,k,j+1}-1)(cos\theta_{i-1,k,j+1}-cos\theta_{0,n,j+1})}$$
$$+\frac{1}{(cos\theta_{2^j+i-1,k,j+1}-1)(cos\theta_{2^j+i-1,k,j+1}-cos\theta_{0,n,j+1})}$$

$$=\frac{1}{(cos\theta_{i-1,k,j+1}-1)(cos\theta_{i-1,k,j+1}-cos\theta_{0,n,j+1})}$$
$$+\frac{1}{(cos\theta_{i-1,k,j+1}+1)(cos\theta_{i-1,k,j+1}+cos\theta_{0,n,j+1})}$$

$$=\frac{2cos^2\theta_{i-1,k,j+1}+2cos\theta_{0,n,j+1}}{(cos^2\theta_{i-1,k,j+1}-1)(cos^2\theta_{i-1,k,j+1}-cos^2\theta_{0,n,j+1})}$$

$$=\frac{8(cos^2\theta_{i-1,k,j+1}+cos\theta_{0,n,j+1})}{(cos2\theta_{i-1,k,j+1}-1)(cos2\theta_{i-1,k,j+1}-cos2\theta_{0,n,j+1})}$$

$$=\frac{2(cos^2\theta_{i-1,k,j+1}+cos\theta_{0,n,j+1})}{(cos\theta_{i-1,k,j}-1)(cos\theta_{i-1,k,j}-cos\theta_{0,n,j})}. \tag{3.12}$$

We are now in a position to compute $h(2^{j+1},n,k)$. Using (3.12), we get that

$$h(2^{j+1},n,k)=\sum_{l=1}^{2^{j+1}}\frac{1}{4(cos\theta_{l-1,k,j+1}-1)(cos\theta_{l-1,k,j+1}-cos\theta_{0,n,j+1})}$$

$$=\sum_{l=1}^{2^j}\frac{2(cos^2\theta_{l-1,k,j+1}+cos\theta_{0,n,j+1})}{(cos\theta_{l-1,k,j}-1)(cos\theta_{i-1,k,j}-cos\theta_{0,n,j})}$$

$$\le 8(cos^2\theta_{0,k,j+1}+cos\theta_{0,n,j+1})h(2^j,n,k). \tag{3.13}$$

We have used the facts that $cos^2\theta_{0,k,j+1}>cos^2\theta_{i-1,k,j+1}$, $i=2,\cdots,2^j$, and that the first term (i=1) of the summation in (3.13) is negative while all the others are positive to justify the inequality in (3.13). It then follows from (3.13) that $h(2^{j+1}, n, k) < 0$. We just complete the proof of the theorem. $\square$

Theorem 3.3. *Let n and j be the block and wavelet dimensions of $C(\alpha, 1)$, respectively. Assume n and j are any positive integers. Let $\lambda_2(\alpha)$ be the second eigencurve of $C(\alpha, 1)$. Then the following hold.*
(i) $\lambda_2(\alpha)$ is a nonincreasing function of α.
(ii) If n is an even number, then $\lambda_2(\alpha) = \lambda_{0,n}$ whenever $\alpha \geq \alpha^*$ for some $\alpha^* > 0$.
(iii) If n is an odd number, then $\lambda_2(\alpha) < \lambda_{0,n}$ whenever $\alpha \geq \overline{\alpha}$ for some $\overline{\alpha} > 0$.

Proof. We first remark that in the case of $\beta = 1$, the set of the indexes $k's$ in (3.1) is $\{0, 2, 4, ..., 2(n-1)\} := I_n$. Suppose n is an even number. Then $n \in I_n$. Thus, $\delta_n = -1$, $\theta_{0,n} = \frac{\pi}{m}$, and $\boldsymbol{p}_0(n) = \left(e^{i\frac{\pi}{m}}, e^{i\frac{2\pi}{m}}, \cdots, e^{i\pi}\right)^T$. Applying Proposition 3.1, we see that $\boldsymbol{p}_0(n) - \boldsymbol{p}_0^s(n)$, an antisymmetric vector, is also an eigenvector of $D_1(n)$. And so $e^T(\boldsymbol{p}_0(n) - \boldsymbol{p}_0^s(n)) = 0$. It then follows from Remark 3.1 that $\lambda_{0,n}$ is an eigenvalue of $\Gamma_n = D_1(n) - \rho(n)ee^T$ for all α. The first and second assertions of the theorem now follow from Theorems 3.1 and 3.2. Let n be an odd number. Then $\delta_i \cdot \delta_i \neq 1$ for any $i \in I_n$. Thus, if the pair $(\delta_i, \boldsymbol{v}_i)$ satisfy (2.3), then $\boldsymbol{v}_i \neq -\boldsymbol{v}_i^s$. Otherwise, the pair $(\delta_i, \boldsymbol{v}_i - (-\boldsymbol{v}_i)^s) = (\delta_i, \boldsymbol{v}_i + \boldsymbol{v}_i^s)$ also satisfy (2.3). This is a contradiction to the last assertion in Proposition 2.1. Thus, $\boldsymbol{v}_i^H \cdot e \neq 0$ for any $i \in I_n$. We then conclude, via Proposition 3.2-(iii) and Theorem 3.2, that the last assertion of the theorem holds. $\square$

Remark 3.2. (i)Let the number of uncoupled (chaotic) oscillators be $N = 2^j n$. If n is an odd number, then the wavelet method for controlling the coupling chaotic oscillators work even better in the sense that the critical coupling strength ϵ can be made even smaller. (ii)For n being a multiple of 4 and $j \in \mathbb{N}$, the assertions in Theorem 3.3 was first proved in [6] by a different method.

Theorem 3.4. *Let n and j be the block and wavelet dimensions of $C(\alpha, 0)$, respectively. Assume n and j are any positive integers. Let $\lambda_2(\alpha)$ be the second eigencurve of $C(\alpha, 0)$. Then for any n, there exists a $\widetilde{\alpha}$ such that $\lambda_2(\alpha) = \lambda_{0,n}$ whenever $\alpha \geq \widetilde{\alpha}$.*

Remark 3.3. For $n \in \mathbb{N}$ and $j = 1$, the explicit formulas for the eigenvalues of $C(\alpha, 0)$ was obtained in [4]. Such results are possible due to the fact that the dimension of the matrices in (2.4) is 2×2.

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Partial-State Coupling, Nonlinearities, and Synchronization

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Abstract: Partial-state coupling plays a surprising role in determining if a coupled system can achieve synchronization. For instance, synchronization for the single z_i-component coupled Lorenz equations can not occur, while synchronization for other partial-state couplings can be realized. The purpose of this paper is to address these coupling issues in a general framework. Some sharp conditions are given on the nonlinearities of the subsystem and on the coupling scheme to ensure the synchronization. We apply our theorems to three examples: coupled Lorenz-equations [10]; coupled chaotic works [12]; coupled Duffing oscillators [16]. The results on the latter two examples appear to be new.

keywords: Synchronization, partial-state coupling, bounded dissipative.

1. Introduction

Coupled dynamical systems are typically synthesized from simpler, low dimensional systems to form new and more complex systems for which their analysis and/ or control remains tractable. These and other motivations have led to numerous studies of coupled systems in a wide range of disciplines. For instance, it has been observed that coupling allows cells to synchronize to each other. Indeed, synchronization in coupled systems has been observed in many diverse areas, coupled mechanical and electrical systems [3, 9], laser systems [6, 7], biological systems [2, 11] and Josephson junctions [14]. Other than dissipation and the type of nonlinearities of chaotic subsystems, the coupling rule plays a very important role in any discussion of synchronization. Two types of coupling rules need to be specified. One is the coupling scheme between the subsystems. The other is the coupling rule within the subsystems. The rule of the latter plays a surprising role. In [9], the lattices of coupled Rössler-like equations with diffusive coupling between the subsystems and a single x_i, y_i or z_i- component's coupling were considered. It was numerical reported there that synchronization occurs for either x_i or y_i- component's coupling with strong enough mutual diffusive coupling. However, the lack of synchronization in the numerical z_i-coupling was also reported there. Similar results were also stated in [10] for the lattices of coupled Lorenz equations with the rigorous proof for either x_i or y_i-component's coupling. No explanations for why the lack of synchronization in the z_i-coupling for both cases or x_i-coupling for the first case were offered. The purpose of this paper is to address these coupling issues in a general framework. Specifically, we give rather general conditions on the nonlinearities of the subsystem and on the coupling schemes so that the coupled system can achieve synchronization. Some comparison principles on the coupling scheme are also derived. We note that our main results can be applied to many coupled systems. In particular, we apply our theorems to three examples here: coupled Lorenz equations [10]; coupled chaotic works [12]; coupled Duffing oscillators [16]. For the coupled Lorenz equation, we see that x_i and y_i component's couplings satisfy our sufficient conditions, while the z_i-component's coupling fails to satisfy our sufficient condition, which, in turn, illustrates the sharpness of our conditions and sheds some light on why the lack of synchronization. The results on the latter two examples appear to be new (see our Remarks 4.2-(i) and (ii)). We also note that different partial-state couplings are considered in [8]. As commented in [8], additional difficulties arise for the formulation considered here.

We organize the paper as follows. Section 2 is to lay down the foundation of our work. To do so, we introduce the notion of dual system of the coupled oscillator system and the notion of self-synchronization of the dual system. Some various notions of synchronization are also recorded there. The relationship between bounded dissipation and Lyapunov function is also explored in this section. Our main results are contained in Section 3. The concept of matrix measures, which find successful applications in nonlinear control system, is introduced to obtain the sufficient conditions on synchronization of the coupled oscillator systems. Some comparison principles for the coupling systems are also given there. Three examples mentioned earlier are given in Section 4 to illustrate the effectiveness of our main results.

