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Müntz linear transforms of Brownian motion

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ABSTRACT

A class of Volterra kernels of Goursat type is considered. Calculations are shown to be more explicit for the class of Müntz transforms such as the kernels, the control of the order to be infinite and the ergodic properties.

Keywords: Brownian motion; canonical decomposition; enlargement of filtrations; ergodicity; Goursat kernels; Gramian matrix; Müntz polynomials; harmonic functions; self-reproducing kernels; strong mixing; reproducing Hilbert spaces; Volterra transform.

AMS 2000 subject classification: 26C05; 60J65.

1. Introduction and preliminaries

Let B be a standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P}_0)$ and denote by $\{\mathcal{F}_t^B, t \geq 0\}$ the natural filtration it generates. Take $f = (f_1, \cdots, f_n)^*$, where $*$ stands for the transpose operator, to be a fixed vector of $L^2_{loc}(\mathbb{R}_+)$ functions and n is some positive integer or $+\infty$. We know that there exists a unique Volterra transform

$$\begin{aligned} \Sigma : C\left(\mathbb{R}^+, \mathbb{R}\right) &\rightarrow C\left(\mathbb{R}^+, \mathbb{R}\right) \\ X &\rightarrow X - \int_0^{\cdot} \int_0^u k(u,v)\, dX(v) \end{aligned}$$

preserving the Wiener measure such that the orthogonal decomposition

$$\mathcal{F}_t^B = \mathcal{F}_t^{\Sigma(B)} \otimes \sigma\left(\int_0^t f(u)\, dB_u\right) \tag{1}$$

holds for any $t \geq 0$, where by $\mathcal{F} \otimes \mathcal{G}$ we mean $\mathcal{F} \vee \mathcal{G}$ with independence between $\mathcal{F}$ and $\mathcal{G}$. Because we are in the Gaussian setting we observe that if $H_t(X)$ stands for

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the closed linear hull of $\{X_s; s \leq t\}$ then the decomposition (1) is equivalent to

$$H_t(B) = H_t(\Sigma(B)) \oplus \operatorname{span}\left(\int_0^t f(u)\, dB_u\right)$$

where $\oplus$ stands for direct sum. It is known that the kernel k takes the form $k(t,s) = \phi^*(t).f(s)$, where $\phi(t) = \alpha_t.f(t)$ and α_t stands for the inverse of the covariance function M_t of the Gaussian vector $\int_0^t f(s)dB_s$.

In this paper we are interested in cases when the self-reproducing space is a Müntz Gaussian Hilbert space. That corresponds to $f_i(x) = x^{\lambda_i}$, $i = 1, 2, \ldots, n$, where $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ is a sequence of reals such that $\lambda_i \neq \lambda_j$ for $i \neq j$ and $\lambda_i > -1/2$, where n is a positive integer which may be $+\infty$. In this case the Gramian matrix $M_t = \int_0^t f(s) \cdot f^*(s)\, ds$ has entries

$$(2) \qquad (M_t)_{ij} = \frac{t^{\lambda_i+\lambda_j+1}}{\lambda_i+\lambda_j+1}, \quad i,j = 1, \cdots, n.$$

We shall demonstrate in Theorem 2.1 that, for $n < \infty$, the corresponding volterra transform Σ_n has the Müntz kernel

$$(3) \qquad k_n(t,s) = \frac{1}{t}\sum_{m=1}^{n}(2\lambda_m+1)(s/t)^{\lambda_m}\prod_{j=1, j\neq m}^{n}\frac{\lambda_m+\lambda_j+1}{\lambda_m-\lambda_j}.$$

