PAPER

Applying Output Feedback Integral Sliding Mode Controller to Time-Delay Systems

Huan-Chan TING†, *Nonmember*, **Jeang-Lin CHANG**††a), *and* **Yon-Ping CHEN**†, *Members*

SUMMARY For time-delay systems with mismatched disturbances and uncertainties, this paper developed an integral sliding mode control algorithm using output information only to stabilize the system. An integral sliding surface is comprised of output vectors and an auxiliary full-order compensator. The proposed output feedback sliding mode controller can satisfy the reaching and sliding condition and maintain the system on the sliding surface from the initial moment. When the specific linear matrix inequality has a solution, our method can guarantee the stability of the closed-loop system and satisfy the property of disturbance attenuation. Moreover, the design parameters of the controller and compensator can be simultaneously determined by the solution to the linear matrix inequality. Finally, a numerical example illustrated the applicability of the proposed scheme.
key words: *output feedback, full-order compensator, sliding mode, LMI*

1. Introduction

Sliding mode control method [1]–[12] is one of efficient robust control approaches to stabilize systems in the presence of external disturbances and interior uncertainties and is well known for the complete robustness to matched disturbances and uncertainties. Its simple and explicit design procedure is another advantage. Based on Utkin's research [1], the relative papers are continuing presented over the decade. Edward and Spurgeon [2] contributed the analysis and complete introduction of sliding mode methods, state and output feedback controllers, observers, and other applications. When the system states are all available, researchers have proposed many significant reports [3]–[6]. Chiang and Chiu [5] presented a novel TS recurrent fuzzy neural network based sliding mode control to stabilize the time-delay systems and compensate system uncertainties effectively. In the field of output feedback sliding mode control methods [7]–[12], many contributions stabilizing the disturbed and uncertain systems are published. In 2004, Niu et al. [9] extended an observer-based sliding mode control using LMI technique to regulate uncertain time-delay systems. Yan et al. [10] applied an effective sliding mode design technique using output only to control the time-delay systems with disturbances.

Adopting the output feedback sliding mode approach, this paper designs a set of control algorithm of time-delay systems with unknown disturbances and uncertainties. Time delay phenomenon exists in states, inputs, and outputs of many real systems, such as chemical process, electrical networks, nuclear reactors, biological systems, and economic models, etc. Because delay terms usually cause the system's instability and worse performance, the control of time-delay systems [3], [4], [9]–[19] has been one of the most popular issues recently. Focusing on systems with state delay terms, Fridman and Shaked [13] presented an explicit and significant H_∞ control method using the descriptor system transformation for time-delay systems with mismatched external disturbances and measurement noises. The descriptor system transformation can simplify the analysis of time-delay systems and effectively perform the disturbance attenuation.

There exists a synthesis problem in the design of output feedback sliding mode control for time-delay systems. Synthesizing a control law using the outputs only is significant since the derivative of the sliding surface always involves the unmeasured system states. For resolving this problem, a normal strategy is to add an extra constraint on the designed controller. Unfortunately, the strategy maybe has no solutions to controller parameters due to this extra constraint. In this paper, an output feedback integral sliding mode controller combining with a full-order compensator is proposed to improve the synthesis problem for time-delay systems with mismatched disturbances and uncertainties. The novel features of the proposed method are that the controller design involves no extra condition, and the parameters of controller and compensator as well as the integral sliding surface are all found from an LMI. Although mismatched disturbances cannot be eliminated completely even the system is in the sliding mode, the disturbance attenuation technique [20]–[22] can robustly reduce the effect of disturbances acting on a system to an acceptable level. The disturbance attenuation involving H_∞ robust control method is a successful strategy to minimize the gain from external disturbances to the controlled output over all frequencies. It also helps the proposed controller guarantee the robust stability of the closed-loop system and accomplish the property of disturbance attenuation.

This paper is organized as follows. The description of time-delay systems and the problem formulation are given in Sect. 2. Section 3 is divided into three parts. In Sect. 3.1, the integral sliding surface is designed and the output feedback sliding mode control law is used to satisfy the reaching and sliding condition. Section 3.2 designs the full-order compensator and Sect. 3.3 presents that solution an LMI guarantee the robust stability once the system is in the slid-

Manuscript received April 22, 2010.
†The authors are with the Dep. Electrical & Control Eng., NCTU, Hsinchu, 300 Taiwan.
††The author is with the Dep. Electrical Eng., OIT, Banchiao, Taipei County, 220 Taiwan.
a) E-mail: jlchang@ee.oit.edu.tw
DOI: 10.1587/transfun.E94.A.1051

ing mode, which detailed proof is shown in Appendix. The feasibility of the proposed method is illustrated in Sect. 4 with a numerical example. Conclusions are given in Sect. 5.

2. Problem Formulation

Consider a continuous-time time-delay system described by the state-space form as

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= (\boldsymbol{A} + \Delta\boldsymbol{A}(t))\,\boldsymbol{x}(t) + (\boldsymbol{A}_d + \Delta\boldsymbol{A}_d(t))\,\boldsymbol{x}(t-\tau)\\ &\quad + \boldsymbol{B}(\boldsymbol{u}(t) + \boldsymbol{f}(\boldsymbol{x},\boldsymbol{u},t)) + \boldsymbol{E}\boldsymbol{d}(t)\\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t)\\ \boldsymbol{x}(t) &= \phi(t),\ t \in [-\tau, 0] \qquad (1)\end{aligned}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the system state vector, $\boldsymbol{y} \in \mathbb{R}^l$ is the system output vector, $\boldsymbol{u} \in \mathbb{R}^m$ is the control input vector, and $\boldsymbol{d} \in \mathbb{R}^p$ is the mismatched disturbance vector. The function $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u},t) \in \mathbb{R}^m$ represents the unknown matched uncertainty. The constant τ is an unknown delay time but bounded by a known constant τ^*, where $\tau \leq \tau^*$. The vector $\phi(t)$ is a continuous initial function. The real constant matrices $\boldsymbol{A}$, $\boldsymbol{A}_d$, $\boldsymbol{B}$, $\boldsymbol{E}$, and $\boldsymbol{C}$ are known and have appropriate dimensions with $l \geq m$. The structure uncertainties $\Delta\boldsymbol{A}(t)$ and $\Delta\boldsymbol{A}_d(t)$ satisfy $\Delta\boldsymbol{A} = \boldsymbol{D}\Phi(t)\boldsymbol{H}$ and $\Delta\boldsymbol{A}_d = \boldsymbol{D}_d\Phi_d(t)\boldsymbol{H}_d$, where $\boldsymbol{D}$, $\boldsymbol{D}_d$, $\boldsymbol{H}$, and $\boldsymbol{H}_d$ are non-unique known constant matrices with appropriate dimensions. Moreover, the matrices $\Phi(t)$ and $\Phi_d(t)$ are unknown, satisfying $\Phi^T(t)\Phi(t) \leq \boldsymbol{I}$ and $\Phi_d^T(t)\Phi_d(t) \leq \boldsymbol{I}$ for all t, respectively. The controlled plant (1) can be rewritten as

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= (\boldsymbol{A} + \boldsymbol{D}\Phi(t)\boldsymbol{H})\,\boldsymbol{x}(t) + \boldsymbol{B}(\boldsymbol{u}(t) + \boldsymbol{f}(\boldsymbol{x},\boldsymbol{u},t))\\ &\quad + (\boldsymbol{A}_d + \boldsymbol{D}_d\Phi_d(t)\boldsymbol{H}_d)\,\boldsymbol{x}(t-\tau) + \boldsymbol{E}\boldsymbol{d}(t)\\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t). \qquad (2)\end{aligned}$$

Suppose that the triple $(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C})$ is completely controllable and observable. Spurgeon and Edwards [2] have shown that there exists a stable static output feedback sliding mode controller if

(C1) rank $(\boldsymbol{CB})$ = rank $(\boldsymbol{B}) = m$,

(C2) The triple $(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C})$ is minimum phase.

