行政院國家科學委員會專題研究計畫　成果報告

使用階梯格子點模型評價支付股息的股票選擇權

計畫類別：個別型計畫
計畫編號：NSC94-2213-E-009-153-
執行期間：94 年 08 月 01 日至 95 年 07 月 31 日
執行單位：國立交通大學資訊與財金管理學系

計畫主持人：戴天時

報告類型：精簡報告
報告附件：出席國際會議研究心得報告及發表論文
處理方式：本計畫可公開查詢

中　華　民　國 95 年 10 月 26 日

中文摘要

在學術界中，如何評價配離散股息的股票選擇權這個問題尚未完全解決。Roll (1977) 假定股票的價格減去未來股息的現值服從對數常態分配，另一方面，Musiela and Rutkowski (1997)則假定股票的價格加上股息的未來值服從對數常態分配。因為在兩種假定下，可以很容易地推出封閉公式解，所以這兩種假定廣為多數的學術論文採用。然而，Frishling (2002)指出，使用這兩種假定下產生的選擇權評價結果會互相矛盾。他指出應該假定股票在配息日所配的股息會反映到股價的減少。然而，這種假定雖然貼近市場真實的狀況，但是卻不易推導出封閉公式解，而有些數值方法例如樹狀結構，也無法有效率地處理評價問題。在本次研究計畫中，我成功推導出一個新的樹狀結構—階梯格子樹結構，該結構忠實模擬股價因配息而出現不連續的現象。該樹狀結構十分簡易、易於建構、而且評價效率也甚高。同時我也證明該樹狀結構會收斂到配股息的股價隨機過程。根據數值實驗的結果，階梯格子樹結構的精確度、評價速度都較現有方法還要好。此外，階梯格子樹模型可以處理更一般化的假定，例如在時間點 t 的股票配息可寫成時間點 t 之前的股價以及股息的函數。這種特性讓本模型能夠更貼近真實市場狀況。

關鍵字:階梯格子樹模型、股息、股票選擇權、評價。

英文摘要

Pricing options on a stock that pays discrete dividends has not been satisfactorily settled in the literature. Roll (1997) assumes that the stock price minus the present value of future dividends follows a lognormal diffusion process. On the other hand, Musiela and Rutkowski (1997) assume that the stock price plus the forward value of future dividends follows the lognormal diffusion. Both assumptions are widely accepted in academic literature since analytical formulas can be easily derived. Unfortunately, Frishling (2002) shows that these two assumptions produce inconsistent option prices. Frishling shows it is more reasonable to assume that the stock price jumps down with the amount of dividend paid at the exdividend date. However, it is hard to derive analytical pricing formulas for this assumption and some numerical methods like lattices are inefficient when used to implement this assumption. My project successfully constructs a new lattice, the stair lattice, which does not change the dividend-paying regime. The stair lattice is simple to construct, easy to understand, and efficient. It is furthermore guaranteed to converge to the price process of the stock that pays multiple discrete dividends. Numerical experiments confirm the stair lattice's superior performance to existing methods in terms of accuracy, speed, and/or generality. Besides, the stair lattice can be extended so a future dividend paid at time *t* depends on the stock prices and the dividends prior to time *t* in arbitrary ways. This extension makes our model more realistic and flexible.

Keywords: stair lattice, dividend, stock option, pricing

1 Preface

By assuming that the stock price follows the lognormal diffusion, Black and Scholes (1973) arrive at their ground-breaking option pricing model for non-dividend-paying stocks. Merton (1973) extends the model to the case where the underlying stock pays a non-stochastic continuous dividend yield. In reality, however, almost all stock dividends are paid discretely rather than continuously. For simplicity, this dividend setting is called the discrete dividend setting if the amounts of future dividends are known today. Pricing options on a stock that pays discrete dividends seems to be investigated first in Black (1975). The discrete-dividend option pricing problem has drawn a lot of attention in the literature. Frishling (2002) summarizes that there are three different ways to model the stock price process with discrete dividends. But only one model that assumes the stock price jumps down with the amount of dividend paid at the exdividend date can produce reasonable option prices. However, it is hard to derive analytical pricing formulas for this model and some numerical methods like lattices are inefficient when used to implement it. My project will provide a novel tree model for solving this problem.

2 The Goal of This Research Project

My research project intends to construct a new lattice, the stair lattice, that simulate the stock price process with discrete dividends. It can successfully model the jumping of stock price at the exdividend date. It is also guaranteed to converge to the price process of the stock that pays multiple discrete dividends. And the stair lattice is proved to be constructable and efficient. Besides, the stair lattice can be extended so a future dividend paid at time t depends on the stock prices and the dividends prior to time t in arbitrary ways. This extension makes the stair lattice more realistic and flexible.

3 Literature Review

Frishling (2002) summarizes that there are three different ways to model the stock price process with discrete dividends. These three different models are introduced as follows:
Model 1. Roll (1977) suggests that the stock price is divided into two parts: the stock price minus the present value of future dividends over the life of the option and the present value of future dividends. The former part (call it net-of-dividend stock price) is assumed to follow a lognormal diffusion process, whereas the latter part is assumed to grow at the risk-free rate. Thus vanilla options can be computed by applying the Black-Scholes formula with the stock price replaced by the net-of-dividend stock price. Cox and Rubinstein (1985) also call this model the ad hoc adjustment.
Model 2. Musiela and Rutkowski (1997), following Heath and Jarrow (1988), suggests that the cum-dividend stock price, defined as the stock price plus the forward values of the dividends paid from today up to maturity, follows a lognormal diffusion process. Thus vanilla options can be computed by the Black-Scholes formula by replacing the stock price with the cum-dividend stock price and by adding the forward values of the dividends prior to maturity to the strike price.

Model 3. The stock price jumps down with the amount of dividend paid at the exdividend date, and follows lognormal price process between two exdividend dates.

Although above three models all address the discrete-dividend problem, Frishling (2002) shows that these three models generates very different option prices since these three stock price models are very different. A simple argument is given to explain the differences among these models. Assume that the volatility input to these three models is σ. Then Model 1 sets the volatility of the net-of-dividend stock price as σ while Model 3 sets the volatility of the stock price as σ. The volatility of the stock price in Model 1 is lower than that in Model 3 since the volatility of the present value of future dividends, a component of stock price, is assumed to be 0 in Model 1. Model 1 therefore produces lower option prices and the price difference between these two models becomes larger as the volatility of the stock price becomes larger. Similarly, Model 2 produces higher option prices than Model 3 since Model 2 assigns the volatility of the forward values of the dividends, which is not a part of stock price, to be σ.