Müntz kernels are homogeneous of degree -1 in the sense that for any sequence Λ and a positive integer n, we have $k_n(\alpha t, \alpha s) = \alpha^{-1}k_n(t,s)$, for any $\alpha > 0$. This implies that the associated Volterra transforms have a close connection to stationary processes. To explain the link let us introduce the isometry $U : L^2(\mathbb{R}_+, dx) \to L^2(\mathbb{R}, dx)$ defined by $U(f)(t) = e^{-t/2}f(e^t)$, $t \geq 0$, and set $\tilde{\Sigma} = U \circ \Sigma$. Now, Σ preserves the Wiener measure if and only if $\tilde{\Sigma}(B)$ is a stationary Brownian motion or Ornstein Uhlenbeck process. A necessary and sufficient condition for this to happen is that the spectral measure of $\tilde{\Sigma}(B)$ with respect to the Lebesgue measure is $(2/\pi)(1+4x^2)^{-1}$ on $\mathbb{R}$, see Lemma 1 in Jeulin-Yor [21]. We shall show that $\tilde{\Sigma}_n(B)$ has the moving average representation

$$(4) \qquad \tilde{\Sigma}_n(B)_t = \int_{-\infty}^{t} \eta_n(t-r)\, d\beta_r$$

where β is a Brownian motion indexed by $\mathbb{R}$, $\eta_n(t) = \mathbf{1}_{t>0}U \circ \rho_n(t)$ and

$$\rho_n(t) = 1 - \int_1^t k_n(r,1)\, dr.$$

The Fourier transform of η_n is given by

$$\hat{\eta}_n(\xi) = \frac{2}{1-2i\xi} \prod_{j=1}^{n} \frac{\xi - i(\frac{1}{2}+\lambda_j)}{\xi + i(\frac{1}{2}+\lambda_j)}. \tag{5}$$

Thanks to the characterization given in the pioneering Karahunen [24], a glance at eq. (17) allows to detect an inner factor which implies, as expected, that the representation (16) is noncanonical with respect to β.

A natural question is to know whether there exists Müntz transforms of infinite orders. A partial answer is given by Hibin-Muraoka [13]. They established the existence of infinite order kernels and showed that for a given infinite sequence Λ, the condition $\sup \lambda_j < \infty$ is a necessary condition for the existence of kernels with infinite order. Furthermore, they proved that if $0 < \lambda_1 < \lambda_2 < ...$ then there exists no corresponding Müntz transform of infinite order. We establish in Theorem 2.2 that the condition $\sum_{j=1}^{\infty}(2\lambda_j+1) < \infty$ is necessary and sufficient for the existence. Next, we need to introduce iteration notations $\Sigma^{(0)} = Id$, $\Sigma^{(1)} = \Sigma$ and $\Sigma^{(m)} = \Sigma^{(m-1)} \circ \Sigma$, for $m \geq 2$, where $\circ$ stands for the composition rule. Similarly we denote by $k^{(m)}$ the kernel corresponding to $\Sigma^{(m)}$. In Theorem 2.4 we shall shaw that the orthogonal decomposition

$$\mathcal{F}_t^B = \bigotimes_{m=0}^{\infty} \sigma\left(\int_0^t u^{\lambda_j}\, d\Sigma^{(m)}(B)_u, 1 \leq j \leq m\right) \tag{6}$$

holds for any $t \geq 0$. As a consequence, we conclude that Müntz transforms are strongly mixing.

2. Müntz Gaussian Hilbert spaces and transforms

Let $\Lambda = \{\lambda_1, \lambda_2, \cdots\}$ be a sequence of reals such that $\lambda_i \neq \lambda_j$ for $i \neq j$. Consider the set of functions $\{x^{\lambda_1}, x^{\lambda_2}, \cdots\}$ defined on $[0,1]$. A Müntz polynomial is an element of $\mathrm{span}\{x^{\lambda_1}, \cdots, x^{\lambda_n}\}$ for some $n \in \mathbb{N}$. The latter span the linear spaces

$$M_n(\Lambda) = \mathrm{span}\{x^{\lambda_1}, \cdots, x^{\lambda_n}; x \in [0,1]\}$$

and

$$M(\Lambda) = \bigcup_{n=1}^{\infty} M_n(\Lambda) = \mathrm{span}\{x^{\lambda_1}, x^{\lambda_2}, \cdots; x \in [0,1]\}$$

usually called Müntz spaces. For convenience, we assume that $-1/2 < \lambda_i$, $i = 1, 2, ...$, so that the above polynomials lie in $L^2([0,1])$. Recall that $M(\Lambda)$ is dense in