In the case of time-delay systems satisfying the conditions (C1) and (C2), Castanos and Fridman [7] mentioned the state-dependent integral sliding surface design for linear systems with mismatched disturbances to ensure the robust disturbance attenuation. Niu et al. [9] proposed the observer-based sliding mode controller involving a synthesis condition to stabilize uncertain time-delay systems. Since the output is the only available signal, this paper presents the output-dependent integral sliding surface applying the full-order compensator in which the proposed control algorithm can guarantee the performance bound of robust disturbance attenuation [20], [21] once the system is in the sliding mode. The control algorithm involved the information of outputs and the compensator is designed to satisfy the reaching and sliding condition and force the controlled system to the sliding mode without any synthesis condition. Before introducing the main results, the following three assumptions are fulfilled throughout this paper.

Assumption 1: The matched term $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u},t)$ and mismatched disturbance $\boldsymbol{d}(t)$ are norm-bounded as

$$\|\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u},t)\| \leq \eta(t,\boldsymbol{y}) + \chi\|\boldsymbol{u}(t)\| \text{ and } \|\boldsymbol{d}(t)\| \leq \bar{d}$$

where $0 < \chi < 1$, $\eta(t,\boldsymbol{y})$ and $\bar{d}$ are known positive constants, respectively. The symbol $\|\bullet\|$ denotes the 2-norm of $\bullet$.

Assumption 2: The triple $(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C})$ is minimum phase.

Assumption 3: rank $(\boldsymbol{CB})$ = rank $(\boldsymbol{B}) = m$.

3. Integral Sliding Mode Controller Design

In this section, the output feedback integral sliding mode controller is first proposed by employing the full-order compensator. Then the controller using the output only is proposed to force system (2) in the sliding mode from the initial moment. Once the system is in the sliding mode, the proposed algorithm can guarantee the stability of the closed-loop system and sustain the nature of disturbance attenuation when an LMI has the solution.

3.1 Integral Sliding Surface and Sliding Mode Controller

Since Assumption 3 holds, we design the output-dependant integral sliding surface as

$$\boldsymbol{s}(t) = (\boldsymbol{GCB})^{-1}\boldsymbol{G}(\boldsymbol{y}(t) - \boldsymbol{y}(0)) - \int_0^t \boldsymbol{v}(q)\,dq \qquad (3)$$

where $\boldsymbol{G} \in \mathbb{R}^{m\times l}$ is chosen such that $\boldsymbol{GCB}$ is invertible and $\boldsymbol{v} \in \mathbb{R}^m$ is designed in the latter. Substituting system (2) into the derivative of $\boldsymbol{s}(t)$ with respect to time can obtain

$$\begin{aligned}\dot{\boldsymbol{s}}(t) = &\ \boldsymbol{f}(\boldsymbol{x},\boldsymbol{u},t) - \boldsymbol{v}(t) + \bar{\boldsymbol{G}}((\boldsymbol{A} + \boldsymbol{D}\Phi(t)\boldsymbol{H})\,\boldsymbol{x}(t)\\ &\ \boldsymbol{u}(t) + (\boldsymbol{A}_d + \boldsymbol{D}_d\Phi_d(t)\boldsymbol{H}_d)\,\boldsymbol{x}(t-\tau) + \boldsymbol{E}\boldsymbol{d}(t)). \qquad (4)\end{aligned}$$

where $\bar{\boldsymbol{G}} = (\boldsymbol{GCB})^{-1}\boldsymbol{GC}$. Referring to [12], define two regions Ω_1 and Ω_2 as

$$\begin{aligned}\Omega_1 &:= \left\{\boldsymbol{x}(t) \,|\, \left\|\bar{\boldsymbol{G}}(\boldsymbol{A} + \boldsymbol{D}\Phi(t)\boldsymbol{H})\,\boldsymbol{x}(t)\right\| \leq \sigma_1\right\} \subset \Omega\\ \Omega_2 &:= \left\{\boldsymbol{x}(t-\tau) \,|\, \left\|\bar{\boldsymbol{G}}(\boldsymbol{A}_d + \boldsymbol{D}_d\Phi_d(t)\boldsymbol{H}_d)\,\boldsymbol{x}(t-\tau)\right\|\right.\\ &\quad \left.\leq \sigma_2\right\} \subset \Omega \qquad (5)\end{aligned}$$

where $\sigma_1 > 0$ and $\sigma_2 > 0$ are known and bounded constants, and the region $\Omega \subset \mathbb{R}^n$ is a neighborhood of the origin. Consider system (2) in $\Omega_1 \times \Omega_2$ and design the control input as

$$\boldsymbol{u}(t) = \boldsymbol{v}(t) - \kappa(t)\,\boldsymbol{s}(t)/\|\boldsymbol{s}(t)\| \qquad (6)$$

where σ_1 and σ_2 are positive constants and

$$\kappa(t) = \left(\sigma_1 + \sigma_2 + \eta(t,\boldsymbol{y}) + \chi\|\boldsymbol{v}(t)\| + \psi\bar{d} + \mu\right)/(1-\chi).$$

The remaining control parameters $\psi = \left\|\bar{\boldsymbol{G}}\boldsymbol{E}\right\|$ and μ are also positive constants. Through straightforward calculation, we know that

$$\kappa(t) = \chi\kappa(t) + \chi\|\boldsymbol{v}(t)\| + \sigma_1 + \sigma_2 + \eta(t,\boldsymbol{y}) + \psi\bar{d} + \mu$$

$$\geq \chi\|\boldsymbol{u}(t)\| + \sigma_1 + \sigma_2 + \eta(t,\boldsymbol{y}) + \psi\bar{d} + \mu.$$

Substituting (6) into (4) can attain the following approaching and sliding condition:

$$\begin{aligned}
&\boldsymbol{s}^T(t)\,\dot{\boldsymbol{s}}(t)\\
&\leq \|\boldsymbol{s}(t)\|\Big(\bar{\boldsymbol{G}}\left((\boldsymbol{A}+\boldsymbol{D}\Phi(t)\boldsymbol{H})\boldsymbol{x}(t)+\boldsymbol{E}\boldsymbol{d}(t)\right)\\
&\quad+(\boldsymbol{A}_d+\boldsymbol{D}_d\Phi_d(t)\boldsymbol{H}_d)\boldsymbol{x}(t-\tau)+\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u},t)-\kappa(t)\Big)\\
&\leq \Big(\alpha\|\boldsymbol{x}(t)\|+\beta\|\boldsymbol{x}(t-\tau)\|+\eta(t,y)\\
&\quad+\chi\|\boldsymbol{u}(t)\|+\psi\bar{d}-\kappa(t)\Big)\|\boldsymbol{s}(t)\|\\
&\leq (\alpha\|\boldsymbol{x}(t)\|+\beta\|\boldsymbol{x}(t-\tau)\|-\sigma_1-\sigma_2-\mu)\|\boldsymbol{s}(t)\|\\
&\leq -\mu\|\boldsymbol{s}(t)\|
\end{aligned}$$

where $\alpha = \|\bar{\boldsymbol{G}}(\boldsymbol{A}+\boldsymbol{D}\Phi(t)\boldsymbol{H})\|$ and $\beta = \|\bar{\boldsymbol{G}}(\boldsymbol{A}_d+\boldsymbol{D}_d\Phi_d(t)\boldsymbol{H}_d)\|$. Since $\boldsymbol{s}(0)=\boldsymbol{0}$, the control input (6) can guarantee the following identities:

$$\boldsymbol{s}(t) = \dot{\boldsymbol{s}}(t) = \boldsymbol{0} \quad \forall\, t \geq 0.$$

Therefore, the design of integral sliding surface (3) can shorten the transient time that the system entered the sliding mode efficiently. Subsequently, this paper focuses on the stability analysis when the system is in the sliding mode.