2. Basic Framework

In this paper, we will denote scalar variables in lower case, matrices in bold type upper case, and vectors (or vector-valued functions) in bold type lower case. We consider an array of m cells, coupled linearly together, with each cell being an n- dimensional system. The entire array is a system of nm ordinary differential equations. In particular, the state equations are

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{f}(\mathbf{x}_i, t) + d \cdot \sum_{j=1}^{m} g_{ij}\, \mathbf{D}\mathbf{x}_j, \quad i = 1, 2, \ldots, m, \tag{2.1}$$

where $\mathbf{x}_i \in \mathbb{R}^n$, $\mathbf{f} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ and $\mathbf{D}$ is an $n \times n$ real matrix. Let

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_m \end{pmatrix}, \; \mathbf{x}_i = \begin{pmatrix} x_{i,1} \\ \vdots \\ x_{i,n} \end{pmatrix}, \text{ and } \mathbf{G} = (g_{ij})_{m\times m.} \tag{2.2}$$

Then (2.1) can be written as

$$\dot{\mathbf{x}} = \begin{pmatrix} \mathbf{f}(\mathbf{x}_1, t) \\ \vdots \\ \mathbf{f}(\mathbf{x}_m, t) \end{pmatrix} + d(\mathbf{G} \otimes \mathbf{D})\mathbf{x} =: \mathbf{F}(\mathbf{x}, t) + d(\mathbf{G} \otimes \mathbf{D})\mathbf{x}, \tag{2.3a}$$

where $\otimes$ is the Kronecker product, and

$$\mathbf{f}(\mathbf{x}_i, t) = \begin{pmatrix} f_1(\mathbf{x}_i, t) \\ \vdots \\ f_n(\mathbf{x}_i, t) \end{pmatrix}. \tag{2.3b}$$

From time to time, we will refer system (2.3) as the coupled system $(\mathbf{D}, \mathbf{G}, \mathbf{F}(\mathbf{x}, t))$. Suppose the state variables are permuted in the following way:

$$\tilde{\mathbf{x}}_i = \begin{pmatrix} x_{1,i} \\ \vdots \\ x_{m,i} \end{pmatrix}, \text{ and } \tilde{\mathbf{x}} = \begin{pmatrix} \tilde{\mathbf{x}}_1 \\ \vdots \\ \tilde{\mathbf{x}}_n \end{pmatrix}. \tag{2.4}$$

Then (2.3) can be written as

$$\dot{\tilde{\mathbf{x}}} = \begin{pmatrix} \tilde{\mathbf{f}}_1(\tilde{\mathbf{x}}, t) \\ \vdots \\ \tilde{\mathbf{f}}_n(\tilde{\mathbf{x}}, t) \end{pmatrix} + d(\mathbf{D} \otimes \mathbf{G})\tilde{\mathbf{x}} =: \tilde{\mathbf{F}}(\tilde{\mathbf{x}}, t) + d(\mathbf{D} \otimes \mathbf{G})\tilde{\mathbf{x}}, \tag{2.5a}$$

where

$$\tilde{\mathbf{f}}_i(\tilde{\mathbf{x}}, t) = \begin{pmatrix} f_i(\mathbf{x}_1, t) \\ \vdots \\ f_i(\mathbf{x}_m, t) \end{pmatrix}. \tag{2.5b}$$

Such reformulation is certainly not new (see e.g., [10, 15]). From here on, we will treat $\tilde{}$ as a function that take $\mathbf{x}$ into $\tilde{\mathbf{x}}$ or $\mathbf{x}_i$ into $\tilde{\mathbf{x}}_i$.

Definition 2.1. System (2.5) is called the dual system of (2.3).

We assume the system of ordinary differential equations under consideration has a unique solution for all time and for each initial condition. We write $\mathbf{x}(t, \mathbf{x}_0, t_0)$ for the unique solution at time t where $\mathbf{x}_0$ is the initial condition at time t_0. This will sometimes be simplified as $\mathbf{x}(t)$. Let $B_k(\alpha)$ be the ball in $\mathbb{R}^k$ with center at 0 and radius α. We define the system to be synchronized if the trajectories of all the cells approach each other. We define the system to be self-synchronized if the components $x_{i,k}$ of each subsystem $\mathbf{x}_i$ approach each other. Various notions of synchronization and self-synchronization are given in the following.

Definition 2.2. (see e.g., Definition 1 of [15]) Let a ball $B_n(\alpha)$ be given. System (2.3) is uniformly (resp., self-) synchronized if for each $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that if $\|\mathbf{x}_i(t_0) - \mathbf{x}_j(t_0)\| \le \delta(\epsilon)$ (resp., $|x_{i,k}(t_0) - x_{j,k}(t_0)| \le \delta(\epsilon)$), and

$\mathbf{x}_i(t_0)$ and $\mathbf{x}_j(t_0) \in B_n(\alpha)$ for all i, j (resp., i, j, k), then $\|\mathbf{x}_i(t) - \mathbf{x}_j(t)\| \leq \epsilon$ (resp., $|x_{i,k}(t) - x_{j,k}(t)| \leq \epsilon$) for all $t \geq t_0$ and for all i, j (resp., i, j, k).

Definition 2.3. (see e.g., Definition 2 of [15]) Let a ball $B_n(\alpha)$ be given. System (2.3) is uniformly asymptotically (resp., self-) synchronized if the followings hold:

(i) It is uniformly synchronized.
(ii) There exists a $\delta > 0$ such that for all $\epsilon > 0$ there exists a $t_\epsilon \geq 0$ such that if $\|\mathbf{x}_i(t_0) - \mathbf{x}_j(t_0)\| \leq \delta$ (resp., $|x_{i,k}(t_0) - x_{j,k}(t_0)| \leq \delta$), and $\mathbf{x}_i(t_0)$ and $\mathbf{x}_j(t_0) \in B_n(\alpha)$ for all i, j (resp., i, j, k) and $t \geq t_0 + t_\epsilon$, then $\|\mathbf{x}_i(t) - \mathbf{x}_j(t)\| \leq \epsilon$. (resp., $|x_{i,k}(t) - x_{j,k}(t)| \leq \epsilon$) for all i, j (resp., i, j, k).

Definition 2.4. Let a ball $B_n(\alpha)$ be given. System (2.3) is globally (resp., self-) synchronized if for all $\epsilon > 0$, there exists a $t_\epsilon \geq 0$ such that $\|\mathbf{x}_i(t) - \mathbf{x}_j(t)\| \leq \epsilon$ (resp., $|x_{i,k}(t) - x_{j,k}(t)| \leq \epsilon$) for all i, j (resp., i, j, k), all $\mathbf{x}_i(t_0)$ and $\mathbf{x}_j(t_0) \in B_n(\alpha)$, and all $t \geq t_0 + t_\epsilon$.

Proposition 2.1. *If a system is globally (resp., self-) synchronized, then it is uniformly asymptotically (resp., self-) synchronized.*

Proof. If a system is as assumed, then given $\epsilon > 0$, there exists a t' such that for all i, j and all $\mathbf{x}_i(t_0)$ and $\mathbf{x}_j(t_0) \in B_n(\alpha)$, we have $\|\mathbf{x}_i(t) - \mathbf{x}_j(t)\| \leq \epsilon$ for $t \geq t'$. Letting $t_0 = t'$ and $\delta = \epsilon$, we see immediately that the corresponding system is uniformly synchronized. Obviously, the assumption in Definition 2.3-(ii) can be fulfilled by choosing any $\delta > 0$. The other assertion in the proposition can be similarly proved. $\square$

Theorem 2.1. *System (2.3) is synchronized if and only if its dual system (2.5) is self-synchronized.*

We skip the proof of the theorem due to its triviality. We next give the definition of the bounded dissipation of a system.

Definition 2.5. A system of n ordinary differential equations is called bounded dissipative with respect to (α, β) provided that (i) for any initial conditions $\mathbf{x}_0$, there exists a time $t^* \geq t_0$ such that $\mathbf{x}(t^*) \in B_n(\alpha)$; (ii) $\mathbf{x}(t) \in B_n(\beta)$ for all $\mathbf{x}_0 \in B_n(\alpha)$ and all time $t \geq t_0$. If no confusion arise, we shall just say the system is bounded dissipative.

Proposition 2.2. *System (2.3) is bounded dissipative if and only if its dual system is bounded dissipative.*

Proof. It is clear since system (2.5) is derived from system (2.3) by some permutation. $\square$

To prove the bounded dissipation of the system, it often requires to construct an approximate Lyapunov function. The following proposition gives the type of Lyapunov functions that would ensure the bounded dissipation of the system.

Proposition 2.3. *Let a system of n ordinary differential equations be given. Let V be a continuous real-valued function $V : \mathbb{R}^n \to \mathbb{R}^+$ so that V is strictly decreasing along the solution of the system on $\mathbb{R}^n - \Gamma$, where Γ is homeomorphic to an open ball in $\mathbb{R}^n$. Suppose*

$$\lim_{\|\mathbf{x}\| \to \infty} V(\mathbf{x}) = \infty. \tag{2.6}$$

Then the system is bounded dissipative.

Proof. For any $\mathbf{x}_0 \in \mathbb{R}^n$, we first prove that $\mathbf{x}(t)$ must enter Γ at a certain time. Otherwise, the values of V at the points of the ω-limit set of $\mathbf{x}(t)$ must be the same, a contradiction. The contradiction comes from the facts that the ω-limit set is closed and invariant and V is strictly decreasing along the solution trajectory, which stays in $\mathbb{R}^n - \Gamma$. We then find a ball $B_n(\alpha)$ so that $B_n(\alpha) \supset \Gamma$. Let $k_1 = \max_{\mathbf{x} \in \bar{B}_n(\alpha)} V(\mathbf{x})$, and $B_n(\beta)$ be a ball satisfying $V(\mathbf{x}) > k_2$ whenever $\mathbf{x} \in \mathbb{R}^n - B_n(\beta)$, where $k_2 > k_1$. Then we conclude that if $\mathbf{x}_0 \in B_n(\alpha)$, $\mathbf{x}(t)$ stays in $B_n(\beta)$ for all time t. We just complete the proof of the proposition. □

3. Main Results

For completeness and ease of references, we begin with recalling the following definitions and the results (see e.g., [4, 13]).

Definition 3.1. Let $\| \cdot \|_i$ be an induced matrix norm on $\mathbb{C}^{n \times n}$. The matrix measure $\mu_i(\mathbf{A})$ of a matrix on $\mathbb{C}^{n \times n}$ is defined to be $\mu_i(\mathbf{A}) = \lim_{\epsilon \to 0^+} \dfrac{\|I + \epsilon \mathbf{A}\|_i - 1}{\epsilon}$.