$L^2([0,1])$ if and only if

$$\sum_{i=1}^{\infty} \frac{(2\lambda_i+1)}{(2\lambda_i+1)^2+1} = \infty \tag{7}$$

This result is known as Müntz Theorem, see Borwein-Erdélyi [6]. Now, the procedure of Gram-Schmidt allows to construct an orthogonal basis, denoted by $\{L_k^{(\Lambda)}(x); 1 \leq k \leq n\}$, for the Hilbert space $M_n(\Lambda)$. That is specified by

$$L_n^{(\Lambda)}(s) = \sum_{k=1}^{n} c_{k,n} s^{\lambda_k}, \quad c_{k,n} = \frac{\prod_{j=1}^{n-1}(\lambda_k+\lambda_j+1)}{\prod_{j=1, j\neq k}^{n}(\lambda_k-\lambda_j)}. \tag{8}$$

These generalized polynomials are usually called Müntz-Legendre polynomials. For related topics we refer to Borwein-Erdelyi [6] and Borowein-Erdelyi-Zhang [7]. Next, to the linear spaces $M_n(\Lambda)$ and $M(\Lambda)$ we associate, respectively, the Müntz Gaussian spaces

$$M_n^G(\Lambda) = \left\{ \int_0^1 p(s)\, dB_s, p \in M_n(\Lambda) \right\} \quad \text{and} \quad M_\infty^G(\Lambda) = \left\{ \int_0^1 p(s) dB_s, p \in M(\Lambda) \right\}.$$

Introduce the first Wiener chaos of B on $[0,1]$ specified by

$$\Gamma = \left\{ \int_0^1 f(u)\, dB_u : f \in L^2([0,1]) \right\}.$$

The orthogonal spaces of $M_n^G(\Lambda)$ and $M_\infty^G(\Lambda)$ in Γ are respectively given by

$$M_n^{G,\perp}(\Lambda) = \left\{ \int_0^1 f(u)\, dB_u : f \in L^2([0,1]), \int_0^1 f(s)p(s)\, ds = 0, p \in M_n(\Lambda) \right\}$$

and

$$M_\infty^{G,\perp}(\Lambda) = \left\{ \int_0^1 f(u)\, dB_u : f \in L^2([0,1]), \int_0^1 f(s)p(s)\, ds = 0, p \in M(\Lambda) \right\}.$$

By Müntz Theorem if the condition (7) holds then $M^G(\Lambda)$ is total in Γ which implies that $M^{G,\perp}(\Lambda) = \{0\}$. Otherwise, $M^{G,\perp}(\Lambda)$ is a nontrivial set. Now, it is natural to give the following definition.

Definition 2.1. A Müntz transform is a Goursat-Volterra transform having a Müntz-Gaussian space as a reproducing space. Its order is defined to be the dimension of the reproducing space. We call the associated Goursat-Volterra kernel a Müntz kernel.

We are now ready to state the following result.

Theorem 2.1. *Let $\lambda_1, \lambda_2, \cdots$ be a sequence of reals such that $\lambda_i \neq \lambda_j$ for $i \neq j$ and $\lambda_i > -\frac{1}{2}$ for $i = 1, 2, \cdots$. For a fixed $n < \infty$, the kernel $k_n(t,s) = t^{-1}k_n(1, s/t)$ where*

$$k_n(1,s) = \sum_{j=1}^{n} a_{j,n} s^{\lambda_j}, \quad a_{j,n} = \frac{\prod_{i=1}^{n}(\lambda_i + \lambda_j + 1)}{\prod_{i=1, i\neq j}^{n}(\lambda_i - \lambda_j)}, \quad j = 1, ..., n, \tag{9}$$

is a Müntz kernel of order n. Its Gaussian self-reproducing space is $M_n^G(\Lambda)$.