Model 1 and Model 2 are widely accepted in academic literature (see Geske (1979), Whaley (1981, 1982), Carr (1998), and Chance *et al.* (2002)) in solving the discrete-dividend problem, since closed-form option pricing formulas can be easily derived. However, these two models suffer from the following problems:

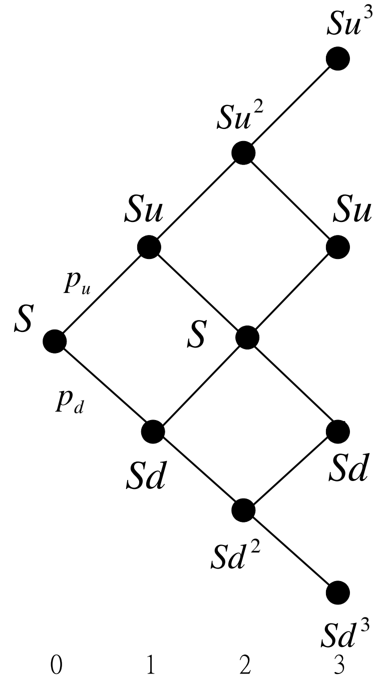
• These two models can not provide consistent and reasonable option prices. For example, Frishling (2002) shows that Model 1 could incorrectly renders a down-and-out barrier option worthless simply because the net-of-dividend stock price reaches the barrier when the dividend(s) are big enough. In reality, the option has a reasonable chance to survive since the dividend(s) are paid later than the option initial date. Similar problem could occur for pricing a up-and-out barrier option with Model 2. Besides, Bender and Vorst (2001) also show that arbitrage opportunities exist in Model 1 if the volatility surface is continuously interpolated around exdividend date.

• Bos and Vandermark (2002) show that both models violate a perfectly reasonable continuity requirement. A dividend paid just before maturity should be equivalent to an increase in the strike price by the dividend amount. Model 1 subtract the present value of the dividend from the stock price input to the Black-Scholes formula is not equivalent to adding the dividend amount to the strike price. Similarly, a dividend paid just after option initial date should be equivalent to a decrease in the stock price. Model 2 add the forward value of the dividend to the stock price input to the Black-Scholes formula also cause the same problem.

• The volatilities of the net-of-dividend stock price (see Model 1) and the cum-dividend stock price (see Model 2) relate awkwardly to the historical volatility, which is commonly calculated according to historical stock price.

Although Model 3 is much closer to reality than the other two models, there is no exact closed-form solution for pricing European-style options. Hull (2000) recommends that an approximation pricing formula can be obtained by adjusting the volatility parameter input to Model 1 with a simple formula. Numerical experiments show that the Hull's volatility adjustment performs poorly. Bos and Vandermark (2002) present an approach that is a mixture between the stock and the strike price adjustment or, in other words, Model 1 and Model 2. Bos and Shepeleva (2002) claim that this approach results in some inaccuracies, especially for in- and out-of-the-money options. They suggest another pricing formula by adjusting the volatility input to Model 1 with a complex formula. But their approach can

Figure 1: **The CRR Binomial Lattice.**



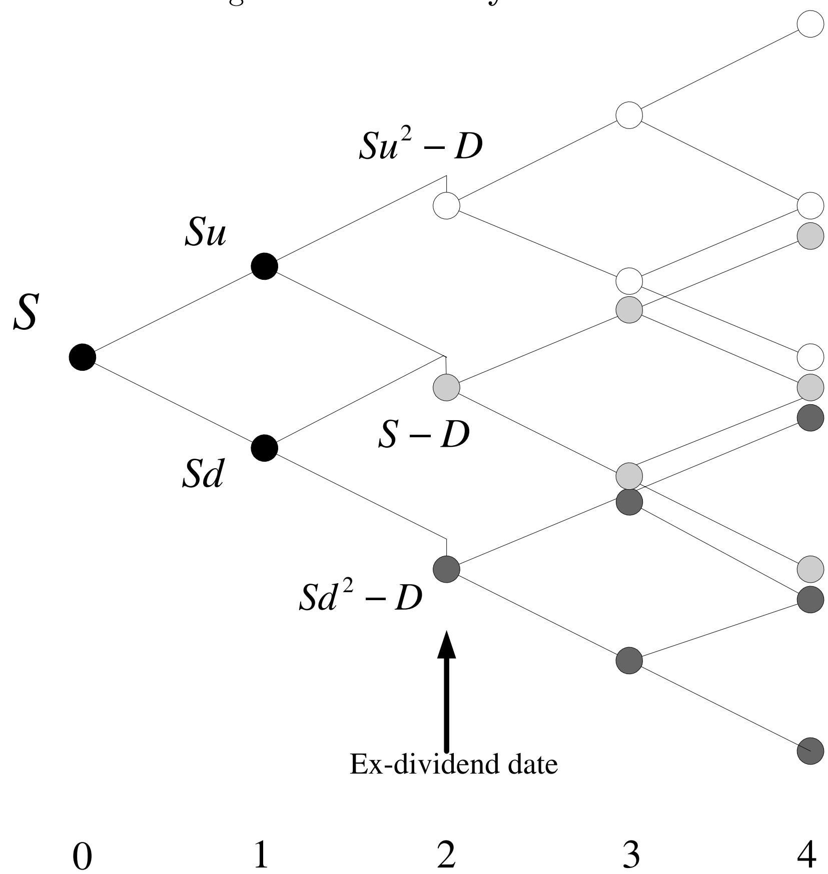
The initial stock price is S. The upward and downward multiplicative factors for the stock price are u and d, respectively. The upward and downward branching probabilities are p_u and p_d, respectively.

not be easily extended for pricing American options.

Model 3 can also be implemented by the lattice or the related PDE method. But a naive application of these methods results in combinatorial explosion. Take the well-known Cox-Ross-Rubinstein (CRR) binomial lattice as an example. Assume that the lattice starts at time step 0 and ends at time step n. Let R stand for the gross risk-free return per time step. When the stock does not pay dividends, in one time step the price S becomes Su (the up move) with probability p_u and Sd (the down move) with probability $p_d \equiv 1 - p_u$, where $p_u \equiv (R - d)/(u - d)$. The relation $ud = 1$ is enforced by the CRR binomial lattice. A simple 3-time-step binomial lattice is illustrated in Fig. 1. The binomial lattice recombines; thus the size of the lattice is only quadratic in n. Unfortunately, the recombination property disappears if the stock pays discrete dividends. Consider the 4-time-step binomial lattice with an ex-dividend date at time step 2 in Fig. 2. This lattice splits into 3 lattices after the ex-dividend date. Each such lattice will be split further into more separate lattices for each subsequent ex-dividend date. As a result, the lattice size grows exponentially with the number of exdividend dates. It will be referred to as the bushy lattice. The bushy lattice implements the Model 3, but the exponential complexity renders it impractical.