Lemma 3.1. *Let $\|\cdot\|_k$ be an induced k-norm on $\mathbb{R}^{n \times n}$, where $k = 1, 2, \infty$. Then the matrix measure $\mu_k(\mathbf{A})$, $k = 1, 2, \infty$ of a matrix $\mathbf{A} = (a_{ij})$ on $\mathbb{R}^{n \times n}$ is, respectively,*

$$\mu_\infty(\mathbf{A}) = \max_i \{a_{ii} + \sum_{j \neq i} |a_{ij}|\}, \tag{3.1a}$$

$$\mu_1(\mathbf{A}) = \max_j \{a_{jj} + \sum_{i \neq j} |a_{ij}|\}, \tag{3.1b}$$

and

$$\mu_2(\mathbf{A}) = \lambda_{\max}(\mathbf{A}^H + \mathbf{A})/2. \tag{3.1c}$$

Here $\lambda_{\max}(\mathbf{A})$ is the maximum of the eigenvalues of $\mathbf{A}$.

Theorem 3.1. *(see e.g., 3.5.32 of* [13]*) Consider the differential equation $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{v}(t)$, $t \geq 0$, where $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{A}(t) \in \mathbb{R}^{n \times n}$, and $\mathbf{A}(t), \mathbf{v}(t)$ are piecewise-continuous. Let $\| \cdot \|_i$ be a norm on $\mathbb{R}^n$, and $\| \cdot \|_i$, μ_i denote, respectively, the corresponding induced norm and matrix measure on $\mathbb{R}^{n \times n}$. Then whenever $t \geq t_0 \geq 0$, we have*

$$\|\mathbf{x}(t_0)\| \exp\left\{\int_{t_0}^t -\mu_i(-\mathbf{A}(s))ds\right\} - \int_{t_0}^t \exp\left\{\int_s^t -\mu_i(-\mathbf{A}(\tau))d\tau\right\} \|\mathbf{v}(s)\|ds \leq \|\mathbf{x}(t)\|$$

$$\leq \|\mathbf{x}(t_0)\| \exp\left\{\int_{t_0}^t \mu_i(\mathbf{A}(s))ds\right\} + \int_{t_0}^t \exp\left\{\int_s^t \mu_i(\mathbf{A}(\tau))d\tau\right\} \|\mathbf{v}(s)\|ds. \tag{3.2}$$

We next impose conditions on coupling matrices $\mathbf{G}$ and $\mathbf{D}$. We assume that the coupling matrix $\mathbf{G}$ satisfies the following

$$\text{(i) all eigenvalues of } \mathbf{G} \text{ have nonpositive real parts.} \tag{3.3a}$$

$$\text{(ii) } \lambda = 0 \text{ is a simple eigenvalue of } \mathbf{G} \text{ and its corresponding eigenspace is } span(\mathbf{e}), \text{where } \mathbf{e} = [1, 1, \ldots, 1]^T_{1\times m}. \tag{3.3b}$$

We further assume that the matrix $\mathbf{D}$ is, without loss of generality, of the form

$$\mathbf{D} = \begin{pmatrix} \mathbf{I}_k & \mathbf{0} \\ \mathbf{0} & 0 \end{pmatrix}_{n\times n.} \tag{3.3c}$$

The index k, $1 \leq k \leq n$, means that the first k components of the subsystem are coupled. If $k \neq n$, then the system is said to be partial-state coupled. Otherwise, it is said to be full-state coupled.

To study the self-synchronization of (2.5), we first make a coordinate change. Let $\mathbf{A}$ be an $m \times m$ matrix of the form

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & -1 \\ 1 & \cdots & \cdots & 1 & 1 \end{pmatrix}_{m\times m} =: \begin{pmatrix} \mathbf{C} \\ \mathbf{e}^T \end{pmatrix}, \tag{3.4a}$$

where $\mathbf{e}$ is given as in (3.3b). It is then easy to see that $\mathbf{C}\mathbf{C}^T$ is invertible and that

$$\mathbf{A}^{-1} = \left(\mathbf{C}^T(\mathbf{C}\mathbf{C}^T)^{-1} | \frac{\mathbf{e}}{m}\right). \tag{3.4b}$$

Setting

$$\mathbf{E} = \mathbf{I}_n \otimes \mathbf{A}, \tag{3.4c}$$

we see that

$$\begin{aligned} \mathbf{E}(\mathbf{D} \otimes \mathbf{G})\mathbf{E}^{-1} &= (\mathbf{I}_n \otimes \mathbf{A})(\mathbf{D} \otimes \mathbf{G})(\mathbf{I}_n \otimes \mathbf{A}^{-1}) \\ &= \mathbf{D} \otimes \mathbf{A}\mathbf{G}\mathbf{A}^{-1} = \mathbf{D} \otimes \begin{pmatrix} \mathbf{C}\mathbf{G}\mathbf{C}^T(\mathbf{C}\mathbf{C}^T)^{-1} & \mathbf{0} \\ * & 0 \end{pmatrix} \\ &=: \mathbf{D} \otimes \begin{pmatrix} \bar{\mathbf{G}} & \mathbf{0} \\ * & 0 \end{pmatrix}. \end{aligned} \tag{3.4d}$$

We remark, via (3.4d), that $\sigma(\mathbf{G}) - \{0\} = \sigma(\bar{\mathbf{G}})$, where $\sigma(\mathbf{A})$ is the spectrum of the matrix $\mathbf{A}$. Multiplying $\mathbf{E}$ to the both side of equation (2.5a), we get

$$\begin{aligned}\dot{\tilde{\mathbf{y}}} =: \mathbf{E}\dot{\tilde{\mathbf{x}}} &= \mathbf{E}\tilde{\mathbf{F}}(\tilde{\mathbf{x}},t) + d\mathbf{E}(\mathbf{D}\otimes\mathbf{G})\mathbf{E}^{-1}\tilde{\mathbf{y}} \\ &= \mathbf{E}\tilde{\mathbf{F}}(\mathbf{E}^{-1}\tilde{\mathbf{y}},t) + d(\,\mathbf{D}\otimes\begin{pmatrix}\bar{\mathbf{G}} & \mathbf{0}\\ * & 0\end{pmatrix})\tilde{\mathbf{y}}.\end{aligned} \tag{3.5}$$

Let $\tilde{\mathbf{y}} = \begin{pmatrix}\tilde{\mathbf{y}}_1\\ \vdots\\ \tilde{\mathbf{y}}_n\end{pmatrix}$. Then $\tilde{\mathbf{y}}_i = \begin{pmatrix}x_{1,i}-x_{2,i}\\ \vdots\\ x_{m-1,i}-x_{m,i}\\ \sum_{j=1}^m x_{j,i}\end{pmatrix}$. Setting $\tilde{\mathbf{y}}_i = \begin{pmatrix}\bar{\mathbf{y}}_i\\ \sum_{j=1}^m x_{j,i}\end{pmatrix}$, and $\bar{\mathbf{y}} = \begin{pmatrix}\bar{\mathbf{y}}_1\\ \vdots\\ \bar{\mathbf{y}}_n\end{pmatrix}$, we have that the dynamics of $\bar{\mathbf{y}}$ is satisfied by the following equation

$$\dot{\bar{\mathbf{y}}} = d(\mathbf{D}\otimes\bar{\mathbf{G}})\bar{\mathbf{y}} + \bar{\mathbf{F}}(\bar{\mathbf{y}},t), \tag{3.6}$$

Here $\bar{\mathbf{F}}$ is obtained from $\mathbf{E}\tilde{\mathbf{F}}(\mathbf{E}^{-1}\tilde{\mathbf{y}},t)$ accordingly.

We next give conditions on the nonlinearities $\bar{\mathbf{F}}$. Let k be given as in (3.3c). Write $\bar{\mathbf{y}}$ and $\bar{\mathbf{F}}(\bar{\mathbf{y}},t)$ as

$$\bar{\mathbf{y}} = \begin{pmatrix}\bar{\mathbf{y}}_c\\ \bar{\mathbf{y}}_u\end{pmatrix}, \text{ and } \bar{\mathbf{F}}(\bar{\mathbf{y}},t) = \begin{pmatrix}\bar{\mathbf{F}}_c(\bar{\mathbf{y}},t)\\ \bar{\mathbf{F}}_u(\bar{\mathbf{y}},t)\end{pmatrix}, \text{ respectively.} \tag{3.7}$$

Here $\bar{\mathbf{y}}_c = \begin{pmatrix}\bar{\mathbf{y}}_1\\ \vdots\\ \bar{\mathbf{y}}_k\end{pmatrix}$, and $\bar{\mathbf{y}}_u = \begin{pmatrix}\bar{\mathbf{y}}_{k+1}\\ \vdots\\ \bar{\mathbf{y}}_n\end{pmatrix}$. We assume that $\bar{\mathbf{F}}_c(\bar{\mathbf{y}},t)$ satisfies a dual-Lipschitz condition with a dual-Lipschitz constant b. That is,

$$\|\bar{\mathbf{F}}_c(\bar{\mathbf{y}},t)\|_i \le b\|\bar{\mathbf{y}}\|_i \tag{3.8a}$$

whenever $\bar{\mathbf{y}}$ in $B_{(m-1)n}(\alpha)$, and for all time t. Suppose that $\bar{\mathbf{F}}_u(\bar{\mathbf{y}},t)$ can be written as

$$\bar{\mathbf{F}}_u(\bar{\mathbf{y}},t) = \mathbf{U}(t)\bar{\mathbf{y}}_u + \bar{\mathbf{R}}_u(\bar{\mathbf{y}},t). \tag{3.8b}$$

Here $\mathbf{U}(t)$ is a block diagonal matrix of the form $\mathbf{U}(t) = \mathrm{diag}(\mathbf{U}_1(t),\cdots,\mathbf{U}_l(t))$ where $\mathbf{U}_j(t)$, $j=1,\ldots,l$, are matrices of size $(m-1)k_j \times (m-1)k_j$. Here $\sum_{j=1}^{l} k_j = n-k$, and $k_j \in \mathbb{N}$. We assume further that the followings hold.