Proof. We set $k_n \equiv k$ and assert that $k_n(t,s) = t^{-1}\sum_{j=1}^{n} a_{j,n}(s/t)^{\lambda_j}$ for some sequence of reals $a_{1,n}, a_{2,n}, ..., a_{n,n}$. The self-reproducing condition implies the linear system

$$\sum_{i=1}^{n} \frac{a_{i,n}}{\lambda_i + \lambda_j + 1} = 1, \qquad j = 1, \cdots, n. \tag{10}$$

To solve the latter let us consider the n-degree polynomial

$$p(x) = \prod_{j=1}^{n}(x + \lambda_j + 1) - \sum_{i=1}^{n} a_{i,n} \prod_{j=1, j\neq i}^{n} (x + \lambda_j + 1) \tag{11}$$

which has at most n roots. We observe the equivalence

$$p(x) = 0 \qquad \Longleftrightarrow \qquad \sum_{i=1}^{n} \frac{a_{i,n}}{x + \lambda_i + 1} = 1$$

which readily implies $p(x) = \prod_{j=1}^{n}(x - \lambda_j)$. Next, choose $m \in \{1, \cdots, n\}$ and substitute $p(x)$ in eq. (11). Dividing then both sides by $\prod_{j\neq m}(x + \lambda_j + 1)$ and rearranging terms we get

$$x + \lambda_m + 1 - \sum_{i=1}^{n} a_{i,n} \frac{x + \lambda_m + 1}{x + \lambda_i + 1} = (x - \lambda_m) \prod_{j=1, j\neq m}^{n} \frac{x - \lambda_j}{x + \lambda_j + 1}.$$

By letting x go to $-(\lambda_m + 1)$ we obtain eq. (9). It remains to determine the corresponding reproducing Gaussian Hilbert space associated to the transform Σ with the kernel k_n. But that is equivalent to solving $f(t) = \int_0^t k_n(t,u) f(u)\, du$, $t \geq 0$, that is easily seen to be an ordinary linear differential equation of degree n. Because the functions t^{λ_j}, $j = 1, \cdots, n$, are n linearly independent solutions, we conclude that $\Gamma^{(k_n)}$ coincides with $M_n^G(\Lambda)$. This completes the proof. □

Going back to the Gramian matrix $(m_t, t \geq 0)$, observe that, in this setting, it has the entries $(m_t)_{ij} = (\lambda_i + \lambda_j + 1)^{-1} t^{\lambda_i + \lambda_j + 1}$, for $i, j = 1, \cdots, n$. Thus, M_1 is a Cauchy matrix and when $\lambda_i = ci$, for some constant $c \neq 0$, and $n = \infty$, M_1 is the well-known Hilbert matrix. Note that because $||f_i|| = +\infty$, $i = 1, \cdots, n$, we have $\alpha_\infty \equiv 0$. The remaining ingredients ϕ and $\alpha_.$ are given in the following result.

Corollary 2.1. *We have* $\phi_i(t) = a_{i,n} t^{-\lambda_i - 1}$, $i = 1, 2, \cdots, n$. *Furthermore, the entries of* α_t *are given by* $(\alpha_t)_{i,j} = a_{i,n} a_{j,n} (\lambda_i + \lambda_j + 1)^{-1} t^{-\lambda_i - \lambda_j - 1}$.

Proof. This follows from the expression of the kernel given in Theorem 2.1. $\square$

Remark 2.1. Another way to solve (10) is to use the obvious decomposition

$$\prod_{j=1}^{n} \frac{x + \lambda_j + 1}{x - \lambda_j} = 1 + \sum_{j=1}^{n} \frac{a_{j,n}}{x - \lambda_j}, \quad x \neq \lambda_j, j = 1, 2, \cdots, n.$$

Replacing x by $-x + 1$, the latter can be rewritten as

$$\prod_{j=1}^{n} \frac{x - \lambda_j}{x + \lambda_j + 1} = 1 - \sum_{j=1}^{n} \frac{a_{j,n}}{x + \lambda_j + 1}, \quad x \neq -\lambda_j - 1, j = 1, 2, \cdots, n.$$