From (4), once the system is in the sliding mode, $\boldsymbol{s}(t) = \dot{\boldsymbol{s}}(t) = \boldsymbol{0}$, the corresponding equivalent control [2] is given by

$$\begin{aligned}
\boldsymbol{u}_{eq}(t)+\boldsymbol{f}\left(\boldsymbol{x},\boldsymbol{u}_{eq},t\right) &= -\bar{\boldsymbol{G}}\left((\boldsymbol{A}+\boldsymbol{D}\Phi(t)\boldsymbol{H})\boldsymbol{x}(t)\right.\\
&\quad\left.+(\boldsymbol{A}_d+\boldsymbol{D}_d\Phi_d(t)\boldsymbol{H}_d)\boldsymbol{x}(t-\tau)+\boldsymbol{E}\boldsymbol{d}(t)\right)+\boldsymbol{v}(t). \quad (7)
\end{aligned}$$

Deriving the closed-loop system dynamics in the sliding mode from substituting (7) into system (2) can obtain

$$\begin{aligned}
\dot{\boldsymbol{x}}(t) &= \left(\boldsymbol{I}_n-\boldsymbol{B}\bar{\boldsymbol{G}}\right)\left((\boldsymbol{A}+\boldsymbol{D}\Phi(t)\boldsymbol{H})\boldsymbol{x}(t)+\boldsymbol{E}\boldsymbol{d}(t)\right.\\
&\quad\left.+(\boldsymbol{A}_d+\boldsymbol{D}_d\Phi_d(t)\boldsymbol{H}_d)\boldsymbol{x}(t-\tau)\right)+\boldsymbol{B}\boldsymbol{v}(t)\\
&= \boldsymbol{N}(\boldsymbol{A}+\boldsymbol{D}\Phi(t)\boldsymbol{H})\boldsymbol{x}(t)+\boldsymbol{N}\boldsymbol{E}\boldsymbol{d}(t)\\
&\quad+\boldsymbol{N}(\boldsymbol{A}_d+\boldsymbol{D}_d\Phi_d(t)\boldsymbol{H}_d)\boldsymbol{x}(t-\tau)+\boldsymbol{B}\boldsymbol{v}(t) \quad (8)
\end{aligned}$$

where $\boldsymbol{N} = \boldsymbol{I}_n - \boldsymbol{B}\bar{\boldsymbol{G}}$. Since system (2) is controllable, it can imply that the pair $(\boldsymbol{N}\boldsymbol{A},\boldsymbol{B})$ is also controllable. Referring to [20], [21], the robust disturbance attenuation for system (8) is to design a control input $\boldsymbol{v}(t)$ such that the system is stable and satisfies the following inequality:

$$\int_0^t \left(\boldsymbol{y}^T\boldsymbol{y}+\boldsymbol{v}^T\boldsymbol{R}\boldsymbol{v}\right)dq \leq \gamma^2\int_0^t\left(\boldsymbol{d}^T\boldsymbol{d}\right)dq \quad \forall\, t\geq 0 \quad (9)$$

where $0 \leq \gamma < \infty$ and $\boldsymbol{R} > 0$ is a weighting matrix. Next subsection will utilize a full-order compensator to complete the design of $\boldsymbol{v}(t)$ fulfilling the robust disturbance attenuation.

Remark 1: In [7], the integration term in the sliding manifold can be thought as a trajectory of the system in the absence of perturbations and in the presence of the nominal control, that is, as a nominal trajectory for a given initial condition. In this paper, adding the integration term $\boldsymbol{v}(t)$ into the sliding surface (3) can compensate the degree of freedom to attenuate the effects of disturbances and uncertainties in the closed-loop system. Involving the integrator is also helpful to analyze the stability and robustness of the closed-loop system.

3.2 Full-Order Compensator

Before designing the compensator, define the following matrices as

$$\begin{aligned}
\boldsymbol{U} &= -\boldsymbol{B}(\boldsymbol{C}\boldsymbol{B})^+ + \boldsymbol{Y}\left(\boldsymbol{I}_l-\boldsymbol{C}\boldsymbol{B}(\boldsymbol{C}\boldsymbol{B})^+\right)\\
\boldsymbol{M} &= \boldsymbol{I}_n+\boldsymbol{U}\boldsymbol{C}\\
\Gamma &= \boldsymbol{L}(\boldsymbol{I}_l+\boldsymbol{C}\boldsymbol{U})-\boldsymbol{M}\boldsymbol{A}\boldsymbol{U}
\end{aligned}$$

where $(\boldsymbol{C}\boldsymbol{B})^+ = \left((\boldsymbol{C}\boldsymbol{B})^T\boldsymbol{C}\boldsymbol{B}\right)^{-1}(\boldsymbol{C}\boldsymbol{B})^T$, $\boldsymbol{Y}\in\mathbb{R}^{n\times l}$ is an arbitrary matrix, and $\boldsymbol{L}$ is a gain matrix designed in the latter. Notice that $\boldsymbol{M}\boldsymbol{B}=\boldsymbol{0}$ and $\text{rank}(\boldsymbol{M}) = n-m$ from Assumption 3. Then the control input $\boldsymbol{v}(t)$ is generated from the following full-order dynamic compensator:

$$\begin{aligned}
\dot{\xi}(t) &= (\boldsymbol{M}\boldsymbol{A}-\boldsymbol{L}\boldsymbol{C})\xi(t)+\Gamma\boldsymbol{y}(t)\\
\boldsymbol{v}(t) &= -\boldsymbol{K}(\xi(t)-\boldsymbol{U}\boldsymbol{y}(t)) \quad (10)
\end{aligned}$$

where $\xi\in\mathbb{R}^n$ is an available vector of auxiliary states. Moreover, $\boldsymbol{K}\in\mathbb{R}^{m\times n}$ is a gain matrix decided in the latter. According to (10) and $\boldsymbol{M}\boldsymbol{B}=\boldsymbol{0}$, the dynamics of error vector $\boldsymbol{e} = \boldsymbol{M}\boldsymbol{x}-\xi$ can be given by

$$\begin{aligned}
\dot{\boldsymbol{e}}(t) &= -\Gamma\boldsymbol{y}(t)+\boldsymbol{M}\left((\boldsymbol{A}+\boldsymbol{D}\Phi(t)\boldsymbol{H})\boldsymbol{x}(t)+\boldsymbol{E}\boldsymbol{d}(t)\right.\\
&\quad\left.+(\boldsymbol{A}_d+\boldsymbol{D}_d\Phi_d(t)\boldsymbol{H}_d)\boldsymbol{x}(t-\tau)\right)-(\boldsymbol{M}\boldsymbol{A}-\boldsymbol{L}\boldsymbol{C})\xi(t)\\
&= \boldsymbol{M}\boldsymbol{D}\Phi(t)\boldsymbol{H}\boldsymbol{x}(t)+(\boldsymbol{M}\boldsymbol{A}-\boldsymbol{L}\boldsymbol{C})\boldsymbol{e}(t)\\
&\quad+\boldsymbol{M}(\boldsymbol{A}_d+\boldsymbol{D}_d\Phi_d(t)\boldsymbol{H}_d)\boldsymbol{x}(t-\tau)+\boldsymbol{M}\boldsymbol{E}\boldsymbol{d}(t). \quad (11)
\end{aligned}$$