In additional to the first two models mentioned in Frishling (2002), efficient numerical algorithms and simple formulas can also result by approximating the discrete dividend with either (1) a fixed dividend yield on each dividend-paying date or (2) a fixed continuous dividend yield. The first approach is followed by Geske and Shastri (1985). They replace the discrete dividends with fixed dividend yields. The resulting lattice hence recombines and

Figure 2: **The Bushy Lattice.**



A dividend D is paid out at time step 2. Three separate lattices beginning at time step 2 are colored in white, light gray, and dark gray, respectively.

is efficient. This approximation approach works well for American options. But it produce significant pricing errors for European options. The second approach is followed by Chiras and Manaster (1978). They transform the discrete dividends into a fixed continuous dividend yield and then apply the Merton formula. As this approach can be shown to be equivalent to the first approach in pricing European options, it shares the same faults.

4 Research Method: Construction of the Stair Lattice

4.1 The Mathematical Assumptions and Required Financial Background

In Model 3, the stock price is assumed to follow the lognormal diffusion process:

$$\frac{dS(t)}{S(t)} = r\,dt + \sigma d\,\omega_t,$$

where $S(t)$ denotes the stock price at year t, r denotes the annual risk-free interest rate, σ denotes the volatility, and ω_t denotes the standard Brownian motion. In the discrete-time lattice model, it is assumed that there are n equal time steps between year 0 and year T. The length of each time step Δt is equal to T/n. Thus, time step i in the discrete-time model corresponds to year $i\Delta t$ in the continuous-time model. The upward and downward multiplicative factors u and d for the stock price equal $e^{\sigma\sqrt{\Delta t}}$ and $e^{-\sigma\sqrt{\Delta t}}$, respectively, for the CRR and stair lattices. S_i denotes the stock price at year $i\Delta t$ (or, equivalently, time step i for a lattice). The stock is assumed to pay m dividends $D_{t_1}, D_{t_2}, \ldots, D_{t_m}$, where D_{t_i} is paid out at time step t_i. Assume that $t_1 < t_2 < \cdots < t_m$ for convenience. Under the discrete dividend assumption, any arbitrary dividend D_{t_i} is already known at time step 0. In general, D_{t_i} can be determined by a function of stock prices and/or the dividends paid up to time step t_i under the path-dependent dividends assumption. The stock price simultaneously falls by the amount αD_{t_i}. For simplicity, α is assumed to be 1 throughout the paper, but a general α poses no difficulties to the stair lattice. When the ex-dividend stock price becomes negative, it is assumed to stay at zero from that point onward. Harvey and Whaley (1992), in contrast, assume that the dividend is not paid if its amount exceeds the prevailing stock price. The stair lattice can easily incorporate their assumption, too.

The option is assumed to start at time step 0 and mature at time step n. The strike price for this option is K. Define $(A)^+$ to denote $\max(A, 0)$ for simplicity. The payoff for a European option at maturity is

$$\text{final payoff} = \begin{cases} (S_n - K)^+, & \text{for a call,} \\ (K - S_n)^+, & \text{for a put.} \end{cases}$$

An American option gives the holder the right to exercise the option before maturity. The exercise value for an American option at a non-dividend-paying time step i is

$$\text{exercise value} = \begin{cases} S_i - K, & \text{for a call,} \\ K - S_i, & \text{for a put.} \end{cases}$$

The exercise strategy for an American option at an ex-dividend date is only slightly more complicated. It is never optimal to exercise an American call immediately after the underlying stock pays a dividend because it is dominated by the strategy of exercising the call immediately before. Similarly, it is never optimal to exercise a put before the stock pays a dividend. Consequently, the exercise value for an option at a dividend-paying time step i is

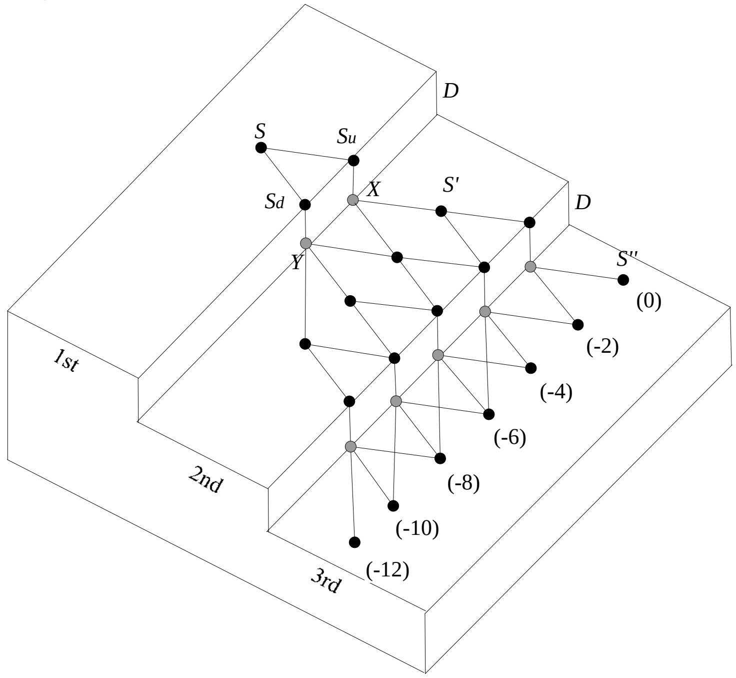
$$\text{exercise value} = \begin{cases} S_i^* - K, & \text{for a call,} \\ K - S_i, & \text{for a put,} \end{cases}$$

where S_i^* and S_i denote the stock price immediately before paying the dividend, and the price immediately after paying the dividend, respectively. An option will be exercised early by the owner if the option's continuation value (i.e., the value to hold the option) is smaller than its exercise value.

4.2 Construction of the Stair Lattice

The main ideas are illustrated by the 4-time-step lattice in Fig. 3. This 4-time-step stair lattice contains two ex-dividend dates: one at time step 1 and the other at time step 3. For simplicity, the same D-dollar dividend is paid at each ex-dividend date. The price drop

Figure 3: **The Structure of the Stair Lattice.**



The gray nodes are the nodes right after the dividend is paid. S' and S'' denote the largest stock price at time step 2 and time step 4, respectively. The stock price for each node on the third tread is represented as $S''u^k = S''e^{k\sigma\sqrt{\Delta t}}$, where k is parenthesized.