These are to distinguish from the decomposition

$$\prod_{j=1}^{n} \frac{1}{\lambda_j - x} = \sum_{j=1}^{n} \frac{b_{j,n}}{\lambda_j - x}, \quad b_{j,n} = \prod_{i=1, i \neq j}^{n} \frac{1}{\lambda_i - \lambda_j}, \quad x \neq \lambda_j, j = 1, 2, \cdots, n,$$

usually used to invert the moment generating function of the sum of n independent exponentially distributed random variables with different means $\lambda_1, \lambda_2, \cdots, \lambda_n$.

Remark 2.2. $k_n(.)$ is a generalized Müntz polynomilal. Its value at 0 is finite only when $\lambda_k \geq 0$ for $k = 1, \cdots, n$. In that case, if there exists k_0 such that $\lambda_{k_0} = 0$ then $k_n(0) = a_{k_0,n}$ if not then $k_n(0) = 0$.

The next result is devoted to some properties of the studied kernels. First, we shall show that Müntz kernels can be expressed in terms of Müntz-Legendre polynomials and then exploit these facts. We write $k_n(.)$ for $k_n(1, .)$.

Theorem 2.2. *Hold the following assertions.*

1) *For any fixed positive integer n, we have*

$$k_n(1, x) = x^{-\lambda_n} \frac{\partial}{\partial x} \left(x^{\lambda_n + 1} L_n^{\Lambda}(x) \right) \quad or \quad L_n^{\Lambda}(x) = x^{-\lambda_n - 1} \int_0^x s^{\lambda_n} k_n(1, s)\, ds,$$

for $x \in \mathbb{R}$. Note that unlike Müntz-Legendre polynomials, the kernel k_n does not depend of the order of $\lambda_1, \lambda_2, \cdots, \lambda_n$.

2) *The sequence $k_.$ satisfies the differential-difference equations*

$$x \left(k_n'(x) - k_{n-1}'(x) \right) = (\lambda_n + 1) k_{n-1}(x) + \lambda_n k_n(x). \tag{12}$$

Proof. 1) For a fixed $n \geq 1$ observe that $(\lambda_j + \lambda_n + 1)c_{j,n} = a_{j,n}$ for any $j \leq n$. gives The first assertion follows by integration. 2) We know that

$$\begin{aligned} x^{\lambda_n+\lambda_{n-1}+1}\left(x^{-\lambda_n}L_n(x)\right)' &= \left(x^{\lambda_{n-1}+1}L_{n-1}(x)\right)' \\ &= x^{\lambda_{n-1}}k_{n-1}(x). \end{aligned}$$

Now, we have

$$\begin{aligned} \left(x^{-\lambda_n}L_n(x)\right)' &= x^{-\lambda_n-1}K_n(x) - (2\lambda_n+1)x^{-\lambda_n-1}L_n(x) \\ &= x^{-\lambda_n-1}k_n(x) - (2\lambda_n+1)x^{-2\lambda_n-2}\int_0^x s^{\lambda_n}k_n(s)\,ds. \\ &= x^{-\lambda_n-1}k_{n-1}(x). \end{aligned}$$

We extract from the last equality

$$x^{\lambda_n+1}k_n(x) - (2\lambda_n+1)\int_0^x x^{\lambda_n}k_n(s)\,ds = x^{\lambda_n+1}k_{n-1}.$$

Differentiating and simplifying yields

$$-\lambda_n k_n(x) + xk_n'(x) = (\lambda_n+1)k_{n-1} + xk_{n-1}'.$$

□

The latter being well studied a number of their properties can be translated to new properties for Müntz kernels. Next, we continue discussing a question tackled by Hibino-Muraoka [13] consisting on determining the cases when k_n converges as $n \to \infty$ to a Müntz kernel.