(i) The matrix measures $\mu_i(\mathbf{U}_j(t))$ are less than $-r$ for all t and all j, where $r > 0$. (3.8c)

(ii) Let $\bar{\mathbf{R}}_u(\bar{\mathbf{y}}, t) = \begin{pmatrix} \mathbf{R}_{u1}(\bar{\mathbf{y}}, t) \\ \vdots \\ \mathbf{R}_{ul}(\bar{\mathbf{y}}, t) \end{pmatrix}.$ Then $\mathbf{R}_{uj}(\bar{\mathbf{y}}, t)$, $j = 1, \ldots, l$, satisfy a strong dual-Lipschitz condition with a strong dual-Lipschitz constant b. Specifically, we assume that

$$\|\mathbf{R}_{uj}(\bar{\mathbf{y}}, t)\|_i \leq b \left\| \begin{pmatrix} \bar{\mathbf{y}}_c \\ \bar{\mathbf{y}}_{u1} \\ \vdots \\ \bar{\mathbf{y}}_{u\,j-1} \end{pmatrix} \right\|_i \tag{3.8d}$$

whenever $\bar{\mathbf{y}}$ in $B_{(m-1)n}(\alpha)$, and for all $j = 1, \ldots, l$ and all time t.

Theorem 3.2. *Let* $\mathbf{G}$ *and* $\mathbf{D}$ *be given as in (3.3). Assume that* $\bar{\mathbf{F}}$ *satisfies (3.8a-d), and system (2.5a) is bounded dissipative with respect to* $(\tau, \frac{\alpha}{2})$. *If* d *is sufficiently large, then* $\bar{\mathbf{y}}(t)$ *must eventually stay in* $B_{(m-1)n}(\alpha)$. *Moreover,* $\lim_{t\to\infty} \bar{\mathbf{y}}(t) = 0$.

Proof. In the following proof, we just consider the case of l_2-norm. The other norm can also be done in the similar way. Since system (2.5a) is bounded dissipative with respect to $(\tau, \frac{\alpha}{2})$, the first assertion of the theorem is obvious. Without loss of generality, we may assume that $\|\bar{\mathbf{y}}(t)\| \leq \alpha$ for all time $t \geq t_0$. Using (3.8b), we write (3.6) as

$$\begin{pmatrix} \dot{\bar{\mathbf{y}}}_c \\ \dot{\bar{\mathbf{y}}}_u \end{pmatrix} = \begin{pmatrix} d(\mathbf{I}_k \otimes \bar{\mathbf{G}}) & \mathbf{0} \\ \mathbf{0} & \mathbf{U}(t) \end{pmatrix} \begin{pmatrix} \bar{\mathbf{y}}_c \\ \bar{\mathbf{y}}_u \end{pmatrix} + \begin{pmatrix} \bar{\mathbf{F}}_c(\bar{\mathbf{y}}, t) \\ \bar{\mathbf{R}}_u(\bar{\mathbf{y}}, t) \end{pmatrix}. \tag{3.9}$$

Since $\mathbf{U}(t) = \mathrm{diag}(\mathbf{U}_1(t), \ldots, \mathbf{U}_l(t))$ is a block diagonal matrix, we will write $\bar{\mathbf{y}}_u$ as $\begin{pmatrix} \bar{\mathbf{y}}_{u1} \\ \vdots \\ \bar{\mathbf{y}}_{ul} \end{pmatrix}$ accordingly. Applying the variation of constant formula to (3.9) on $\bar{\mathbf{y}}_c$, we get

$$\bar{\mathbf{y}}_c(t) = e^{(t-t_0)d(\mathbf{I}_k \otimes \bar{\mathbf{G}})} \bar{\mathbf{y}}_c(t_0) + \int_{t_0}^{t} e^{(t-s)d(\mathbf{I}_k \otimes \bar{\mathbf{G}})} \bar{\mathbf{F}}_c(\bar{\mathbf{y}}(s), s) ds.$$

Let $\lambda_1 = \max\{\lambda_j | \lambda_j \in Re(\sigma(\bar{\mathbf{G}}))$, the set of the real parts of the spectrum of $\bar{\mathbf{G}}\}$. Then $\lambda_1 < 0$. Note that

$$\|e^{td(\mathbf{I}_k \otimes \bar{\mathbf{G}})}\| \leq c_1 e^{td\nu} \tag{3.10}$$

for some constant $c_1 > 0$, and $\nu = \frac{\lambda_1}{2}$. Thus,

$$\begin{aligned}\|\bar{\mathbf{y}}_c(t)\| &\le c_1 e^{(t-t_0)d\nu}\|\bar{\mathbf{y}}_c(t_0)\| + c_1 b\int_{t_0}^{t} e^{d(t-s)\nu}\|\bar{\mathbf{y}}(s)\|ds \\ &\le c_1 e^{(t-t_0)d\nu}\alpha + \frac{c_2}{2}\alpha\end{aligned}$$

for some constant $c_2 = \frac{2c_1 b}{d\,|\nu|}$. Thus

$$\|\bar{\mathbf{y}}_c(t)\| \le c_2\alpha, \tag{3.11a}$$

whenever $t \ge t_{0,1}$ for some $t_{0,1} > 0$. We then apply Theorem 3.1 on $\bar{\mathbf{y}}_{u1}$ and the resulting inequality is

$$\begin{aligned}\|\bar{\mathbf{y}}_{u1}(t)\| &\le \|\bar{\mathbf{y}}_{u1}(t_{0,1})\| \exp\left\{\int_{t_{0,1}}^{t} \mu_i(\mathbf{U}_1(s))ds\right\} \\ &+ \int_{t_{0,1}}^{t} \exp\left\{\int_{s}^{t} \mu_i(\mathbf{U}_1(\tau))d\tau\right\} \|\mathbf{R}_{u1}(\bar{\mathbf{y}}(s), s)\|ds.\end{aligned}$$

It then follows from (3.8c-d) and (3.11a) that

$$\|\bar{\mathbf{y}}_{u1}(t)\| \le \alpha e^{-r(t-t_{0,1})} + \frac{bc_2\alpha}{r},$$

whenever $t \ge t_{0,1}$. Moreover, letting $\omega = \max\{1, \frac{2b}{r}\}$, we have

$$\|\bar{\mathbf{y}}_{u1}(t)\| \le \omega c_2\alpha \tag{3.11b}$$

whenever $t \ge t_{1,1}$ for some $t_{1,1} \ge t_{0,1}$. Inductively, we get

$$\|\bar{\mathbf{y}}_{uj}(t)\| \le \sqrt{2^{j-1}}\omega^j c_2\alpha, \quad j = 2, \dots, l, \tag{3.11c}$$

whenever $t \ge t_{j,1} (\ge t_{j-1,1})$. Letting $t_{l,1} = t_1$ and summing up (3.11a), (3.11b) and (3.11c), we get

$$\|\bar{\mathbf{y}}(t)\| = \sqrt{\sum_{j=1}^{l} \|\bar{\mathbf{y}}_{uj}(t)\|^2 + \|\bar{\mathbf{y}}_c(t)\|^2} \le (\sqrt{1 + \sum_{j=1}^{l} 2^{j-1}\omega^{2j}}\ c_2)\alpha =: h\alpha,$$

whenever $t \ge t_1$. Choosing $d > \frac{2c_1 b}{|\nu|}\sqrt{1 + \sum_{j=1}^{l} 2^{j-1}\omega^{2j}}$, we see that the contraction factor h is strictly less than 1, and $\|\bar{\mathbf{y}}(t)\|$ contracts as time progresses. Moreover, t_1 is independent of the initial conditions $\bar{\mathbf{y}}(t_0)$. We have completed the proof of the theorem. $\square$

Corollary 3.1. *Suppose $\bar{\mathbf{F}}$ and $\mathbf{G}$ are assumed as in Theorem 3.2. Assume, in addition, that all eigenvalues of $\mathbf{G}$ are nonpositive. Let*

$$\mathbf{D} = \begin{pmatrix} \bar{\mathbf{D}}_{k\times k} & \mathbf{0} \\ \mathbf{0} & 0 \end{pmatrix}_{n\times n,} =: \mathbf{D}_{new} \text{ where } Re\,\sigma(\bar{\mathbf{D}}) > 0. \tag{3.12}$$

If the corresponding coupled system $(\mathbf{D}_{new}, \mathbf{G}, \mathbf{F}(\mathbf{x}, t))$ is also bounded dissipative, then assertions in Theorem 3.2 still hold true.

Proof. Assumption (3.12) is to ensure that (3.10) is still valid. Other parts of the proof are similar to those in Theorem 3.2 and are thus omitted. □

We next turn our attention to finding conditions on the nonlinearities $f_i(\mathbf{u}, t)$, $i = 1, \ldots, n$, $\mathbf{u} \in \mathbb{R}^n$, so that assumptions (3.8a-d) are satisfied. To this end, we need the following notations. Let $\tilde{\mathbf{x}}_i$ and $\tilde{\mathbf{x}}$ be given as in (2.4). Define

$$[\tilde{\mathbf{x}}_i]^- = \begin{pmatrix} x_{1,i} \\ \vdots \\ x_{m-1,i} \end{pmatrix}_, \text{ and } [\tilde{\mathbf{x}}]^- = \begin{pmatrix} [\tilde{\mathbf{x}}_1]^- \\ \vdots \\ [\tilde{\mathbf{x}}_n]^- \end{pmatrix}_. \tag{3.13}$$

We then break $\tilde{\mathbf{F}}$ as given in (2.5a) into two parts so that the breaking is in consistent with $\bar{\mathbf{y}}$ in (3.7). Specifically, we shall write

$$\tilde{\mathbf{F}}(\tilde{\mathbf{x}}, t) = \begin{pmatrix} \tilde{\mathbf{F}}_c(\tilde{\mathbf{x}}, t) \\ \tilde{\mathbf{F}}_u(\tilde{\mathbf{x}}, t) \end{pmatrix}_. \tag{3.14}$$

Proposition 3.1. *Suppose that $f_i(\mathbf{x}, t)$, $i = 1, 2, \ldots, k$ satisfy a Lipschitz condition on $B_n(\frac{\alpha}{2})$ with a Lipschitz constant b. That is*

$$|f_i(\mathbf{u}, t) - f_i(\mathbf{v}, t)| \leq b\|\mathbf{u} - \mathbf{v}\|, i = 1, 2, \ldots, k, \tag{3.15}$$

for all $\mathbf{u}$, $\mathbf{v}$ in $B_n(\frac{\alpha}{2})$ and all time t. Then (3.8a) holds true.