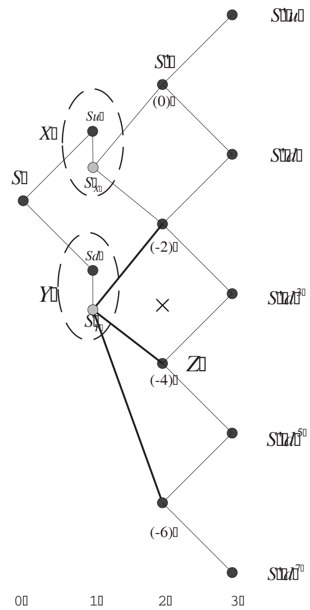
due to the dividend payout is represented by a riser. Each tread covers a time interval between two adjacent ex-dividend dates except the first tread, which covers the time interval between time step 0 and the first ex-dividend date. The branches follow the CRR lattice structure except those from the nodes at ex-dividend dates. For example, the stock price at the root node is S. The stock prices for its two successor nodes are Su and Sd, where $ud = 1$. Because of the CRR lattice structure, the stock prices on the same tread are Pu^k, where P is the stock price of some specific node on the tread and k is an even integer. For example, the stock price for each node on the third tread can be represented as $S''u^k$, where S'' denotes the largest stock price at time step 4 and k is parenthesized. Technically, any node's stock price can be picked for P because the stock prices on the same tread are part of the geometric sequence

$$\ldots, Pu^{-4}, Pu^{-2}, P, Pu^2, Pu^4, \ldots$$

Note that the first tread contains a single, complete CRR lattice. The lattice structure on each subsequent tread is composed of a CRR binomial lattice with the initial part truncated.

The next step is to construct the branches out of the gray nodes at an ex-dividend date to complete the stair lattice. Fig. 4 illustrates what happens at an ex-dividend date by zooming in the first three time steps of the stair lattice in Fig. 3. Nodes X and Y are from the first ex-dividend date. The ex-dividend stock price at node X is $S_X = Su - D$. The two branches from X follow the CRR lattice structure. S', the stock price for the top node at time step 2, therefore equals $S_X u$. Define the V-log-price of stock price V' as $\ln(V'/V)$;

Figure 4: **Branching Scheme at the Ex-Dividend Date.**



Nodes X and Y are at the first ex-dividend date (time step 1). Both nodes are represented by dotted ellipses. The cum-dividend stock prices at X and Y are Su and Sd, respectively, whereas the net-of-dividend stock prices at X and Y are $S_X(\equiv Su - D)$ and $S_Y(\equiv Sd - D)$, respectively. The stock price for the top node at time step 2 is S' $(= S_X u)$. The integer k in parentheses for each node at time step 2 means the stock price equals $S' e^{k\sigma\sqrt{\Delta t}}$. The cross right above Z denotes the mean of the stock price process one time step from Y, or $S_Y e^{(r-\sigma^2/2)\,\Delta t}$. The three branches of Y are marked with thick solid lines.

a V-log-price of z implies a stock price of Ve^z. Since the stock price for each node on the second tread can be expressed in terms of $S'u^k$ for some even integer k, the S'-log-prices for nodes at time step 2 in Fig. 4 are integral multiples of $2\sigma\sqrt{\Delta t}$.

The branches from node Y are constructed as follows. Let the ex-dividend stock price for node Y be S_Y. At least three branches are required for node Y to match the first two moments of the logarithmic stock price process for degrees-of-freedom considerations. Three nodes at time step 2 follow node Y. By the lognormality of the stock price, the mean and the variance of the S_Y-log-prices at these nodes equal

$$\begin{aligned} \mu &\equiv (r - \sigma^2/2)\,\Delta t, \\ \text{Var} &\equiv \sigma^2 \Delta t. \end{aligned}$$

Note that the distance between two adjacent nodes' S_Y-log-prices at time step 2 is $2\sigma\sqrt{\Delta t}$. Thus there exists a unique node Z at time step 2 whose S_Y-log-price $\hat{\mu}$ lies in the interval

$$[\,\mu - \sigma\sqrt{\Delta t}, \mu + \sigma\sqrt{\Delta t}).$$

In other words, the S_Y-log-price of node Z, i.e., $\hat{\mu}$, is closest to μ among the S_Y-log-prices of the nodes at time step 2. Use $\hat{\mu}$ to denote the mean tracker of node Y. The middle branch from node Y will be connected to node Z. Figure 4 illustrates the case where $\hat{\mu} = \ln(S'/S_Y) - 4\sigma\sqrt{\Delta t}$ (or, 2 nodes below S').

In general, the S_Y-log-prices of the two nodes connected by the upper and lower branches from node Y can be expressed as $\hat{\mu}+\ell_u\sigma\sqrt{\Delta t}$ and $\hat{\mu}-\ell_d\sigma\sqrt{\Delta t}$ for some even positive integers ℓ_u and ℓ_d. It is clear that the jump sizes ℓ_u and ℓ_d should be as small as possible to minimize the size of the stair lattice. And ℓ_u and ℓ_d should also be properly selected to make the branching probabilities of node Y valid. Let p_Y^u, p_Y^m, and p_Y^d denote the probabilities for the upper, middle, and lower branches from node Y, respectively. Define

$$\begin{aligned} \beta &\equiv \hat{\mu} - \mu, \\ \alpha &\equiv \beta + \ell_u\sigma\sqrt{\Delta t}\,, \\ \gamma &\equiv \beta - \ell_d\sigma\sqrt{\Delta t}\,. \end{aligned}$$

Note that the first equation implies that $\beta \in [-\sigma\sqrt{\Delta t}, \sigma\sqrt{\Delta t})$. Note also that $\alpha > \beta > \gamma$. The probabilities can be derived by solving

$$p_Y^u\alpha + p_Y^m\beta + p_Y^d\gamma = 0, \tag{1}$$
$$p_Y^u\alpha^2 + p_Y^m\beta^2 + p_Y^d\gamma^2 = \mathrm{Var}, \tag{2}$$
$$p_Y^u + p_Y^m + p_Y^d = 1. \tag{3}$$

Equations (1) and (2) match the first two moments of the logarithmic stock price, and Eq. (3) ensures that p_Y^u, p_Y^m, p_Y^d as probabilities sum to one. The three equations do not automatically guarantee $0 < p_Y^u, p_Y^m, p_Y^d < 1$. A proof to show that they actually do with $\ell_u = \ell_d = 2$ is given in section 4.3. The stair lattice hence does not lead to branches with huge jump sizes. This finding is essential to the efficiency of the algorithm. The same procedure can be repeated for nodes below Y. To handle multiple dividends, just apply the procedure to each ex-dividend date.