On the other hand, $\boldsymbol{v}(t)$ can be rewritten as

$$\boldsymbol{v}(t) = -\boldsymbol{K}\boldsymbol{x}(t)+\boldsymbol{K}\boldsymbol{e}(t). \quad (12)$$

Substituting (12) into (8) can obtain the system dynamics in the sliding mode as

$$\begin{aligned}
\dot{\boldsymbol{x}}(t) &= (\boldsymbol{N}(\boldsymbol{A}+\boldsymbol{D}\Phi(t)\boldsymbol{H})-\boldsymbol{B}\boldsymbol{K})\boldsymbol{x}(t)+\boldsymbol{B}\boldsymbol{K}\boldsymbol{e}(t)\\
&\quad+\boldsymbol{N}(\boldsymbol{A}_d+\boldsymbol{D}_d\Phi_d(t)\boldsymbol{H}_d)\boldsymbol{x}(t-\tau)+\boldsymbol{N}\boldsymbol{E}\boldsymbol{d}(t). \quad (13)
\end{aligned}$$

Combining (11) with (13), the overall closed-loop system is shown as below:

$$\begin{aligned}
\begin{bmatrix}\dot{\boldsymbol{x}}(t)\\ \dot{\boldsymbol{e}}(t)\end{bmatrix} &= \begin{bmatrix}\boldsymbol{N}\boldsymbol{A}-\boldsymbol{B}\boldsymbol{K}+\boldsymbol{N}\boldsymbol{D}\Phi(t)\boldsymbol{H} & \boldsymbol{B}\boldsymbol{K}\\ \boldsymbol{M}\boldsymbol{D}\Phi(t)\boldsymbol{H} & \boldsymbol{M}\boldsymbol{A}-\boldsymbol{L}\boldsymbol{C}\end{bmatrix}\begin{bmatrix}\boldsymbol{x}(t)\\ \boldsymbol{e}(t)\end{bmatrix}\\
&\quad+\begin{bmatrix}\boldsymbol{N}\boldsymbol{A}_d+\boldsymbol{N}\boldsymbol{D}_d\Phi_d(t)\boldsymbol{H}_d & \boldsymbol{0}\\ \boldsymbol{M}\boldsymbol{A}_d+\boldsymbol{M}\boldsymbol{D}_d\Phi_d(t)\boldsymbol{H}_d & \boldsymbol{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{x}(t-\tau)\\ \boldsymbol{e}(t-\tau)\end{bmatrix}+\begin{bmatrix}\boldsymbol{N}\boldsymbol{E}\\ \boldsymbol{M}\boldsymbol{E}\end{bmatrix}\boldsymbol{d}(t)\\
&= \boldsymbol{A}_w\begin{bmatrix}\boldsymbol{x}(t)\\ \boldsymbol{e}(t)\end{bmatrix}+\boldsymbol{B}_w\begin{bmatrix}\boldsymbol{x}(t-\tau)\\ \boldsymbol{e}(t-\tau)\end{bmatrix}+\boldsymbol{G}_w\boldsymbol{d}(t) \quad (14)
\end{aligned}$$

where

$$\boldsymbol{A}_w = \begin{bmatrix}\boldsymbol{N}\boldsymbol{A}-\boldsymbol{B}\boldsymbol{K}+\boldsymbol{N}\boldsymbol{D}\Phi(t)\boldsymbol{H} & \boldsymbol{B}\boldsymbol{K}\\ \boldsymbol{M}\boldsymbol{D}\Phi(t)\boldsymbol{H} & \boldsymbol{M}\boldsymbol{A}-\boldsymbol{L}\boldsymbol{C}\end{bmatrix},$$

$$\boldsymbol{B}_w = \begin{bmatrix}\boldsymbol{N}\boldsymbol{A}_d+\boldsymbol{N}\boldsymbol{D}_d\Phi_d(t)\boldsymbol{H}_d & \boldsymbol{0}\\ \boldsymbol{M}\boldsymbol{A}_d+\boldsymbol{M}\boldsymbol{D}_d\Phi_d(t)\boldsymbol{H}_d & \boldsymbol{0}\end{bmatrix}, \text{ and } \boldsymbol{G}_w = \begin{bmatrix}\boldsymbol{N}\boldsymbol{E}\\ \boldsymbol{M}\boldsymbol{E}\end{bmatrix}.$$

Moreover, we define the controlled output $z \in \mathbb{R}^{l+m}$ as

$$z(t) = \begin{bmatrix} C \\ 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ C_v \end{bmatrix} v(t) = \begin{bmatrix} C \\ -C_v K \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ C_v K \end{bmatrix} e(t)$$
$$= \bar{C} x(t) + \bar{C}_v e(t) = C_w \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} \quad (15)$$

where $C_w = \begin{bmatrix} \bar{C} & \bar{C}_v \end{bmatrix} = \begin{bmatrix} C & 0 \\ -C_v K & C_v K \end{bmatrix}$ and $C_v^T C_v = R$. Next subsection will design the matrices K and L, and analyze the robust stability of the closed-loop system (14).

3.3 Robust Disturbance Attenuation

Define $q(t) = \begin{bmatrix} x^T(t) & e^T(t) \end{bmatrix}^T$ and a quadratic energy function as

$$E_n(q) = q^T P q + \int_{t-\tau}^{t} q^T(\alpha) Q q(\alpha) d\alpha > 0 \quad (16)$$

where the matrices $P > 0$ and $Q > 0$ are determined in a latter. Then define the Hamiltonian function as

$$H[d] = z^T z - \gamma^2 d^T d + dE_n/dt \quad (17)$$

where dE_n/dt is the derivative of E_n along the trajectory of the closed-loop system (14). A sufficient condition satisfying the robust disturbance attenuation is that

$$H[d] < 0, \text{ for all } d \in L_2 \quad (18)$$

Since (18) holds, $E_n(q)$ is a strict radially unbounded Lyapunov function of the closed-loop system (14), and hence the robust stability can be guaranteed [20]. Notice that (18) is equivalent to $\sup_{d \in L_2} H[d] < 0$. As $z(t) = C_w q(t)$, (17) can be rewritten as

$$\begin{aligned} H[d] &= z^T z - \gamma^2 d^T d + dE_n/dt \\ &= q^T(t) \left(A_w^T P + P A_w + Q + C_w^T C_w \right) q(t) - \gamma^2 d(t)^T d(t) \\ &+ q^T(t) P B_w q(t-\tau) - q^T(t-\tau) Q q(t-\tau) \\ &+ q^T(t) P G_w d(t) + q^T(t-\tau) B_w^T P q(t) + d^T(t) G_w^T P q(t). \end{aligned}$$

Based on the above equation, the worst case $\sup_{d \in L_2} H[d]$ occurs when $d(t) = \gamma^{-2} G_w^T P q(t)$, and it follows that

$$\begin{aligned} H[d] &\le q^T(t-\tau) B_w^T P q(t) + q^T(t) P B_w q(t-\tau) \\ &+ q^T(t) \left(A_w^T P + P A_w + Q + C_w^T C_w + \gamma^{-2} P G_w G_w^T P \right) q(t) \\ &- q^T(t-\tau) Q q(t-\tau) \end{aligned} \quad (19)$$

The following lemma is introduced to obtain the bound of the uncertainty variations in (19) by known quantities.