Proof. Note that $\mathbf{E}\tilde{\mathbf{F}}(\tilde{\mathbf{x}}, t) = \begin{pmatrix} \mathbf{A}\tilde{\mathbf{f}}_1(\tilde{\mathbf{x}}, t) \\ \vdots \\ \mathbf{A}\tilde{\mathbf{f}}_n(\tilde{\mathbf{x}}, t) \end{pmatrix}_,$ so that

$$[\mathbf{A}\tilde{\mathbf{f}}_i(\tilde{\mathbf{x}}, t)]^- = \begin{pmatrix} f_i(\mathbf{x}_1, t) - f_i(\mathbf{x}_2, t) \\ \vdots \\ f_i(\mathbf{x}_{m-1}, t) - f_i(\mathbf{x}_m, t) \end{pmatrix}_, \quad i = 1, 2, \ldots, n.$$

Since

$$\bar{\mathbf{F}}_c(\bar{\mathbf{y}},t) = \left(\begin{array}{c} [\mathbf{A}\tilde{\mathbf{f}}_1(\tilde{\mathbf{x}},t)]^- \\ \vdots \\ [\mathbf{A}\tilde{\mathbf{f}}_k(\tilde{\mathbf{x}},t)]^- \end{array} \right),$$

we conclude that (3.8a) holds. □

The following proposition is very useful in the sense that by checking how each component f_i of the nonlinearity $\mathbf{f}$ is formed, one would then be able to conclude whether (3.8c-d) are satisfied.

Proposition 3.2. *Let $\mathbf{u} = (u_1, \ldots, u_n)^T$ and $\mathbf{v} = (v_1, \ldots, v_n)^T$ be vectors in $B_n(\frac{\alpha}{2})$. Let $w_p = \sum_{i=0}^{p} k_i$, $p = 1, \ldots, l$, where $k_0 = k$, and $k_1, \ldots, k_l$, l are given as in (3.8c). Assume that for $i = w_{p-1}+1, \ldots, w_p$,*

$$f_i(\mathbf{u},t) - f_i(\mathbf{v},t) = \sum_{j=w_{p-1}+1}^{w_p} v_{i,j}(\mathbf{u},\mathbf{v},t)(u_j - v_j) + r_i(\mathbf{u},\mathbf{v},t). \tag{3.16a}$$

We further assume that the followings are true.

(i) *For $p = 1, \ldots, l$, let $\mathbf{V}_p = (v_{i,j}(\mathbf{u},\mathbf{v},t))$, where $w_{p-1}+1 \le i,j \le w_p$. Then $\mu_*(\mathbf{V}_p) < -r$ for all $\mathbf{u}$, $\mathbf{v}$, t and p, where $* = 1, 2, \infty$.* *(3.16b)*

(ii) *Let $\mathbf{r}_p = \left(r_{w_{p-1}+1}(\mathbf{u},\mathbf{v},t), \ldots, r_{w_p}(\mathbf{u},\mathbf{v},t)\right)^T$. We have that*

$$\|\mathbf{r}_p\| \le b \left\| \left(\begin{array}{c} u_1 - v_1 \\ \vdots \\ u_{w_{p-1}} - v_{w_{p-1}} \end{array} \right) \right\| \tag{3.16c}$$

for all $\mathbf{u}, \mathbf{v}, t, p$ and some constant b.

Then (3.8c) and (3.8d) hold true for $* = 1, 2, \infty$.

Proof. Let $\mathbf{u}_w = (\mathbf{x}_w, \mathbf{x}_{w+1}, t)$, $w = 1, \ldots, m-1$, and let $\mathbf{U}_{i,j,p}(t)$ be diagonal matrices of the form

$$\mathbf{U}_{i,j,p}(t) = \left(\begin{array}{ccc} v_{w_{p-1}+i,w_{p-1}+j}(\mathbf{u}_1) & & 0 \\ & \ddots & \\ 0 & & v_{w_{p-1}+i,w_{p-1}+j}(\mathbf{u}_{m-1}) \end{array} \right), \tag{3.17a}$$

where $p = 1, \ldots, l$. It then follows from (3.17a) that $\bar{\mathbf{F}}_u(\bar{\mathbf{y}},t)$ can be written as the form in (3.8b). In particular, $\mathbf{U}_p(t)$, $p = 1, 2, \ldots, l$, can be chosen as

$$\mathbf{U}_p(t) = \left(\mathbf{U}_{i,j,p}(t)\right), \quad w_{p-1}+1 \le i,j \le w_p. \tag{3.17b}$$

For fixed p and $w = 1, \ldots, m-1$, we define the matrices $\mathbf{M}_p^w(t)$ as follows

$$\mathbf{M}_p^w(t) = \left(v_{w_{p-1}+i,w_{p-1}+j}(\mathbf{u}_w)\right)_{k_p \times k_p}, \quad 1 \le i,j \le k_p. \tag{3.18a}$$

By assumption, we see that $\mu_*(\mathbf{M}_p^w(t)) < -r$. Now,

$$\mathbf{U}_p(t) = \sum_{w=1}^{m-1} \mathbf{M}_p^w(t) \otimes \mathbf{D}_w, \tag{3.18b}$$

where

$$(\mathbf{D}_w)_{ij} = \begin{cases} 1 & i = j = w, \\ 0 & otherwise, \end{cases} \quad 1 \leq i, j \leq m-1.$$

It then follows from (3.1a,b), (3.17) and (3.18) that $\mu_*(\mathbf{U}_p(t)) < -r$ for $* = 1$ or ∞. For $* = 2$, we have that

$$\begin{aligned} \bigcup_{w=1}^{m-1} \sigma\{\mathbf{M}_p^w(t) + \left(\mathbf{M}_p^w(t)\right)^T\} &= \sigma \left\{ \sum_{w=1}^{m-1} \left(\mathbf{M}_p^w(t) \otimes \mathbf{D}_w + \left(\mathbf{M}_p^w(t)\right)^T \otimes \mathbf{D}_w \right) \right\} \\ &= \sigma \left(\mathbf{U}_p(t) + \mathbf{U}_p^T(t) \right), \end{aligned}$$

where $\sigma(\mathbf{A})$ is the set of eigenvalues of $\mathbf{A}$. We remark that the first equality above can be verified by the definition of eigenvalues due to the structure of $\mathbf{U}_p(t)$. It then follows from (3.1c) that $\mu_2(\mathbf{U}_p(t)) < -r$. The remainder of the proof is similar as Proposition 3.1, and is thus omitted. □

We are now ready to state the main theorem of the paper.

Theorem 3.3. *Assume that (2.3) is bounded dissipative. Let the coupling matrices* $\mathbf{G}$ *and* $\mathbf{D}$ *satisfy (3.3) and the nonlinearities* $f_i(\mathbf{x}, t)$, $i = 1, 2, \ldots, n$, *satisfy (3.15) and (3.16). Suppose* d *is chosen sufficiently large. Then (2.3) and (2.5) are, respectively, globally synchronized and self-synchronized.*

Proof. The proof is direct consequences of Propositions 3.1 and 3.2, and Theorem 3.2. □

Remark 3.1. (i) From here on, we will refer the assumptions in Theorem 3.3 as synchronization hypotheses. (ii) An example in Section 4 (i)-c will illustrate the sharpness of our sufficient conditions here.

Corollary 3.2. *(Comparison Principle for synchronization) Let* $(\mathbf{D}, \mathbf{G}, \mathbf{F}(\mathbf{x}, t))$ *satisfies synchronization hypotheses. Furthermore, we assume that all eigenvalues of* $\mathbf{G}$ *are nonpositive. Assume* $\bar{\mathbf{D}}$ *satisfies (3.12). Then the coupled system* $(\mathbf{D}_{new}, \mathbf{G}, \mathbf{F}(\mathbf{x}, t))$ *is also globally synchronized.*

4. Applications

To see the effectiveness of our main results, we consider three examples in this section. These are coupled Lorenz equations [10], coupled chaotic works [12], and coupled Duffing oscillators [16].

(i) We shall begin with Lorenz equations. Let $\mathbf{x} = (x_1, x_2, x_3)^T$,

$$\begin{aligned}\mathbf{f}(\mathbf{x}, t) = \mathbf{f}(\mathbf{x}) &= (\sigma(x_2 - x_1),\, rx_1 - x_2 - x_1x_3,\, -bx_3 + x_1x_2)^T \\ &=: (f_1(\mathbf{x}, t),\ f_2(\mathbf{x}, t),\ f_3(\mathbf{x}, t))^T. \end{aligned} \tag{4.1}$$

For $\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} =: \mathbf{D}_1$ or $\mathbf{D} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} =: \mathbf{D}_2$ or $\mathbf{D} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} =:$ $\mathbf{D}_3$, it was shown (see e.g., [10]) that system (2.3) is bounded dissipative. We remark that in our formulation, we may rearrange the components of the nonlinearity $\mathbf{f}$ so that $\mathbf{D}$ always has the form $\mathbf{D} = \mathbf{D}_1$. Instead, in this section, we will fix the representation of the nonlinearity $\mathbf{f}$ just as given in (4.1).

Three types of coupling within the subsystems are considered in the following.