Because the first and second moments are matched via Eqs. (1)–(3), the stair lattice converges to the stock price process with discrete dividends. The option price computed on the lattice therefore converges to the option price under the lognormal diffusion. This guarantee solves the accuracy problem affecting many other approximation schemes.

4.3 Proof of Valid Branching Probabilities

Define

$$\begin{aligned} \det &= (\beta - \alpha)(\gamma - \alpha)(\gamma - \beta), \\ \det{}_u &= (\beta\gamma + \mathrm{Var})(\gamma - \beta), \\ \det{}_m &= (\alpha\gamma + \mathrm{Var})(\alpha - \gamma), \\ \det{}_d &= (\alpha\beta + \mathrm{Var})(\beta - \alpha). \end{aligned}$$

Then Cramer's rule applied to Eqs. (1)–(3) gives $p_Y^u = \det_u/\det$, $p_Y^m = \det_m/\det$, and $p_Y^d = \det_d/\det$. Note that $\det < 0$ because $\alpha > \beta > \gamma$. To ensure that the branching probabilities are valid, it suffices to show that p_Y^u, p_Y^m, $p_Y^d \geq 0$. As $\det < 0$, it is sufficient to show $\det_u, \det_m, \det_d \leq 0$ instead. Finally, as $\alpha > \beta > \gamma$, it suffices to show that

$\beta\gamma + \mathrm{Var} \geq 0$, $\alpha\gamma + \mathrm{Var} \leq 0$, and $\alpha\beta + \mathrm{Var} \geq 0$ under the premise $\beta \in [-\sigma\sqrt{\Delta t}, \sigma\sqrt{\Delta t})$. Indeed,

$$\begin{aligned}
\beta\gamma + \mathrm{Var} &= \beta^2 - 2\beta\sigma\sqrt{\Delta t} + \sigma^2\Delta t = (\beta - \sigma\sqrt{\Delta t})^2 \geq 0, \\
\alpha\gamma + \mathrm{Var} &= \beta^2 - 4\sigma^2\Delta t + \sigma^2\Delta t = \beta^2 - 3\sigma^2\Delta t < 0, \\
\alpha\beta + \mathrm{Var} &= \beta^2 + 2\beta\sigma\sqrt{\Delta t} + \sigma^2\Delta t = (\beta + \sigma\sqrt{\Delta t})^2 \geq 0,
\end{aligned}$$

as desired.

4.4 Extend the Stair Lattice Model to Price Options on a Stock That Pays Path-Dependent Dividends

The stochastic dividend assumption is more general and realistic than the discrete dividend assumption. However, the option can only be hedged if the dividends are known exogenously or completely determined by the stock price process as argued in Cox and Rubinstein (1985) unless one adds nonstandard derivatives such as the forward contracts on dividends in Chance et al. (2002). We call Cox and Rubinstein's assumption the path-dependent dividends assumption since the future dividend, says D_{t_i}, completely depends on the stock prices and the dividends prior to time step t_i. To be more precise, D_{t_i} can be expressed as

$$D_{t_i} \equiv f(S_0, S_1, S_2 \ldots, S_{t_i}, D_{t_{i-1}}, D_{t_{i-2}} \ldots)$$

for some function f. In reality, dividends can be explained by a variety of factors such as the net operating profits, long-run sustainable (or permanent) earnings, and so on. If the stock prices and the dividends paid previously serve as good proxies for these factors, a dividend function that fit the path-dependent dividends assumption can be constructed. Indeed, some empirical dividend models do attempt to fit path-dependent dividends assumptions with slight modification. I will first review one of such dividend models proposed by Marsh and Merton (1987), and then show how the stair lattice can incorporate their dividend model.

Marsh and Merton (1987) derive a dividend model by following Lintner's (1962) stylized facts established by Linter in a classic set of interviews with managers about their dividend policies. Their dividend model can be expressed by a regression formula of the permanent earnings and the dividends paid previously. They argue that their formula can not be directly estimated because management assessments of changes in a firm's permanent earnings are not observable. Thus they assume the permanent earing to cum-dividend stock price ratio is a positive constant. Under this assumption, their dividend model can be expressed by a regression formula of the net-of-dividend stock prices and the dividends paid previously. To illustrate the main idea briefly on how the stair lattice incorporate Marsh and Merton's dividend model, their dividend model is expressed by assuming that the length between two ex-dividend dates is two time steps as follows:

$$\log\left[\frac{D_{t+2}}{D_t}\right] + \frac{D_t}{S_{t-2}} = a_0 + a_1 \log\left[\frac{S_t + D_t}{S_{t-2}}\right] + a_2 \log\left[\frac{D_t}{S_{t-2}}\right] + u(t+2), \quad (4)$$

where $u(t+2)$ denotes the disturbance term at time step $t+2$. One of their empirical studies uses the value-weighted NYSE index over the period 1926–81 to estimate the parameters a_0,

Figure 5: **A 4-Time-Step Stair Lattice that Incorporate the Marsh and Merton's Dividend Model.**

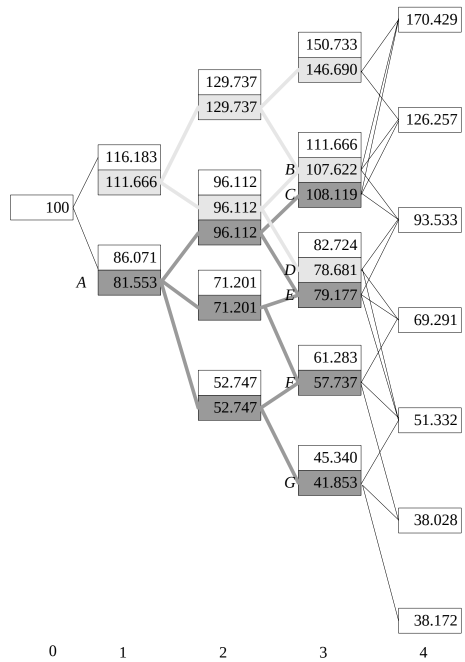


Table 1: **The Branching Probabilities for the States with Trinomial Branches in Fig. 5.**

	A	B	C	D	E	F	G
Upper	0.4983	0.0001	0.0004	0.4699	0.4905	0.4397	0.3744
Middle	0.5017	0.5133	0.5280	0.5296	0.5095	0.5584	0.6165
Lower	1.4×10^{-6}	0.4866	0.4715	0.0005	4.6×10^{-5}	0.0019	0.0091

a_1, and a_2 by ordinary least squares method and obtains $a_0 = -0.101$, $a_1 = 0.437$, and $a_2 = -0.042$. By assuming that the disturbance term $u(t+2) = 0$, Eq. (4) can be rewritten as

$$D_{t+2} = 10^{a_0 + a_1 \log\left[\frac{S_t + D_t}{S_{t-2}}\right] + a_2 \log\left[\frac{D_t}{S_{t-2}}\right] - \frac{D_t}{S_{t-2}} + \log D_t} \equiv f(S_{t-2}, S_t, D_t) \tag{5}$$

under the path-dependent dividends framework.