Lemma 1: Given real matrices D, $\Phi(t)$, and H of appropriate dimensions, suppose $\Phi^T(t)\Phi(t) \le I$, for any positive scalar ρ, then

(i) [16] $D\Phi(t) H + H^T \Phi^T(t) D^T \le \rho D D^T + \rho^{-1} H^T H$;

(ii) [8]

$$\begin{bmatrix} 0 & H^T \Phi^T(t) D^T \\ D \Phi(t) H & 0 \end{bmatrix} \le \begin{bmatrix} \rho H^T H & 0 \\ 0 & \rho^{-1} D D^T \end{bmatrix}.$$

□

Moreover, the following theorem transfers (19) into an LMI using Shur decomposition and demonstrates the designs of K and L which guarantee the robust disturbance attenuation.

Theorem: Consider system (8) with the full-order compensator (10). Given λ, $\rho_i > 0$, $i = 1, 2, \cdots, 10$, and a positive definite matrix R, if there exist $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} > 0$, $Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} > 0$, $P_{22} \ge I$, $Q_{11} > \left(\rho_6^{-1} + \rho_7^{-1} + \rho_8^{-1} + \rho_9^{-1} \right) H_d^T H_d$, and a scalar $\gamma > 0$, satisfying the following LMIs

$$\begin{bmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} & 0 & \Pi_{15} & \Pi_{16} & \Pi_{17} & 0 \\ * & \Pi_{22} & \Pi_{23} & 0 & \Pi_{25} & \Pi_{26} & 0 & \Pi_{28} \\ * & * & \Pi_{33} & \Pi_{34} & 0 & 0 & 0 & 0 \\ * & * & * & \Pi_{44} & 0 & 0 & 0 & 0 \\ * & * & * & * & \Pi_{55} & 0 & 0 & 0 \\ * & * & * & * & * & \Pi_{66} & 0 & 0 \\ * & * & * & * & * & * & \Pi_{77} & 0 \\ * & * & * & * & * & * & * & \Pi_{88} \end{bmatrix} < 0 \quad (20)$$

where

$$\begin{aligned} \Pi_{11} &= (NA)^T P_{11} + P_{11} NA + \left(\rho_1^{-1} + \rho_2 + \rho_3 + \rho_4^{-1} \right) H^T H \\ &\quad + C^T C + Q_{11} \\ \Pi_{12} &= (NA)^T P_{12} + P_{12} MA + Q_{12} \\ \Pi_{13} &= P_{11} N A_d + P_{12} M A_d \\ \Pi_{22} &= P_{22} MA + (MA)^T P_{22} + Q_{22} \\ &\quad - \lambda C^T C + \lambda \rho_{10}^{-1} C^T C C^T C / 2 \\ \Pi_{23} &= P_{12}^T N A_d + P_{22} M A_d \\ \Pi_{33} &= -\left(Q_{11} - \left(\rho_6^{-1} + \rho_7^{-1} + \rho_8^{-1} + \rho_9^{-1} \right) H_d^T H_d \right) \\ \Pi_{34} &= -Q_{12} \\ \Pi_{44} &= -Q_{22} \\ \Pi_{15} &= P_{11} N E + P_{12} M E \\ \Pi_{16} &= -P_{11} B \\ \Pi_{17} &= [P_{12} M D \;\; P_{11} N D \;\; P_{11} N D_d \;\; P_{12} M D_d \;\; P_{12}] \\ \Pi_{25} &= P_{12}^T N E + P_{22} M E \\ \Pi_{26} &= P_{12}^T B \\ \Pi_{28} &= \left[P_{12}^T N D \;\; P_{22} M D \;\; P_{11} B R^{-1} B^T \;\; P_{12}^T \right. \\ &\qquad \left. P_{11} B \;\; P_{12}^T N D_d \;\; P_{22} M D_d \right] \\ \Pi_{55} &= -\gamma^2 I \\ \Pi_{66} &= -R \\ \Pi_{77} &= -\text{diag}\left(\rho_1^{-1} I, \rho_4^{-1} I, \rho_6^{-1} I, \rho_7^{-1} I, 2\lambda^{-1} \rho_{10}^{-1} I \right) \\ \Pi_{88} &= -\text{diag}\left(\rho_2 I, \rho_3 I, \rho_5^{-1} I, \rho_5 I, R, \rho_8^{-1}, \rho_9^{-1} I \right), \end{aligned}$$

then robust disturbance attenuation (9) can be guaranteed. Furthermore, matrices K and L are given by

$$\boldsymbol{K} = \boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}_{11} \text{ and } \boldsymbol{L} = \lambda \boldsymbol{P}_{22}^{-1}\boldsymbol{C}^T/2.$$

Proof: The detail is in Appendix. □

4. Numerical Example

To illustrate the proposed controller design, consider the real example of chemical reactor system [15] within the corresponding form of system (2) with delay time $\tau = 1$ as

$$\boldsymbol{A} = \begin{bmatrix} -4.93 & -1.01 & 0 & 0 \\ -3.20 & -5.30 & -12.8 & 0 \\ -6.40 & 0.347 & -32.5 & -1.04 \\ 0 & 0.833 & 11.0 & -3.96 \end{bmatrix}, \ \boldsymbol{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\boldsymbol{C} = \begin{bmatrix} 1 & 0 & -0.2 & 0 \\ 0 & 1 & 0 & 0.1 \end{bmatrix}, \quad \boldsymbol{E} = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}^T,$$

and $\boldsymbol{A}_d = \text{diag}(1.92,\ 1.92,\ 1.87,\ 0.724)$. The known parts of uncertainties in the system are given by

$$\boldsymbol{D} = \begin{bmatrix} -0.47 & 1.01 & 0 & 0 \\ -0.22 & -0.17 & 1.21 & 0 \\ 0.63 & 0.347 & 0.91 & -1.04 \\ 0 & 0 & 0.14 & -0.96 \end{bmatrix},$$

$$\boldsymbol{H} = \begin{bmatrix} -0.55 & -0.02 & 0 & 0 \\ 0.78 & -0.35 & 0 & 0 \\ 0 & -0.72 & -0.49 & 0 \\ 0 & 0.33 & -0.54 & -0.39 \end{bmatrix},$$

$\boldsymbol{D}_d = \text{diag}(0.47, 0.26, -0.85, 1.53)$, and $\boldsymbol{H}_d = \text{diag}(-1.11, -0.21, 1.26, 0.47)$. The external disturbances and unknown parts of uncertainties for system (2) are set as $\Phi(t) = r_1(t)\boldsymbol{I}_4$, $\Phi_d(t) = r_2(t)\boldsymbol{I}_4$, $d(t) = \exp(-0.001t)\sin 2t$, and

$$\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, t) = \begin{bmatrix} 0.12u_1 \sin t + 0.08u_2 \cos 1.3t + 0.2 \sin x_1 \\ 0.07u_1 \cos 3t + 0.03u_2 \sin 5t + 0.3 \cos x_2 \end{bmatrix}$$

where $\boldsymbol{u} = \begin{bmatrix} u_1^T & u_2^T \end{bmatrix}$, $r_1(t)$ and $r_2(t)$ are different random functions with values between −1 and 1. Notice that the triple $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$ has invariant zeros −4.4463 and −33.377, and rank $(\boldsymbol{CB}) = 2$.