(a) $\mathbf{D} = \mathbf{D}_1$.
In this case $k = 1$, and

$$|f_1(\mathbf{x}, t) - f_1(\mathbf{z}, t)| = \sigma|(x_2 - z_2) - (x_1 - z_1)| \le b\|\mathbf{x} - \mathbf{z}\|$$

for some constant b. Hence, condition (3.15) is satisfied. Choosing $l = 1$, we have that $f_2(\mathbf{x}, t) = -x_2 - x_1x_3 + rx_1,\ f_3(\mathbf{x}, t) = x_1x_2 - bx_3$,

$$\begin{aligned} f_2(\mathbf{x}, t) - f_2(\mathbf{z}, t) &= (-x_2 - x_1x_3 + rx_1) - (-z_2 - z_1z_3 + rz_1) \\ &= [-(x_2 - z_2) - x_1(x_3 - z_3)] + (r - z_3)(x_1 - z_1) \end{aligned} \tag{4.2a}$$

and

$$\begin{aligned} f_3(\mathbf{x}, t) - f_3(\mathbf{z}, t) &= (x_1x_2 - bx_3) - (z_1z_2 - bz_3) \\ &= [x_1(x_2 - z_2) - b(x_3 - z_3)] + z_2(x_1 - z_1). \end{aligned} \tag{4.2b}$$

Writing (4.2a,b) in the vector form, we get

$$\begin{aligned} \begin{pmatrix} f_2(\mathbf{x}, t) - f_2(\mathbf{z}, t) \\ f_3(\mathbf{x}, t) - f_3(\mathbf{z}, t) \end{pmatrix} &= \begin{pmatrix} -1 & -x_1(t) \\ x_1(t) & -b \end{pmatrix} \begin{pmatrix} x_2 - z_2 \\ x_3 - z_3 \end{pmatrix} + \begin{pmatrix} (r - z_3)(x_1 - z_1) \\ z_2(x_1 - z_1) \end{pmatrix} \\ &=: \mathbf{V}_1(t) \begin{pmatrix} x_2 - z_2 \\ x_3 - z_3 \end{pmatrix} + \mathbf{r}_1(t). \end{aligned} \tag{4.2c}$$

Clearly, $\mu_2(\mathbf{V}_1(t)) = \max\{-1, -b\} = -1 < 0$, and $\|\mathbf{r}_1(t)\| \le b \cdot |x_1 - z_1|$ for some constant b. Note that such b exists since system (2.3) is bounded dissipative. Then it follows from Theorem 3.3 that the coupled system $(\mathbf{D}_1, \mathbf{G}, \mathbf{F}(x))$ is globally synchronized.

(b) $\mathbf{D} = \mathbf{D}_2$.
In this case $k = 1$, and f_2 is of course Lipschitz in any bounded domain. Choosing $l = 2$, we have that $f_1(\mathbf{x}, t) = -\sigma x_1 + \sigma x_2,$, $f_3(\mathbf{x}, t) = -bx_3 + x_1 x_2$,

$$f_1(\mathbf{x}, t) - f_1(\mathbf{z}, t) = [-\sigma(x_1 - z_1)] + \sigma(x_2 - z_2)$$

and

$$f_3(\mathbf{x}, t) - f_3(\mathbf{z}, t) = [-b(x_3 - z_3)] + x_1(x_2 - z_2) + z_2(x_1 - z_1).$$

Letting $\mathbf{V}_1(t) = (-\sigma)$, $\mathbf{V}_2(t) = (-b)$, $\mathbf{r}_1(t) = (\sigma(x_2 - z_2))$, and $\mathbf{r}_2(t) = (x_1(x_2 - z_2) + z_2(x_1 - z_1))$. We see that $\mu_2(\mathbf{V}_1(t)) = -\sigma < 0$, $\mu_2(\mathbf{V}_2(t)) = -b < 0$, $|\mathbf{r}_1(t)| \le b|x_2 - z_2|$ and $|\mathbf{r}_2(t)| \le b \| \begin{pmatrix} x_2 - z_2 \\ x_1 - z_1 \end{pmatrix} \|$ for some constant b. Thus, the coupled system $(\mathbf{D}_2, \mathbf{G}, \mathbf{F}(\mathbf{x}))$ is globally synchronized.

(c) $\mathbf{D} = \mathbf{D}_3$.
In this case $k = 1$. Moreover, f_1 and f_2 contains the term x_2 and x_1, respectively, the only feasible way to break the uncoupled components is to pick $l = 1$. Otherwise, (3.16c) is violated. For $l = 1$, we have that

$$\mathbf{V}_1(t) = \begin{pmatrix} -\sigma & \sigma \\ r - x_3(t) & -1 \end{pmatrix}.$$

For such $\mathbf{V}_1(t)$, we see that $\mu_i(\mathbf{V}_1(t))$ is not negative for all time t. Here $i = 1, 2, \infty$. As indicated in [10], the numerical results show that for such partial coupling the synchronization fails. All in all, these suggest that our sufficient conditions for the synchronization are quite sharp.

(d) For other $\mathbf{D}$ satisfying (3.3c), it is easy to show that the synchronization can be achieved.

We summarize our results above as follows.

Theorem 4.1. *Let $\mathbf{f}(\mathbf{x})$ be given as in (4.1) and $\mathbf{G}$ be a symmetry matrix satisfying (3.3a, 3.3b). Let $\mathbf{D} = \mathbf{D}_{new}$ be a symmetric matrix satisfying (3.12). Then the coupled system $(\mathbf{D}, \mathbf{G}, \mathbf{F}(\mathbf{x}))$ is globally synchronized provided that d is chosen sufficiently large.*

Remark 4.1. It is a nontrivial task to show the bounded dissipations of the coupled system whenever $\mathbf{D}$ and $\mathbf{G}$ are not symmetric.

`(ii)` For the second example, we consider the subsystem (see e.g., [12]) of chaotic walks. That is,

$$\dot{\mathbf{x}}_1 = \mathbf{f}(\mathbf{x}_1) = (f_1(\mathbf{x}_1), \ldots, f_n(\mathbf{x}_1))^T, \tag{4.3a}$$

where

$$f_i(\mathbf{x}_1) = \sin(x_{1,k}) - bx_{1,i},\ \ i = 1, 2, \ldots, n, \text{ and}$$
$$k = (i \mod n) + 1. \tag{4.3b}$$

Note that in [12], it was demonstrated numerically that subsystems (4.3a) exhibits hyperchaos. We next show that the coupled system (2.3) with the nonlinearities given as in (4.3b) is bounded dissipative provided that $\mathbf{G}$ is a negative semidefinite matrix, and $\mathbf{D}$ is given as in (3.3c). To this end, we introduce a Lyapunov function of the form

$$V(\mathbf{x}) = \sum_{j=1}^{m}\sum_{i=1}^{n} \frac{x_{j,i}^2}{2}.$$

By taking the time derivative of V along solutions of (2.3), one obtains

$$\begin{aligned}\frac{dV}{dt} &= \sum_{j=1}^{m}\sum_{i=1}^{n} x_{j,i}(\sin(x_{j,k}) - bx_{j,i}) + d\sum_{j=1}^{k} < \mathbf{x}_j, \mathbf{G}\mathbf{x}_j > \\ &\leq \sum_{j=1}^{m}\sum_{i=1}^{n} -bx_{j,i}^2 + |x_{j,i}| =: b_{m,n}.\end{aligned}$$

Suppose $\sum_{j=1}^{m}\sum_{i=1}^{n} x_{j,i}^2 \geq mnc_0^2$, where $c_0 > 0$ satisfying

$$-bc_0^2 + c_0 < -\frac{1}{2b}(mn-1). \tag{4.4}$$

Then, we may assume, without loss of generality, that $|x_{1,1}| \geq c_0$. Now,

$$\begin{aligned}b_{m,n} &= -bx_{1,1}^2 + |x_{1,1}| + \left[\left(\sum_{j=1}^{m}\sum_{i=1}^{n} -bx_{j,i}^2 + |x_{j,i}|\right) + bx_{1,1}^2 - |x_{1,1}|\right] \\ &< -\frac{1}{2b}(mn-1) + \frac{1}{4b}(mn-1) = -\frac{1}{4b}(mn-1) < 0.\end{aligned}$$

We have used (4.4) and the fact that $\max(-bx^2 + |x|) = \frac{1}{4b}$ to justify the above inequality. It then follows from Proposition 2.3 that the coupled chaotic walk is bounded dissipative as claimed. By Corollary 3.2 and noting that the permutation symmetry of equation (4.2), we only consider the case that the matrix $\mathbf{D}$ satisfying (3.3d) with $k = 1$. Letting $l = n - k = n - 1$, we see that $\mathbf{V}_p = -b$, $p = 1, 2, \ldots, l$. Thus, their matrix measure $\mu_i(\mathbf{V}_p) = -b < 0$. Moreover, the corresponding remaining terms $\mathbf{r}_i(\mathbf{x}, \mathbf{y}, t)$ satisfy (3.16c). Thus, system (2.3) is globally synchronized. In summary, we have our results in the following.

Theorem 4.2. *Let $\mathbf{f}(\mathbf{x})$ be given as in (4.2) and $\mathbf{G}$ be a symmetry matrix satisfying (3.3a, 3.3b). Let $\mathbf{D}$ be a matrix satisfying (3.12). Then the coupled system $(\mathbf{D}, \mathbf{G}, \mathbf{F}(\mathbf{x}))$ is globally synchronized provided that d is chosen sufficiently large.*

Proof. To complete the proof of the theorem, it suffices to show that the coupled system (2.5) is bounded dissipative. Writing the first k components of the coupled system, we get

$$\dot{\mathbf{z}}_k := \begin{pmatrix} \dot{\tilde{\mathbf{x}}}_1 \\ \vdots \\ \tilde{\mathbf{x}}_k \end{pmatrix} = \begin{pmatrix} -b\tilde{\mathbf{x}}_1 + \tilde{\mathbf{g}}_1(\tilde{\mathbf{x}}, t) \\ \vdots \\ -b\tilde{\mathbf{x}}_k + \tilde{\mathbf{g}}_k(\tilde{\mathbf{x}}, t) \end{pmatrix} + d(\bar{\mathbf{D}} \otimes \mathbf{G}) \begin{pmatrix} \tilde{\mathbf{x}}_1 \\ \vdots \\ \tilde{\mathbf{x}}_k \end{pmatrix}, \tag{4.5}$$

where the components of $\tilde{\mathbf{g}}_i(\tilde{\mathbf{x}}, t)$ have the form of $\sin(*)$. Applying the variation of constant formula to (4.5), we see that

$$\mathbf{z}_k(t) = e^{(-b\mathbf{I}+d\bar{\mathbf{D}}\otimes\mathbf{G})t}\mathbf{z}_k(0) + \int_0^t e^{(-b\mathbf{I}+d\bar{\mathbf{D}}\otimes\mathbf{G})(t-s)}\bar{\mathbf{G}}(\mathbf{x}, s)ds,$$

where $\bar{\mathbf{G}}(\mathbf{x}, t) = \begin{pmatrix} \tilde{\mathbf{g}}_1(\tilde{\mathbf{x}}, t) \\ \vdots \\ \tilde{\mathbf{g}}_k(\tilde{\mathbf{x}}, t) \end{pmatrix}.$ Now,

$$\begin{aligned} \|\mathbf{z}_k(t)\| &\leq c_0 e^{-\frac{b}{2}t}\|\mathbf{z}_k(0)\| + c_0\sqrt{mk}\int_0^t e^{-\frac{b}{2}(t-s)}ds \\ &\leq c_0 e^{-\frac{b}{2}t}\|\mathbf{z}_k(0)\| + \alpha, \end{aligned}$$

for some constant $c_0 > 0$ and $\alpha = 2\frac{c_0}{b}\sqrt{mk}$. Similarly, we have $\|\tilde{\mathbf{x}}_{k+i}(t)\| \leq c_0 e^{-\frac{b}{2}t}\|\tilde{\mathbf{x}}_{k+i}(0)\| + \alpha$ for all $i = 1, \ldots, n-k$. Hence,

$$\|\tilde{\mathbf{x}}(t)\| \leq ce^{-\frac{b}{2}t}\|\tilde{\mathbf{x}}(0)\| + n\alpha$$

for some constant c. Thus, system (2.5) is bounded dissipative with respect to $((n+1)\alpha, ((c+1)n + c)\alpha)$. □