I will show how the stair lattice can incorporate the dividend model mentioned above. Extra states are added to some nodes of the stair lattice to keep the information required for computing future dividends. For example, to compute D_{t+2}, additional states are added to keep the information like the stock prices (S_t and S_{t-2}) and the dividends (D_t). A simple 4-time-step stair lattice in Fig. 5 illustrate how this mechanism works. The top cell of each node (in white color) denotes the stock price (at a non-dividend paying date) or the cum-dividend stock price (at an ex-dividend date) of that node. Some nodes have more cells (in light-gray or dark-gray) that represent additional states for keeping necessary information. The stock price (at a non-dividend paying date) or the net-of-dividend stock price (at an ex-dividend date) of a state is marked directly on that state. The settings are listed as follows. The stair lattice begins at time step 0 with stock price 100. The length of each time step of the stair lattice is 0.25 year, the risk-free interest rate is 10%, and the volatility of the stock price is 30%. Thus the upward multiplication factor (u) and the downward one (d) are about 1.162 and 0.861, respectively. The settings for dividends are given as follows. The dividends are assumed to be paid for each 2 time steps and the last dividend paid at time step -1 is 5. Let the net-of-dividend stock prices at time step -1 and -3 be 110 and 80, respectively. Note that two dividends will be paid at time step 1 and time step 3, respectively. These two dividends are estimated by Eq. (5).

The cum-dividend stock price at time step 1 are $100 \times u \approx 116.183$ and $100 \times d \approx 86.071$, respectively. The dividend paid at time step 1 is obtained by substituting D_{-1} (=5), S_{-1} (=110), and S_{-3} (=80) into Eq. (5) to get 4.518. Thus the net-of-dividend stock prices at time step 1 are $116.183 - 4.518 \approx 111.666$ and $86.071 - 4.518 \approx 81.553$, respectively. By following the method for constructing the stair lattice mentioned before, the stock prices at time step 2 can be represented as $129.737 \times u^k$ for nonpositive even integers k. The binomial and the trinomial branching schemes for the two nodes at time step 1 can also be constructed.

To compute the dividend paid at time step 3 (D_3) by Eq. (5), three parameters S_{-1}, D_1, and S_1 are needed. While S_{-1} and D_1 are known to be 110 and 4.518 respectively, two possible net-of-dividend stock price (111.666 and 81.553) may occur at time step 1. These two possible prices result in two different dividend payouts at time step 3. Thus, additional states are added to the stair lattice to keep the information about S_1. These states are colored in light-gray and the dark-gray to denote the cases that S_1 are 111.666 and 81.553, respectively. The branches that connect these two types of states are also colored in light-gray and the dark-gray, respectively. Since no dividend is paid at time step 2, all the states at the same node at time step 2 have the same stock prices. By substituting $S_1 = 111.666$ and $S_1 = 81.553$ into Eq. (5), we compute the dividend D_3 as 4.043 and 3.547 , respectively. For the top node (with cum-dividend stock price 150.773) at time step 3, the net-of-dividend stock price is $150.773 - 4.043 \approx 146.690$. For the second node (with cum-dividend stock price 111.666) at time step 3, two possible dividend payouts should be considered. The net-of-dividend stock price is $111.666 - 4.043 \approx 107.622$ for the state B that represents the case $S_1 = 111.666$. The net-of-dividend stock price is $111.666 - 3.547 \approx 108.119$ for the state C that represents the case $S_1 = 81.553$. The net-of-dividend stock prices for other nodes at time step 3 can be similarly computed. The stock price for the top node at time step 4 is $146.690 \times u \approx 170.249$. All the stock prices at time step 4 can be represented as $170.249 \times u^k$ for nonpositive even integers k. The branching schemes for all the colored states at time step 3 can be constructed by following the method for constructing stair lattice mentioned before. The trinomial branching probabilities for all the states with trinomial branches are listed in Table 1.

5 Discussions and Conclusions

I will first give some numerical results to verify the superiority of the stair lattice model. A simple conclusion is then given to summarize fruitful results of this project.

Table 2: **Pricing a Call Option with Single Discrete Dividend.**

	0.4						0.5					
X	FDY	Model1	Hull	Model2	Stair	MC	FDY	Model1	Hull	Model2	Stair	MC
95	*16.263	*16.336	17.090	17.112	16.821	16.933	*19.890	*19.969	20.901	20.937	20.570	20.843
100	*14.214	*14.270	15.044	15.048	14.758	14.754	*17.964	*18.003	*18.959	*18.971	18.591	18.584
105	*12.400	*12.439	*13.222	*13.206	12.924	12.989	*16.194	*16.222	17.194	17.182	16.829	16.929

The initial stock price is 100, the risk-free rate is 3%, the time to maturity is 1 year, and a 5-dollar-dividend is paid at year 0.6. The volatilities of the stock price are shown in the first row. The strike prices are listed in the first column. `FDY` denotes the fixed dividend yield approach of Geske and Shastri (1985). `Model1` and `Model2` denote the option prices generated by Model 1 and Model 2, respectively. `Hull` denotes volatility adjustment approach of Hull (2000). `Stair` denotes the stair lattice model. `MC` denotes the prices generated by Model 3 that based on Monte Carlo simulation based with 100,000 trials. Option prices that deviate from `MC` by 0.3 are marked by asterisks.

5.1 Numerical Results

I first compare Geske and Shastri's fixed dividend yield model, Hull's volatility adjustment model, the stair lattice model, and the aforementioned three discrete dividend models for pricing European options. Geske and Shastri (1985) use fixed dividend yields to approximate discrete dividends. The fixed dividend yield is defined as the discrete dividend amount divided by the initial stock price. For example, the dividend yield is 5% if the initial stock price is 100 and the discrete dividend is 5. I use `FDY` to denote their approach. Note that Chiras and Manaster (1978) approximate the discrete-dividend problem by transforming the discrete dividends into a fixed continuous dividend yield. This approach is equivalent to the `FDY` model in pricing a European option. Model 1 generates lower option prices than Model 3 as argued before. To remove this difference, Hull (2000) recommends that the volatility of net-of-dividend stock price be adjusted by the volatility of the stock price multiplied by $S(0)/(S(0) - D)$, where D denotes the present value of future dividends paid between time 0 to time T. We use `Hull` to denote Hull's volatility adjustment approach. Besides, we use `Model1` and `Model2` to denote the option prices generated by Model 1, Model 2, `Stair` to denote the prices generated by the stair lattice model, `MC` to denote the prices generated by Model 3 that based on Monte Carlo simulation with 100,000 trials. We use `MC` to serve as the benchmark.