For solving the LMI (20) of this example, select the parameters as $\lambda = 1000$, $\boldsymbol{R} = 0.002\boldsymbol{I}_2$, $\rho_1 = \rho_4 = 2$, $\rho_2 = \rho_3 = 0.1$, $\rho_6 = \rho_7 = \rho_8 = \rho_9 = 3$, and $\rho_5 = \rho_{10} = 1$. Then the solutions to (20) are given by

$$\boldsymbol{P}_{11} = \begin{bmatrix} 0.0139 & -0.0013 & 0.0668 & -0.0046 \\ -0.0013 & 0.0012 & -0.0127 & 0.0005 \\ 0.0668 & -0.0127 & 0.6492 & -0.1044 \\ -0.0046 & 0.0005 & -0.1044 & 0.1945 \end{bmatrix},$$

$$\boldsymbol{P}_{12} = \begin{bmatrix} 0.0030 & -0.0001 & -0.0028 & -0.0072 \\ -0.0019 & -0.0005 & 0.0003 & 0.0094 \\ -0.0012 & -0.0011 & -0.0841 & -0.0242 \\ 0.0128 & 0.0044 & 0.0080 & -0.0734 \end{bmatrix},$$

$$\boldsymbol{P}_{22} = \begin{bmatrix} 38.6114 & -0.0364 & -4.8448 & 0.4358 \\ -0.0364 & 38.9961 & -0.0304 & 3.4802 \\ -4.8448 & -0.0304 & 17.5268 & 0.4054 \\ 0.4358 & 3.4802 & 0.4054 & 6.0693 \end{bmatrix},$$

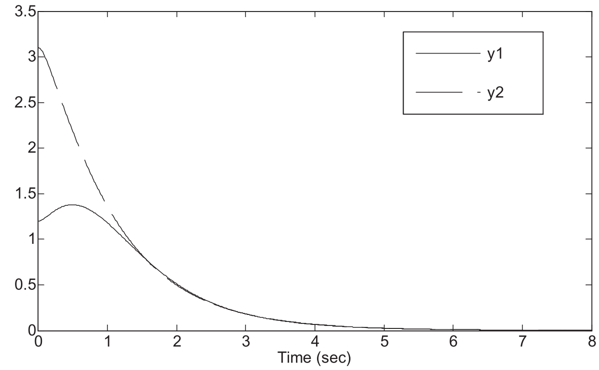

Fig. 1 System outputs.

$$\boldsymbol{Q}_{12} = \begin{bmatrix} 0.1010 & 0.0014 & 0.3402 & -0.1593 \\ -0.0501 & -0.0023 & -0.1692 & 0.0971 \\ -0.0227 & 0.0090 & -0.1059 & -0.1096 \\ -0.0121 & 0.0338 & -0.0276 & -0.4093 \end{bmatrix},$$

$$\boldsymbol{Q}_{22} = \begin{bmatrix} 38.7504 & -0.0190 & 0.0192 & 0.1867 \\ -0.0190 & 38.5812 & 0.1953 & 2.6922 \\ 0.0192 & 0.1953 & 38.7613 & -2.5084 \\ 0.1867 & 2.6922 & -2.5084 & 14.4394 \end{bmatrix},$$

and $\boldsymbol{Q}_{11} = 3\boldsymbol{I}_4$. Hence, we construct the full-order compensator as

$$\dot{\xi}(t) = -\begin{bmatrix} 13.9652 & -0.0678 & 3.9630 & 0.2082 \\ 0.0827 & 12.8218 & 1.0835 & 0.8778 \\ 4.2199 & -0.3461 & 32.9360 & 1.0401 \\ -0.8127 & 0.1007 & -10.8375 & 4.0534 \end{bmatrix}\xi(t) + \begin{bmatrix} -1.28 & 0 & -6.4 & 0 \\ 0.0694 & -0.0833 & 0.3470 & 0.8330 \end{bmatrix}^T \boldsymbol{y}(t)$$

and design the sliding surface as

$$\boldsymbol{s}(t) = \boldsymbol{y}(t) - \boldsymbol{y}(0) - \int_0^t \begin{bmatrix} -6.9387 & 0.6675 \\ 0.6675 & -0.6178 \end{bmatrix} \boldsymbol{y}(q)\,dq - \int_0^t \begin{bmatrix} -6.9387 & 0.6675 & -33.4124 & 2.2778 \\ 0.6675 & -0.6178 & 6.3451 & -0.2742 \end{bmatrix} \xi(q)\,dq.$$

Moreover, in order to avoid the chattering problem, the term $\boldsymbol{s}(t)/\|\boldsymbol{s}(t)\|$ in the controller (6) is replaced by the saturation function [8], and the new version of the controller is given by

$$\boldsymbol{u}(t) = \begin{bmatrix} -6.9387 & 0.6675 & -33.4124 & 2.2778 \\ 0.6675 & -0.6178 & 6.3451 & -0.2742 \end{bmatrix}\xi(t) + \begin{bmatrix} -6.9387 & 0.6675 \\ 0.6675 & -0.6178 \end{bmatrix}\boldsymbol{y}(t) - \text{sat}(\boldsymbol{s}(t), \varepsilon) \times \left(\sigma_1 + \sigma_2 + \eta(t, \boldsymbol{y}) + \chi \|\boldsymbol{v}(t)\| + \psi \bar{d} + \mu\right)/(1-\chi)$$

where $\sigma_1 = \sigma_2 = 5$, $\eta(t, \boldsymbol{y}) = 2$, $\chi = \psi = 0.8$, $\bar{d} = 1$, $\mu = 2.5$, and $\text{sat}(\cdot)$ denotes the saturation function with $\varepsilon = 0.002$. Figures 1–5 chart the simulation results using the initial state $\boldsymbol{x}(0) = [2\ \ 3\ \ 4\ \ 1]^T$ and $\xi(0) = [0\ \ 0\ \ 0\ \ 0]^T$. The time responses of the system outputs are shown in Fig. 1. Figures 2 and 3 show $\boldsymbol{s}(t)$ and $\|\boldsymbol{s}(t)\|$, respectively. In Fig. 3,

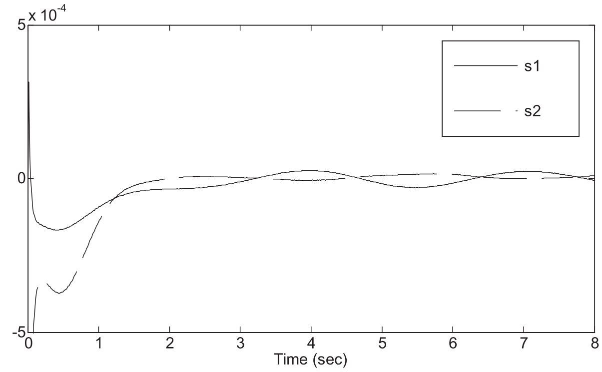

Fig. 2 Sliding surfaces.