Theorem 2.3. *The sequence k_n converges, as n tend to $+\infty$, to a Müntz kernel if and only if $\sum_{j=1}^{\infty}(2\lambda_j+1) < \infty$.*

Proof. Suppose that k is a self-reproducing kernel of the form $k(t,s) = t^{-1}\Sigma_1^{\infty}a_j(s/t)^{\lambda_j}$ for some sequence of reals a. Then it is necessary for a to satisfy the system

$$\sum_{i=1}^{\infty}\frac{a_i}{\lambda_i+\lambda_j+1} = 1, \qquad \text{for all } j \text{ with } a_j \neq 0. \tag{13}$$

First, assume that Λ is such that $(t^{\lambda_n})_{n\geq 1}$ is total in $L^2([0,1])$. Then

$$\int_0^t k(t,u)u^{\lambda_i}\,du = \int_0^t \frac{1}{t}\sum_{j=1}^{\infty}a_j\,(u/t)^{\lambda_j}\,u^{\lambda_i}\,du = t^{\lambda_i}$$

which implies that $\int_0^t u^{\lambda_i}\,dB_u \in \Gamma_t^{(k)}$ for all $\lambda_i \in \Lambda$ and $t \in [0,1]$. Thus, $\sigma\left(\Gamma_\cdot^{(k)}\right) = \mathcal{F}_\cdot^B$ on $[0,1]$ because of the totality property. Hence, $\Sigma(B) = 0$ which is an obvious contradiction. Next, consider the case where $(t^{\lambda_n})_{n\geq 1}$ is not total in $L^2([0,1])$ which is equivalent to

$$\sum_{i=1}^{\infty} \frac{(2\lambda_i+1)}{(2\lambda_i+1)^2+1} < \infty. \tag{14}$$

For the above series to converge it is necessary to have either $\lambda_n \to \infty$ or $2\lambda_n+1 \to 0$ as $n \to \infty$. First, assume that $\lambda_n \to \infty$ and the system (13) has a solution a with $a_j \neq 0$ for infinitely many j's. Splitting $a_i = a_i^+ - a_i^-$ where $a_i^+ := \max\{a_i, 0\}$ and $a_i^- := \max\{-a_i, 0\}$ we can rewrite eq. (13) as

$$\sum_{i=1}^{\infty} \frac{a_i^+}{\lambda_i+\lambda_j+1} = 1 + \sum_{i=1}^{\infty} \frac{a_i^-}{\lambda_i+\lambda_j+1} \geq 1, \quad j \text{ s.t. } a_j \neq 0. \tag{15}$$

Making use of eq. (14), given that a is bounded, we get

$$\sum_{i=1}^{\infty} \frac{a_i^+}{\lambda_i+\lambda_{j^*}+1} = C_{j^*} < \infty, \qquad \text{for some fixed } j^*.$$

Hence, for any given $\varepsilon > 0$, there exists an integer m^* large enough such that $\sum_{i=m^*+1}^{\infty} a_i^+(\lambda_i+\lambda_{j^*}+1)^{-1} < \varepsilon$ which implies that $\sum_{i=1}^{m^*} a_i^+(\lambda_i+\lambda_{j^*}+1)^{-1} < C_{j^*}$. Because $\lim_{n\to\infty} \lambda_n = \infty$, there exists an integer n^* (larger than j^*) such that

$$\sum_{i=1}^{m^*} \frac{a_i^+}{\lambda_i+\lambda_{n^*}+1} < \frac{\lambda_{m^*}+\lambda_{j^*}+1}{\lambda_{m^*}+\lambda_{n^*}+1} \sum_{i=1}^{m^*} \frac{a_i^+}{\lambda_i+\lambda_{j^*}+1} < 1-\varepsilon.$$