(iii) Finally, we explore the example in [16]. Specifically, the subsystem considered is the Duffing oscillation defined by

$$\dot{x}_i = y_i \tag{4.6a}$$

$$\dot{y}_i = -\alpha y_i - x_i^3 + a\cos wt, \tag{4.6b}$$

where α and a are positive constants. Letting $\mathbf{x}_i = (x_{i1}, x_{i2})^T =: (x_i, y_i)^T$, we have

$$\mathbf{f}(\mathbf{x}_i, t) = (f_1(\mathbf{x}_i, t), f_2(\mathbf{x}_i, t)) = (y_i, -\alpha y_i - x_i^3 + a\cos wt). \tag{4.7a}$$

Assume the coupling matrices $\mathbf{D}$ and $\mathbf{G}$ are, respectively,

$$\mathbf{D}(c) = \begin{pmatrix} 0 & 0 \\ c & 1 \end{pmatrix} \tag{4.7b}$$

and

$$\mathbf{G}(\epsilon, r) = \begin{pmatrix} -2\epsilon & \epsilon - r & 0 & \cdots & 0 & \epsilon + r \\ \epsilon + r & -2\epsilon & \epsilon - r & \ddots & & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & -2\epsilon & \epsilon - r \\ \epsilon - r & 0 & \cdots & 0 & \epsilon + r & -2\epsilon \end{pmatrix}, \tag{4.7c}$$

where $\epsilon > 0$ and r are scalar diffusive and gradient coupling parameters, respectively. Setting $\mathbf{x} = (x_1, \ldots, x_m)^T$, $\mathbf{y} = (y_1, \ldots, y_m)^T$, $\mathbf{x}^3 = (x_1^3, \ldots, x_m^3)^T$, and $\mathbf{g}(t) = a\cos(wt)\,(1, \cdots, 1)^T$. We see that (2.5) becomes

$$\dot{\mathbf{x}} = \mathbf{y} \tag{4.8a}$$

$$\dot{\mathbf{y}} = -\alpha\mathbf{y} - \mathbf{x}^3 + \mathbf{g}(t) + dc\mathbf{G}(\epsilon, r)\mathbf{x} + d\mathbf{G}(\epsilon, r)\mathbf{y}. \tag{4.8b}$$

We first study equation (4.8) with $r = 0$. In order to construct an approximate Lyapunov function of the coupled system $(\mathbf{D}(c), \mathbf{G}(\epsilon, 0), \mathbf{F}(\mathbf{x}, \mathbf{y}, t))$, we first find a matrix $\mathbf{L}$ so that

$$\mathbf{L} + \mathbf{L}^T = \mathbf{G}(\epsilon, 0). \tag{4.9}$$

Now, consider the following scalar-valued function

$$U(\mathbf{x}, \mathbf{y}) = \frac{1}{2} < \mathbf{y}, \mathbf{y} > + \sum_{i=1}^{m} \frac{x_i^4}{4} + b < \mathbf{x}, \mathbf{L}\mathbf{x} > + k < \mathbf{x}, \mathbf{y} >, \tag{4.10}$$

where b and k are constants to be determined later. Taking the time derivative of U along solutions of the coupled system $(\mathbf{D}(c), \mathbf{G}(\epsilon, 0), \mathbf{F}(\mathbf{x}, \mathbf{y}, t))$, we have

$$\begin{aligned} \frac{dU}{dt} &= < \mathbf{y}, \dot{\mathbf{y}} > + \sum_{i=1}^{m} x_i^3 y_i + b < \mathbf{y}, \mathbf{G}(\epsilon, 0)\mathbf{x} > + k < \mathbf{y}, \mathbf{y} > + k < \mathbf{x}, \dot{\mathbf{y}} > \\ &= (k - \alpha) < \mathbf{y}, \mathbf{y} > - k\alpha < \mathbf{x}, \mathbf{y} > - k < \mathbf{x}, \mathbf{x}^3 > + < \mathbf{y} + k\mathbf{x}, \mathbf{g}(t) > \\ &\quad + d < \mathbf{y}, \mathbf{G}(\epsilon, 0)\mathbf{y} > + (dc + kd + b) < \mathbf{y}, \mathbf{G}(\epsilon, 0)\mathbf{x} > + kdc < \mathbf{x}, \mathbf{G}(\epsilon, 0)\mathbf{x} >. \end{aligned}$$

Note that

$$< \mathbf{x}, \mathbf{x}^3 >= \sum_{i=1}^{m} x_i^4 \geq \frac{1}{m} \left(\sum_{i=1}^{m} x_i^2 \right)^2 \geq \frac{1}{m} \|\mathbf{x}\|_2^4,$$

and $-\mathbf{G}(\epsilon, 0)$ is a nonnegative definite matrix. We have that, by choosing $b = -d(c+k)$ and $k > 0$,

$$\begin{aligned}
\frac{dU}{dt} &\leq (k-\alpha) < \mathbf{y}, \mathbf{y} > -k\alpha < \mathbf{x}, \mathbf{y} > -k < \mathbf{x}, \mathbf{x}^3 > + < \mathbf{y} + k\mathbf{x}, \mathbf{g}(t) > \\
&\leq (k-\alpha)\|\mathbf{y}\|_2^2 + k\alpha\|\mathbf{x}\|_2\|\mathbf{y}\|_2 - \frac{k}{m}\|\mathbf{x}\|_2^4 + \sqrt{m}a(\|\mathbf{y}\|_2 + k\|\mathbf{x}\|_2) \\
&=: u(\|\mathbf{x}\|_2, \|\mathbf{y}\|_2).
\end{aligned}$$

We are now in a position to show the bounded dissipation of the coupled system $(\mathbf{D}(c), \mathbf{G}(\epsilon, 0), \mathbf{F}(\mathbf{x}, \mathbf{y}, t))$.

Proposition 4.1.

(i) *Let $b = -d(c+k)$, and*

$$0 < k < \min\{\frac{4\alpha}{4+\alpha^2 m}, \alpha\}. \tag{4.11}$$

Then there exists a constant c_0 so that $\frac{dU}{dt} < 0$ for $\|\mathbf{x}\|_2^2 + \|\mathbf{y}\|_2^2 \geq c_0$.

(ii) *If $c = 0$, then the first assertion of the proposition still holds true.*

Proof. Suppose $\|\mathbf{x}\|_2 \geq 1$. Then

$$\begin{aligned}
u(\|\mathbf{x}\|_2, \|\mathbf{y}\|_2) &\leq (k-\alpha)\|\mathbf{y}\|_2^2 + k\alpha\|\mathbf{x}\|_2\|\mathbf{y}\|_2 - \frac{k}{m}\|\mathbf{x}\|_2^2 + \sqrt{m}a(\|\mathbf{y}\|_2 + k\|\mathbf{x}\|_2) \\
&=: \bar{u}(\|\mathbf{x}\|_2, \|\mathbf{y}\|_2).
\end{aligned}$$

It then follows from (4.11) that the the level curve of $\bar{u}$ is a bounded closed curve. We shall call such curve ellipse-like is an elliptic in the plane. Thus, there exists a c_1 so that $\frac{dU}{dt} < 0$ whenever $\|\mathbf{x}\|_2^2 + \|\mathbf{y}\|_2^2 \geq c_1$ and $\|\mathbf{x}\|_2 \geq 1$. Let $\|\mathbf{x}\|_2 < 1$ and $\|\mathbf{x}\|_2^2 + \|\mathbf{y}\|_2^2 \geq c_2$. Here c_2 is a constant to be determined. Then

$$u(\|\mathbf{x}\|_2, \|\mathbf{y}\|_2) \leq (k-\alpha)\|\mathbf{y}\|_2^2 + (k\alpha + \sqrt{m}a)\|\mathbf{y}\|_2 + \sqrt{m}ak =: h(\|\mathbf{y}\|_2).$$

Since $h(\|\mathbf{y}\|_2)$ is a parabola-like curve which is open downward, there exists a $c > 1$ such that $h(\|\mathbf{y}\|_2) < 0$ whenever $\|\mathbf{y}\|_2 \geq c$. Thus, if $c_2 \geq c^2+1$, then $u(\|\mathbf{x}\|_2, \|\mathbf{y}\|_2) < 0$ whenever $\|\mathbf{x}\|_2 < 1$ and $\|\mathbf{x}\|_2^2 + \|\mathbf{y}\|_2^2 \geq c_2$. Picking $c_0 = \max\{c_1, c_2\}$, we have that the assertion of the proposition holds true. $\square$

Proposition 4.2. *Assume (4.11) holds true. Then $\lim_{r\to\infty} U(\mathbf{x}, \mathbf{y}) = \infty$, where $r = \sqrt{\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2}$.*