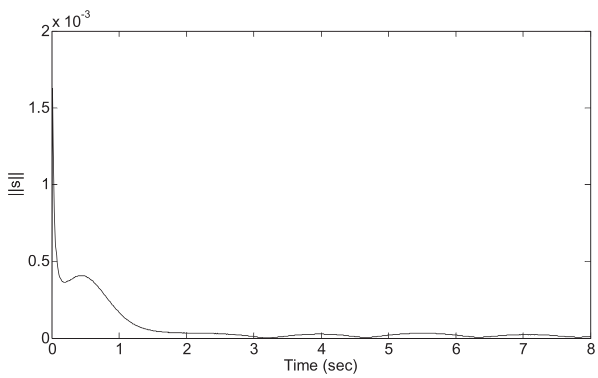

Fig. 3 Response of $||\boldsymbol{s}||$.

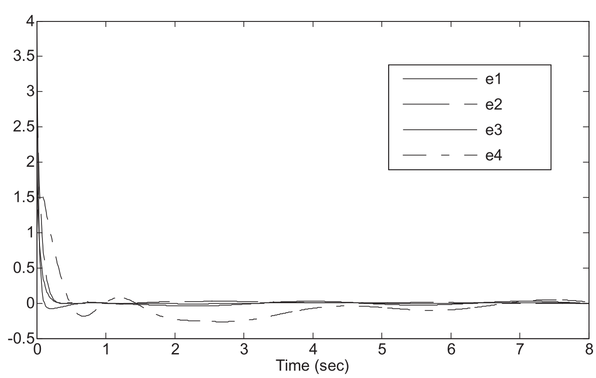

Fig. 4 Trajectories of $\boldsymbol{e}$.

the trajectory representing the controlled system can maintain in the sliding layer in whole time. Figure 4 shows that the trajectories of $\boldsymbol{e}(t)$ are bounded around zero and do not converge to zero because of the mismatched disturbance. The responses of the control inputs $\boldsymbol{u}(t)$ are given in Fig. 5. The replacement of the saturation function eliminates the chattering. From Fig. 1, although the nominal system exists the state delay term and the mismatched disturbance, the system outputs $\boldsymbol{y}(t)$ are finally bounded around zero. The simulation results demonstrate that the proposed controller design can guarantee the robust disturbance attenuation to outputs $\boldsymbol{y}(t)$ once the system is in the sliding mode.

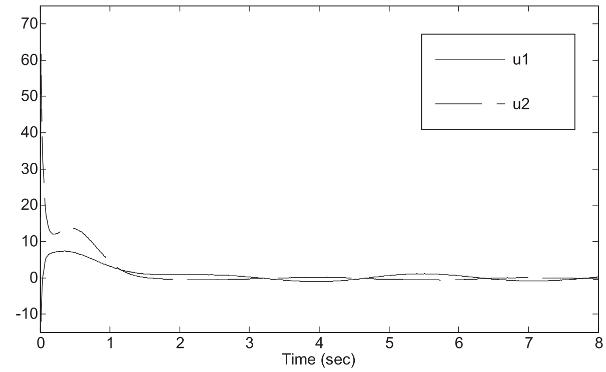

Fig. 5 System inputs.

5. Conclusions

This paper has presented the output feedback integral sliding mode controller for a class of time-delay systems with structure uncertainties and mismatched disturbances. The auxiliary full-order compensator added into the design of the integral sliding surface can improve the synthesis problem of static output feedback sliding mode control. This paper utilizes the disturbance rejection condition in H_∞ theory to derive an LMI comprised of the parameters of the system, controller, and compensator. When the specific LMI have solutions, both the stability of the closed-loop system and the condition of robust disturbance attenuation can be guaranteed. Moreover, the designed controller can maintain that the system is always in the sliding mode from the initial moment. Finally, the simulation results of the real chemical reactor example demonstrated the feasibility of the propose control scheme.

References

[1] V.I. Utkin, Sliding Modes in Control and Optimization, Springer Verlag, New York, 1992.

[2] C. Edwards and S.K. Spurgeon, Sliding Mode Control Theory and Application, Taylor & Francis, London, 1998.

[3] Y. Xia and Y. Jia, "Robust sliding-mode control for uncertain time-delay systems: An LMI approach," IEEE Trans. Autom. Control, vol.48, no.6, pp.1086–1092, 2003.

[4] S. Qu and Y. Wang, "Robust control of uncertain time delay system: A novel sliding mode control design via LMI," J. Systems Engineering and Electronics, vol.17, no.3, pp.624–628, 2006.

[5] T.S. Chiang and C.S. Chiu, "Sliding mode control of a class of uncertain nonlinear time-delay systems using LMI and TS recurrent fuzzy neural network," IEICE Trans. Fundamentals, vol.E92-A, no.1, pp.252–262, Jan. 2009.

[6] C.C. Kung, T.H. Chen, and L.H. Kung, "Modified adaptive fuzzy slidin mode controller for uncertain nonlinear systems," IEICE Trans. Fundamentals, vol.E88-A, no.5, pp.1328–1334, May 2005.

[7] F. Castanos and L. Fridman, "Analysis and design of integral sliding manifolds for systems with unmatched perturbations," IEEE Trans. Autom. Control, vol.51, no.5, pp.853–858, 2006.

[8] J.J.E. Slotine and S.S. Sastry, "Tracking control of nonlinear systems using sliding surfaces with application to robot manipulators," Int. J. Control, vol.38, no.2, pp.465–492, 1983.

[9] Y. Niu, J. Lam, X. Wang, and D.W.C. Ho, "Observer-based sliding

mode control for nonlinear state-delayed systems," Int. J. Systems Science, vol.35, no.2, pp.139–150, 2004.

[10] X.G. Yan, S.K. Spurgeon, and C. Edwards, "Static output feedback sliding mode control for time-varying delay systems with time-delayed nonlinear disturbances," Proc. 17th IFAC World Congress, pp.8642–8647, Seoul, Korea, 2008.

[11] X.R. Han, E. Fridman, S.K. Spurgeon, and C. Edwards, "On the design of sliding mode static output feedback controllers for systems with time-varying delay," Int. Workshop Variable Structure Systems, pp.136–140, 2008.

[12] X.G. Yan, S.K. Spurgeon, and C. Edwards, "Static output feedback sliding mode control for time-varying delay systems with time-delayed nonlinear disturbances," Int. J. Robust Nonlinear Control, published online, 2009.

[13] E. Fridman and U. Shaked, "A descriptor system approach to H_∞ control of linear time-delay systems," IEEE Trans. Autom. Control, vol.47, no.2, pp.253–270, 2002.

[14] C.C. Hua, Q.G. Wang, and X.P. Guan, "Memoryless state feedback controller design for time delay systems with matched uncertain nonlinearities," IEEE Trans. Autom. Control, vol.53, no.3, pp.801–807, 2003.

[15] Y.S. Lee, S.H. Han, and W.H. Kwon, "Receding horizon H_∞ control for systems with a state-delay," Asian J. Control, vol.8, no.1, pp.63–71, 2006.

[16] S.C. Bengea, X. Li, and R.A. DeCarlo, "Combined controller-observer design for uncertain time delay systems with application to engine idle speed control," J. Dynamic Systems, Measurement, and Control, vol.126, pp.772–780, 2004.

[17] Y.I. Son, G.J. Jeong, and I.H. Kim, "A combined simple adaptive control with disturbance observer for a class of time-delay systems," IEICE Trans. Fundamentals, vol.E93-A, no.2, pp.553–556, Feb. 2010.

[18] J.Y. Choi, "A global stability analysis of a class of nonlinear time-delay systems using continued fraction property," IEICE Trans. Fundamentals, vol.E91-A, no.5, pp.1274–1277, May 2008.