Hence for $j = n^*$, the left-hand side of (15) gives

$$\begin{aligned} \sum_{i=1}^{\infty} \frac{a_i^+}{\lambda_i+\lambda_{n^*}+1} &\leq \sum_{i=1}^{m^*} \frac{a_i^+}{\lambda_i+\lambda_{n^*}+1} + \sum_{i=m^*+1}^{\infty} \frac{a_i^+}{\lambda_i+\lambda_{j^*}+1} \\ &< (1-\varepsilon)+\varepsilon = 1 \end{aligned}$$

which contradicts the inequality of eq. (15). It remains to consider the case where eq. (14) holds with $2\lambda_n+1 \to 0$. Note that the combination of the latter conditions is equivalent to saying that $\sum_{j=1}^{\infty}(2\lambda_j+1) < \infty$. We need to recall and use the link to stationary processes. The kernel $k_n(.,.)$ is homogeneous of degree -1 in the sense that $k_n(\alpha t, \alpha s) = \alpha^{-1} k_n(t,s)$ for $\alpha > 0$. Hence

$$\begin{aligned} \Sigma_n(B)_t &= B - \int_0^t \int_0^u k_n(u,v)\, dB_v\, du \\ &= \int_0^t \left(1 - \int_v^t k_n(u,v) du\right) dB_v \\ &= \int_0^t \left(1 - \int_1^{t/v} k_n(vr,v) v\, dr\right) dB_v \\ &= \int_0^t \rho_n(t/v)\, dB_v, \quad t \geq 0, \end{aligned}$$

where

$$\rho_n(x) = 1 - \int_1^x k_n(r,1)\, dr.$$

Thus, with $\eta_n(t) = \mathbf{1}_{t>0} U \circ \rho_n(t)$, we have the moving average representation

$$\tilde{\Sigma}_n(B)_t = \int_{-\infty}^t \eta_n(t-r)\, d\beta_r \tag{16}$$

where β is a Brownian motion indexed by $\mathbb{R}$. The Fourier transform of η_n is given by

$$\hat{\eta}_n(\xi) = \frac{2}{1 - 2i\xi} \prod_{j=1}^n \frac{\xi - i(\frac{1}{2} + \lambda_j)}{\xi + i(\frac{1}{2} + \lambda_j)} \tag{17}$$

and the above condition on Λ insures the convergence of $\hat{\eta}_n(\xi)$ to $\hat{\eta}_\infty(\xi)$ say. It follows that k_n also converges as $n \to \infty$ to a Müntz kernel.

□

We pursue our discussion by studying the ergodic properties of the above transforms for which we need some preparatory lemma.

Lemma 2.1. *We have*

$$\int_0^1 L_i^\Lambda(u)\, d\Sigma^{(n)}(B)_u = \sum_{j=1}^i \sum_{k=0}^n c_{j,i} a_{i,k}^{(n)} \int_0^1 u^{\lambda_j} (\log u)^k\, dB_u, \quad n \geq 1,$$

where $(a_{j,k}^{(n)})$ is some matrix of reals, holds true. Furthermore, the collection $\{x^{\lambda_i}(\log x)^k; 1 \leq i \leq n, k = 0, 1, \cdots\}$ is total in $L^2([0,1])$.

Proof. By definition we have the system

$$d\Sigma^{(m)}(B)_t = dB_t - \int_0^t k^{(m)}(t,u)\, dB_u\, dt, \quad m = 1, 2, \cdots.$$

Now, by using mathematical induction the iterated sequence of kernels $k^{(\cdot)}$ satisfies

$$k^{(p)}(t,s) = k^{(q)}(t,s) + k^{(p-q)}(t,s) - \int_s^t k^{(q)}(t,u)k^{(p-q)}(u,s)\,du \tag{18}$$

for $1 \leq q < p$. By using again induction and taking into account of the first step $k^{(1)} \equiv k_n$, we obtain

$$k^{(m)}(t,s) = \sum_{i=1}^{n} t^{-\lambda_i - 1} s^{\lambda_i} \left(\sum_{p=0}^{m-1} \sum_{k=0}^{p} a_{i,k,p}^{(m)} (\log t)^k (\log s)^{p-k} \right)$$

for some real-valued coefficients $(a_{i,k,p}^{(m)})$. Substituting this result into