Proof. From (4.10), we have that

$$\begin{aligned} U(\mathbf{x},\mathbf{y}) &= \frac{1}{2}\|\mathbf{y}\|^2 + \sum_{i=1}^{m}\frac{x_i^4}{4} + b < \mathbf{x}, \mathbf{L}\mathbf{x} > + k < \mathbf{x}, \mathbf{y} > \\ &\geq \frac{1}{2}\|\mathbf{y}\|^2 + \frac{1}{4m}\|\mathbf{x}\|^4 + b_0\|\mathbf{x}\|^2 - k\|\mathbf{x}\|\cdot\|\mathbf{y}\|, \end{aligned}$$

where $b_0 = b\|\mathbf{L}\|$. Let $b_1 > 0$ satisfying $\frac{1}{4m}b_1^2 + b_0 > k^2$. Then suppose $\|\mathbf{x}\| > b_1$, we have

$$U(\mathbf{x},\mathbf{y}) \geq \frac{1}{2}\|\mathbf{y}\|^2 + k^2\|\mathbf{x}\|^2 - k\|\mathbf{x}\|\|\mathbf{y}\| =: h_1(\|\mathbf{x}\|, \|\mathbf{y}\|).$$

Since the level curve of $h_1(\|\mathbf{x}\|, \|\mathbf{y}\|)$ is elliptic-like in the plane. Thus, for any given $M > 0$, there exists a $d_1 > 0$ such that $U(\mathbf{x},\mathbf{y}) > M$ whenever $\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 \geq d_1^2$ and $\|\mathbf{x}\| > b_1$.
Suppose $\|\mathbf{x}\| \leq b_1$, we then have

$$U(\mathbf{x},\mathbf{y}) \geq \frac{1}{2}\|\mathbf{y}\|^2 - kb_1\|\mathbf{y}\| + b_0 b_1^2 =: h_2(\|\mathbf{x}\|, \|\mathbf{y}\|).$$

Since $h_2(\|\mathbf{x}\|, \|\mathbf{y}\|)$ is parabola-like curve which are open upward in the plane. Thus, for any given $M > 0$, there exists a $d_2 > 0$ such that $U(\mathbf{x},\mathbf{y}) > M$ whenever $\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 \geq d_2^2$ and $\|\mathbf{x}\| \leq b_1$. Picking $\alpha = \max\{d_1, d_2\}$, we have that $U(\mathbf{x},\mathbf{y}) > M$ for all $\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 \geq \alpha^2$. Thus, the assertion of the proposition holds true. □

Theorem 4.3. *Coupled system* $(\mathbf{D}(c), \mathbf{G}(\epsilon, 0), \mathbf{F}(\mathbf{x},\mathbf{y},t))$ *is bounded dissipative, where* $\epsilon > 0$*.*

Proof. The proof is direct consequences of Propositions 2.3, 4.1 and 4.2. □

Theorem 4.4. *Let* $\mathbf{f}$*,* $\mathbf{D}(c)$ *and* $\mathbf{G}(\epsilon, 0)$ *be given as in (4.7a), (4.7b), and (4.7c), respectively. Let* $c \geq 0$*. Then the coupled system* $(\mathbf{D}(c), \mathbf{G}(\epsilon, 0), \mathbf{F}(\mathbf{x},\mathbf{y},t))$ *is globally synchronized provided that* d *is chosen sufficiently large.*

Proof. Letting $w_i = p\,x_i + q\,y_i$, we see that (4.6) becomes

$$\dot{x}_i = -\frac{p}{q}x_i + \frac{1}{q}w_i \tag{4.12a}$$

$$\begin{aligned} \dot{w}_i = {} & (\frac{p}{q} - \alpha)w_i + p(\alpha - \frac{p}{q})x_i - qx_i^3 + qa\cos(wt) \\ & + d(qc - p)(\mathbf{G}(\epsilon, 0)\mathbf{x})_i + d(\mathbf{G}(\epsilon, 0)\mathbf{w})_i \end{aligned} \tag{4.12b}$$

Suppose $c = 0$. Then we pick $q = 1$ and $p = \frac{1}{d}$. In vector form (4.12) becomes

$$\dot{\begin{pmatrix} \mathbf{w} \\ \mathbf{x} \end{pmatrix}} = \begin{pmatrix} \tilde{\mathbf{f}}_1(\mathbf{x},\mathbf{w},t) \\ \tilde{\mathbf{f}}_2(\mathbf{x},\mathbf{w},t) \end{pmatrix} + d\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \mathbf{G}(\epsilon, 0)\right)\begin{pmatrix} \mathbf{w} \\ \mathbf{x} \end{pmatrix}, \tag{4.13}$$

where the ith component of $\tilde{\mathbf{f}}_1(x, w, t)$ is

$$\left(\tilde{\mathbf{f}}_1(\mathbf{x},\mathbf{w},t)\right)_i = (\frac{1}{d}-\alpha)w_i + \frac{1}{d}(\alpha - \frac{1}{d})x_i - qx_i^3 + a\cos\, wt - (\mathbf{G}(\epsilon,0)\mathbf{x})_i$$

and

$$\left(\tilde{\mathbf{f}}_2(\mathbf{x},\mathbf{w},t)\right)_i = -\frac{1}{d}x_i + w_i.$$

It is then clear that (4.13) satisfies the assumptions in Theorem 3.2. Hence, the coupled system $(\mathbf{D}(c), \mathbf{G}(\epsilon,0), \mathbf{F}(\mathbf{x},\mathbf{y},t))$ is synchronized. Suppose $c \neq 0$. Choosing p and q so that $qc = p$, we conclude, again, that new coupled system (4.13) satisfies synchronization hypothesis. □

We now turn our attention to equation (4.8) with $r \neq 0$.

Theorem 4.5. *Assume that $\alpha > c \geq 0$, $\epsilon > 0$ and $r \in \mathbb{R}$. Then coupled system $(\mathbf{D}(c), \mathbf{G}(\epsilon,r), \mathbf{F}(\mathbf{x},\mathbf{y},t))$ is bounded dissipative.*

Proof. Consider the following scalar-valued function

$$U(\mathbf{x},\mathbf{y}) = \frac{1}{2} < \mathbf{y},\mathbf{y} > + \sum_{i=1}^{m}\frac{x_i^4}{4} + < \mathbf{x},\mathbf{K}\mathbf{x} > + c < \mathbf{x},\mathbf{y} >, \tag{4.14a}$$

where c is given as in (4.7b), and the matrix $\mathbf{K}$ satisfies

$$\mathbf{K} + \mathbf{K}^T = -2\epsilon dc\mathbf{G}(1,0) =: \mathbf{K}'. \tag{4.14b}$$

Taking the time derivative of U along solutions of coupled system $(\mathbf{D}(c), \mathbf{G}(\epsilon,r), \mathbf{F}(\mathbf{x},\mathbf{y},t))$, we have

$$\begin{aligned}
\frac{dU}{dt} &= (-\alpha + c)\|\mathbf{y}\|^2 + < \mathbf{g}(t), c\mathbf{x}+\mathbf{y} > - c\alpha < \mathbf{x},\mathbf{y} > - c < \mathbf{x},\mathbf{x}^3 > \\
&\quad + d < \mathbf{G}(\epsilon,r)\mathbf{y},\mathbf{y} > + dc^2 < \mathbf{x},\mathbf{G}(\epsilon,r)\mathbf{x} > + < \mathbf{x},\mathbf{K}'\mathbf{y} > \\
&\quad + dc < \mathbf{x}, [\mathbf{G}^T(\epsilon,r) + \mathbf{G}(\epsilon,r)]\mathbf{y} > \\
&= (-\alpha + c)\|\mathbf{y}\|^2 + < \mathbf{g}(t), c\mathbf{x}+\mathbf{y} > - c\alpha < \mathbf{x},\mathbf{y} > - c < \mathbf{x},\mathbf{x}^3 > \\
&\quad + d\epsilon < \mathbf{G}(1,0)\mathbf{y},\mathbf{y} > + dc^2\epsilon < \mathbf{x},\mathbf{G}(1,0)\mathbf{x} > \\
&\leq (-\alpha + c)\|\mathbf{y}\|^2 + \sqrt{m}a(c\|\mathbf{x}\| + \|\mathbf{y}\|) + c\alpha\|\mathbf{x}\|\|\mathbf{y}\| - \frac{c}{m}\|\mathbf{x}\|^4.
\end{aligned}$$

Following the similar arguments as done in proving the assertions of Proposition 4.1 and 4.2, we conclude that (i) $\frac{dU}{dt} < 0$ whenever $\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 \geq c_0$ for some $c_0 > 0$ and (ii) $\lim_{r\to\infty} U(\mathbf{x},\mathbf{y}) = \infty$, where $r = \sqrt{\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2}$. It then follows from Proposition 2.3 that coupled system (4.8) is bounded dissipative.

□

Theorem 4.6. *Let $\mathbf{f}$, $\mathbf{D}(c)$ and $\mathbf{G}(\epsilon,r)$ be given as in (4.7a), (4.7b) and (4.7c), respectively. Let $0 \leq c < \alpha$. Then the coupled system $(\mathbf{D}(c), \mathbf{G}(\epsilon,r), \mathbf{F}(\mathbf{x},\mathbf{y},t))$ is globally synchronized provided that d is chosen sufficiently large.*

Proof. Since $\mathbf{G}(\epsilon, r)$ is a circulant matrix (see e.g., [5]), the eigenvalues λ_k of $\mathbf{G}(\epsilon, r)$ are

$$\lambda_k = -2\epsilon(1 - \cos\frac{2k\pi}{n}) - i\,2r\sin\frac{2k\pi}{n}, \quad k = 0, \ldots, m-1.$$

Hence $\mathbf{G}(\epsilon, r)$ satisfies assumptions (3.3a) and (3.3b). The proof of the theorem is thus similar to that of Theorem 4.4. $\square$

Remark 4.2. (i) It was shown in [8] that there are positive constants d_0 and c_0 such that, for $d \geq d_0$, $c \geq c_0$, the system $(\mathbf{D}(c), \mathbf{G}(\epsilon, 0), \mathbf{F})$ given in (4.8) is synchronized. Our results also work for the case that $c_0 = 0$ or $\mathbf{G}(\epsilon, r)$, $r \neq 0$. (ii) It was also shown in [1] that there are positive constants d_0 and c_0 such that for $d \geq d_0$, $c \geq c_0$, the system $(\mathbf{D}(c), \mathbf{G}, \mathbf{F})$ is synchronized. Here $-\mathbf{G}$ is a positive definite matrix. (iii) The case that the lattices of coupled Rössler-like equations in [9] is a bit more different, and we will address this issue in a forthcoming paper.

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