The numerical results for these models are listed in Table 2 and 3, where Table 2 focuses on the single-discrete-dividend case and Table 3 focuses on the two-discrete-dividend case. All the prices that deviate from `MC` by 0.3 are marked by asterisks. It is not surprising that the option prices generated by Model 2 are higher than the prices generated by Model 3. On the other hand, Model 1 generates lower option prices than model 3. The difference among these three models becomes larger as volatility increases. `FDY` does not approximate Model 3 well as it produces lower option prices than Model 1. Hull's volatility adjustment approach seems to overprice the options. It can be observed that only the stair lattice model produces options prices that are close to the benchmark.

Table 3: **Pricing a Call Option with Two Discrete Dividends.**

	0.4						0.5					
X	FDY	Model1	Hull	Model2	Stair	MC	FDY	Model1	Hull	Model2	Stair	MC
95	*16.303	*16.336	17.090	17.112	16.806	16.836	*19.931	*19.969	*20.901	*20.937	20.568	20.549
100	*14.250	*14.270	*15.044	*15.048	14.733	14.733	*18.001	*18.003	*18.959	*18.971	18.583	18.621
105	*12.433	*12.439	*13.222	*13.206	12.904	12.883	*16.228	*16.222	*17.194	*17.182	16.826	16.829

The numerical settings are the same as those settings in Table 2 except that a 2.5-dollar-dividend is paid at year 0.4 and year 0.8. Option prices that deviate from MC by 0.3 are marked by asterisks.

Table 4: **Pricing a Call Option with Single Discrete Dividend.**

σ	X	Mix	Vol	Stair	MC
0.4	95	16.802	16.792	16.821	16.933
	100	14.737	14.732	14.758	14.754
	105	12.899	12.899	12.924	12.989
0.5	95	20.550	20.537	20.570	20.843
	100	18.584	18.578	18.591	18.584
	105	16.798	16.798	16.829	16.929
RMSE		0.147	0.152	0.130	
MAE		0.293	0.306	0.272	

The numerical settings are the same as those settings in Table 2. Mix the mixture approach of Bos and Vandermark (2002). Vol denotes the volatility adjustment approach of Bos and Shepeleva (2002). **RMSE** is the root-mean-squared errors. **MAE** is the maximum absolute error.

Note that MC in each two-discrete-dividend case in Table 3 is lower than that in the corresponding case (except one case) in Table 2. The stair lattice model successfully captures this trend, but all other models fail. Note that both Model 1 and Hull's volatility adjustment approach produce similar option prices in the single-discrete-dividend case and the two-discrete-dividend case. This is because the net-of-dividend stock price in the single-discrete-dividend case ($=100-5e^{-0.03\times0.6}$) is almost equal to that in the multiple-discrete-dividend case ($=100-2.5e^{-0.03\times0.4}-2.5e^{-0.03\times0.8}$). Model 2 also produces similar option prices in both cases since the cum-dividend stock prices for both cases are almost equal.

To derive approximation analytical formulas for Model 3, Bos and Vandermark (2002) present an approach (denoted as Mix) that is a mixture between the stock and the strike price adjustment or, in other words, Model 1 and Model 2. Bos and Shepeleva (2002) suggests that the volatility of the net-of-dividend stock price be adjusted by a complex formula. Use Vol to denote their approach. These two approaches and the stair lattice approach are compared in Table 4 and 5. Both the root-mean-squared errors and the maximum absolute errors of the stair lattice model are lower than those of these two approaches. Thus we conclude that the stair lattice provides more accurate values than the other two approaches. Note Model 3 (see MC) produces lower option prices in each two-discrete-dividend case in Table 4 than that in the corresponding case in Table 5. Bos and Vandermark's approach successfully catches this trend, but Bos and Shepeleva's approach fails.

Table 5: **Pricing a Call Option with Two Discrete Dividends.**

σ	X	Mix	Vol	Stair	MC
0.4	95	16.801	16.795	16.806	16.836
	100	14.736	14.734	14.733	14.733
	105	12.898	12.901	12.904	12.883
0.5	95	20.548	20.541	20.568	20.549
	100	18.583	18.581	18.583	18.621
	105	16.797	16.800	16.826	16.829
RMSE		0.026	0.027	0.023	
MAE		0.038	0.041	0.038	

The numerical settings are the same as those settings in Table 3. **RMSE** is the root-mean-squared errors. **MAE** is the maximum absolute error.

Table 6: **Pricing American Calls.**

		$1.0				$2.0				$3.0				$4.0			
X	T	GW	FDY	Stair	B	GW	FDY	Stair	B	GW	FDY	Stair	B	GW	FDY	Stair	B
35	1	*5.07	5.09	5.09	5.09	*5.05	5.09	5.09	5.08	*5.05	5.08	5.09	5.08	*5.05	5.08	5.09	5.08
	4	–	5.41	5.40	5.40	–	5.19	5.18	5.17	–	5.12	5.12	5.11	–	5.10	5.10	5.10
	7	–	*5.79	5.77	5.76	–	*5.29	5.26	5.24	–	*5.15	5.14	5.12	–	5.11	5.11	5.10
40	1	*1.14	1.17	1.17	1.17	*1.03	1.08	1.08	1.07	*1.03	1.04	1.04	1.04	*0.93	1.02	1.03	1.02
	4	–	2.38	2.40	2.39	–	1.91	1.93	1.92	–	1.60	1.60	1.58	–	1.39	1.40	1.38
	7	–	3.05	3.08	3.06	–	2.31	2.33	2.32	–	1.83	1.83	1.81	–	*1.51	*1.51	1.48
45	1	0.08	0.09	0.09	0.09	0.04	0.05	0.06	0.05	0.04	0.04	0.04	0.04	0.02	0.03	0.03	0.03
	4	–	0.87	0.88	0.88	–	0.62	0.64	0.64	–	0.44	0.46	0.46	–	0.31	0.33	0.32
	7	–	*1.47	1.51	1.50	–	*0.99	1.03	1.02	–	*0.66	0.70	0.69	–	*0.43	0.46	0.46

The initial stock price is 40, the risk-free interest rate is 5%, and the volatility is 30%. The ex-dividend dates for the stock are 0.5, 3.5, and 6.5 months. The dividends to be paid at each ex-dividend date are shown in the first row. The strike prices X are listed in the first column. The times to maturity T (in months) are in the second column. The values of American calls priced by the FDY model and the benchmark value are from Geske and Shastri (1985). (GW) denotes the analytical pricing formula of Geske (1979) and Whaley (1981). (Stair) denotes the stair lattice model with 140 time steps. Option prices which deviate from the benchmark values by 0.02 are marked by asterisks.