$$\int_0^1 L_i^\Lambda(u)\,d\Sigma^{(m)}(B)_u = \sum_{j=1}^{i} c_{j,i} \int_0^1 u^{\lambda_j}\,d\Sigma^{(m)}(B)_u$$

we obtain the first part of the Lemma. Now, let g be a square-integrable function on $[0,1]$ satisfying the system $\int_0^1 g(u)u^{\lambda_i}(\log u)^k\,du = 0$, for $1 \leq i \leq n$ and $k \geq 0$. An obvious change of variables yields that, for any positive integer k, we should have

$$(-1)^k \int_0^\infty \left(g(e^{-v})e^{-v\lambda_i} \right) v^k e^{-v}\,dv = 0.$$

Because $\{v^k; k = 0, 1, \cdots\}$ is total with respect to the measure $e^{-v}dv$ we deduce that g must be identical to 0. □

Finally, we are ready to state the following.

Theorem 2.4. *Müntz transforms of finite order are strongly mixing and a fortiori ergodic. Moreover, the orthogonal decomposition (6) holds true.*

Proof. Let Σ be a Müntz transform of order n for some natural number n. Consider the associated sequence of Müntz polynomials $f = \left\{t^{\lambda_i}; 1 \leq i \leq n\right\}$ for some sequence $\Lambda = (\lambda_1, \cdots, \lambda_n)$. For a fixed $m > 1$ we can write

$$\mathcal{F}_t^B = \mathcal{F}_t^{\Sigma^{(m+1)}(B)} \otimes \sigma\left(\int_0^t f(u)\,dB_u\right) \otimes \cdots \otimes \sigma\left(\int_0^t f(u)\,d\Sigma^{(m)}(B)_u\right).$$

Applying Gram-Schmidt procedure yields

$$\begin{aligned} \mathcal{F}_t^B &= \mathcal{F}_t^{\Sigma^{(m+1)}(B)} \otimes \sigma\left(\int_0^t L_i^\Lambda(u)\,dB_u; 1 \leq i \leq m\right) \\ &\qquad \otimes \cdots \otimes \sigma\left(\int_0^t L_i^\Lambda(u)\,d\Sigma^{(n)}(B)_u; 1 \leq i \leq m\right). \end{aligned}$$

Clearly the two families $\left\{x^{\lambda_i}(\log x)^k : 1 \le i \le n, k \ge 0\right\}$ and

$$\left\{\sum_{j=1}^{i}\sum_{p=0}^{K} c_{j,i} a_{i,p}^{(m)} x^{\lambda_j} (\log x)^p; 1 \le i \le n, m \ge 0\right\}$$

span the same space. From Lemma 2.1 we know that the latter is total on $L^2([0,1])$. By letting m tend to ∞ and applying Lemma 2.1 we obtain both eq. (6) and the strongly mixing property of Σ. $\square$

Remark 2.3. The simplest known example of Müntz kernels is $k_1(t,s) = t^{-1}$. For this we have $\Sigma(B_.) = B_. - \int_0^{\cdot} \frac{B_u}{u}\, du$. Furthermore, $\Sigma^{(n)}(B) = \int_0^{\cdot} L_n(\log \frac{t}{s})\, dB_s$ where L_n is the sequence of Laguerre polynomials. This being a basis of the Hilbert space $L^2\left(\mathbb{R}^+, e^{-x}dx\right)$ we get the orthogonal decomposition

$$\mathcal{F}_t^B = \bigotimes_{n=0}^{\infty} \sigma\left(\Sigma^{(n)}(B)_t\right), \quad t \ge 0,$$

see Jeulin-Yor [20].

Remark 2.4. As a by-product of the discussion for the order to be infinite we mention the following result. Let ϕ be a $C^\infty([0,1])$ function satisfying $|\phi^{(k)}| \le$ constant, for all k. Then ϕ is a solution of the integral equation $\phi(u) = \int_0^1 \phi(uv)\phi(v)dv$, defined on $[0,1]$, if and only if $\phi(.) = K_n(1,.)$ where $\lambda_j = j$ for $j \ge 0$ and n is some finite positive integer.

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