For American calls with discrete dividends, I compare the stair lattice with the popular analytical approximation formula of Geske (1979) and Whaley (1981) (abbreviated as GW) and the FDY model of Geske and and Shastri (1985) in Table 6. The parameters are from Cox et al. (1979). The benchmark option prices (B) are from Geske and Shastri (1985). Note that GW bases on Model 1 and thus underprices the options. GW can only price single-dividend cases because the multiple-dividend cases would have required GW to evaluate a multivariate cumulative normal density function, whose deterministic computational cost is prohibitive. This phenomenon is known as the curse of dimensionality (see Lyuu (2002)). Of course, even if the multivariate integral can be computed efficiently, there is no guarantee that the price is numerically accurate. Geske and Shastri (1985) claim that FDY model perform well for pricing American calls. Numerical results in Table 6 show that the stair lattice outperforms the FDY model.

5.2 Conclusions

Traditional approaches to pricing options on discrete-dividend-paying stocks either produce biased results or are inefficient. This project suggests a recombining lattice, the stair lattice, that solves the problem. Numerical results confirm that the stair lattice is both efficient and accurate. When compared with the existing methods, the stair lattice is found to be more efficient, more accurate, and/or more general. Moreover, the stair lattice can be extended so the a future dividend paid at time τ depends on the stock prices and the dividends up to time τ. This extension makes the stair lattice model more realistic and flexible.

6 Self Evaluation on this Project

This project provides a novel, efficient numerical model for pricing options on discrete-dividend-paying stocks. It seems that similar ideas can be applied to model the company asset's process when solving the credit risk problem with structural form. I plan to revise my paper and submit it to an academic journal. I will also study how to apply the stair lattice model to solve the credit risk problem in the future.

References

[1] Black, F. "Fact and Fantasy in the Use of Options." *Financial Analysts Journal*, 31 (1975), pp. 36–41, 61–72.

[2] Black, F., and M. Scholes. "The Pricing of Options and Corporate Liabilities." *Journal of Political Economy*, 81 (1973), pp. 637–659.

[3] Bos, M., and Vandermark, S. (2002) Finessing Fixed Dividends, *Risk*, 157–158.

[4] Bos, M., and Shepeleva, A. (2002) Dealing with Discrete Dividends, *Risk*, 109–112.

[5] Carr, P. "Randomization and the American Put." *The Review of Financial Studies*, 11 (1998), pp. 597–626.

[6] Chance, D.M., R. Kumar, and D. Rich. "European Option Pricing with Discrete Stochastic Dividends." *Journal of Derivatives*, 9 (2002), pp. 39–45.

[7] Chiras, D.P., and S. Manaster. "The Informational Content of Option Prices and a Test of Market Efficiency." *Journal of Financial Economics*, 6 (1978), pp. 213–234.

[8] Cox, J., S. Ross, and M. Rubinstein. "Option Pricing: a Simplified Approach." *Journal of Financial Economics*, 7 (1979), pp. 229–264.

[9] Cox, J.C., and M. Rubinstein. *Options Markets.* Englewood Cliffs, NJ: Prentice-Hall, 1985.

[10] Dai, T.-S., and Y.-D. Lyuu. “An Exact Subexponential-Time Lattice Algorithm for Asian Options.” In *Proceedings of the 15th Annual ACM-SIAM Symposium on Discrete Algorithms*, Philadelphia: Society for Industrial and Applied Mathematics, January 2004, pp. 703–710.

[11] Frishling, V. (2002) A Discrete Question, *Risk*, 115–6.

[12] Gaither, C. “High-Tech Firms Cut Stock Options.” *The Boston Globe*, May 22, 2003, p. D1.

[13] Geske, R. “A Note on an Analytical Valuation Formula for Unprotected American Call Options on Stocks with Known Dividends.” *Journal of Financial Economics*, 7 (1979), pp. 375–380.

[14] Geske, R., and K. Shastri. “Valuation by Approximation: a Comparison of Alternative Option Valuation Techniques.” *Journal of Financial and Quantitative Analysis*, 20 (1985), pp. 45–71.

[15] Harvey, C.R., and R.E. Whaley. “Dividends and S&P 100 Index Option Valuation.” *Journal of Futures Markets*, 12 (1992), pp. 123–137.

[16] Hull, J. *Options, Futures, and Other Derivatives.* 4th ed. Englewood Cliffs, NJ: Prentice-Hall, 2000.

[17] Linter, J. “Dividends, Earnings, Leverage, Stock Prices and the Supply if Capital to Corporations.” *Review of Economics and Statistics*, 64 (1962), pp. 243–269.

[18] Lyuu, Y.-D. *Financial Engineering & Computation: Principles, Mathematics, Algorithms.* Cambridge, U.K.: Cambridge University Press, 2002.

[19] Lyuu, Y.-D., and J.N. Wu. “On Accurate and Provably Efficient GARCH Option Pricing Algorithms.” *Quantitative Finance*, 5 (2005), pp. 181–198.

[20] Marsh, T.A., and Merton, R.C. “Dividend Behavior for the Aggregate Stock Market.” *Journal of Business*, 60 (1987), pp. 1–39.

[21] Merton, R.C. “Theory of Rational Option Pricing”. *Bell Journal of Economics and Management Science*, 4 (1973), pp. 141–183.

[22] Meyer, G.H. “Numerical Investigation of Early Exercise in American Puts with Discrete Dividends.” *Journal of Computational Finance*, 5 (2001/02), pp. 37–53.

[23] Milton, R.C. “Computer Evaluation of the Multivariate Normal Integral.” *Technometrics*, 14 (1972), pp. 811–889.

[24] Roll, R. “An Analytic Valuation Formula for Unprotected American Call Options on Stocks with Known Dividends.” *Journal of Financial Economics*, 5 (1977), pp. 251–258.

[25] Whaley, R.E. “On the Valuation of American Call Options on Stocks with Known Dividends.” *Journal of Financial Economics*, 9 (1981), pp. 207–211.

[26] Whaley, R.E. “Valuation of American Call Options on Dividend-Paying Stocks: Empirical Tests.” *Journal of Financial Economics*, 10 (1982), pp. 29–58.