[19] H.L. Choi and J.T. Lim, "Global asymptotic stabilization of a class of nonlinear time-delay systems by output feedback," IEICE Trans. Fundamentals, vol.E88-A, no.12, pp.3604–3609, Dec. 2005.

[20] L.Y. Wang and W. Zhan, "Robust disturbance attenuation with stability for linear systems with norm-bounded nonlinear uncertainties," IEEE Trans. Autom. Control, vol.41, no.6, pp.886–888, 1996.

[21] L. Zhou, J.C. Doyle, and K. Glover, Robust and Optimal Control, Prentice Hall, New Jersey, 1996.

[22] G. Zhai, X. Chen, S. Takai, and K. Yasuda, "Stability and H_∞ disturbance attenuation analysis for LTI control systems with controllers failures," Asian J. Control, vol.6, no.1, pp.104–111, 2004.

Appendix

If there exists $\boldsymbol{P}_{22} \geq \boldsymbol{I}$, according to lemma 1 (ii), the following inequality is established

$$\boldsymbol{q}^T(t)\begin{bmatrix} \boldsymbol{0} & -\boldsymbol{P}_{12}\boldsymbol{P}_{22}^{-1}\boldsymbol{C}^T\boldsymbol{C} \\ -\boldsymbol{C}^T\boldsymbol{C}\boldsymbol{P}_{22}^{-1}\boldsymbol{P}_{12}^T & \boldsymbol{0} \end{bmatrix}\boldsymbol{q}(t) \leq \boldsymbol{q}^T(t)\begin{bmatrix} \rho_{10}\boldsymbol{P}_{12}\boldsymbol{P}_{12}^T & \boldsymbol{0} \\ \boldsymbol{0} & \rho_{10}^{-1}\boldsymbol{C}^T\boldsymbol{C}\boldsymbol{C}^T\boldsymbol{C} \end{bmatrix}\boldsymbol{q}(t)$$

where $\rho_{10} > 0$. Applying lemma 1 and designing $\boldsymbol{K} = \boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}_{11}$ and $\boldsymbol{L} = \lambda\boldsymbol{P}_{22}^{-1}\boldsymbol{C}^T/2$, the bounded of $H[\boldsymbol{d}]$ can be expressed as

$$H[\boldsymbol{d}] \leq \begin{bmatrix} \boldsymbol{q}(t) \\ \boldsymbol{q}(t-\tau) \end{bmatrix}^T \begin{bmatrix} \Xi_{11} & \Xi_{12} \\ \Xi_{12}^T & \Xi_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}(t) \\ \boldsymbol{q}(t-\tau) \end{bmatrix}$$

where

$$\begin{aligned}\Xi_{11} =& \begin{bmatrix} (\boldsymbol{NA})^T\boldsymbol{P}_{11}+\boldsymbol{P}_{11}\boldsymbol{NA}+\boldsymbol{C}^T\boldsymbol{C} & \boldsymbol{0} \\ \boldsymbol{P}_{12}^T\boldsymbol{NA}+(\boldsymbol{MA})^T\boldsymbol{P}_{12}^T & \boldsymbol{P}_{22}\boldsymbol{MA}+(\boldsymbol{MA})^T\boldsymbol{P}_{22} \end{bmatrix} \\ &+\begin{bmatrix} \frac{\lambda}{2}\rho_{10}\boldsymbol{P}_{12}\boldsymbol{P}_{12}^T & -\boldsymbol{P}_{11}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}_{12} \\ \boldsymbol{0} & \frac{\lambda}{2}\rho_{10}^{-1}\boldsymbol{C}^T\boldsymbol{C}\boldsymbol{C}^T\boldsymbol{C}+\boldsymbol{P}_{12}^T\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}_{12} \end{bmatrix} \\ &+\begin{bmatrix} \boldsymbol{P}_{11}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}_{11}+\rho_1\boldsymbol{P}_{12}\boldsymbol{MDD}^T\boldsymbol{M}^T\boldsymbol{P}_{12}^T+\rho_1^{-1}\boldsymbol{H}^T\boldsymbol{H} & \boldsymbol{0} \\ -\boldsymbol{P}_{12}^T\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}_{11} & \boldsymbol{0} \end{bmatrix} \\ &+\begin{bmatrix} \rho_2\boldsymbol{H}^T\boldsymbol{H}+\rho_3\boldsymbol{H}^T\boldsymbol{H} & \boldsymbol{0} \\ \boldsymbol{0} & \rho_2^{-1}\boldsymbol{P}_{12}^T\boldsymbol{NDD}^T\boldsymbol{N}^T\boldsymbol{P}_{12}+\rho_5^{-1}\boldsymbol{P}_{12}^T\boldsymbol{P}_{12} \end{bmatrix} \\ &+\begin{bmatrix} \rho_4\boldsymbol{P}_{11}\boldsymbol{NDD}^T\boldsymbol{N}^T\boldsymbol{P}_{11}+\rho_4^{-1}\boldsymbol{H}^T\boldsymbol{H} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{P}_{11}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}_{11} \end{bmatrix} \\ &+\begin{bmatrix} \boldsymbol{0} & (\boldsymbol{NA})^T\boldsymbol{P}_{12}+\boldsymbol{P}_{12}\boldsymbol{MA} \\ \boldsymbol{0} & \rho_5\boldsymbol{P}_{11}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}_{11}+\rho_3^{-1}\boldsymbol{P}_{22}\boldsymbol{MDD}^T\boldsymbol{M}^T\boldsymbol{P}_{22} \end{bmatrix} \\ &+\begin{bmatrix} \rho_6\boldsymbol{P}_{11}\boldsymbol{N}\boldsymbol{D}_d\boldsymbol{D}_d^T\boldsymbol{N}^T\boldsymbol{P}_{11}+\rho_7\boldsymbol{P}_{12}\boldsymbol{M}\boldsymbol{D}_d\boldsymbol{D}_d^T\boldsymbol{M}^T\boldsymbol{P}_{12}^T & \boldsymbol{0} \\ \boldsymbol{0} & -\lambda\boldsymbol{C}^T\boldsymbol{C} \end{bmatrix} \\ &+\begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \rho_8\boldsymbol{P}_{12}^T\boldsymbol{N}\boldsymbol{D}_d\boldsymbol{D}_d^T\boldsymbol{N}^T\boldsymbol{P}_{12}+\rho_9\boldsymbol{P}_{22}\boldsymbol{M}\boldsymbol{D}_d\boldsymbol{D}_d^T\boldsymbol{M}^T\boldsymbol{P}_{22} \end{bmatrix} \\ &+\gamma^{-2}\boldsymbol{P}\boldsymbol{G}_w\boldsymbol{G}_w^T\boldsymbol{P}+\boldsymbol{Q},\end{aligned}$$

$$\Xi_{22} = \begin{bmatrix} \left(\rho_6^{-1}+\rho_7^{-1}+\rho_8^{-1}+\rho_9^{-1}\right)\boldsymbol{H}_d^T\boldsymbol{H}_d & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} - \boldsymbol{Q},$$

and $\Xi_{12} = \begin{bmatrix} \boldsymbol{P}_{11}\boldsymbol{N}\boldsymbol{A}_d + \boldsymbol{P}_{12}\boldsymbol{M}\boldsymbol{A}_d & \boldsymbol{0} \\ \boldsymbol{P}_{12}^T\boldsymbol{N}\boldsymbol{A}_d + \boldsymbol{P}_{22}\boldsymbol{M}\boldsymbol{A}_d & \boldsymbol{0} \end{bmatrix}$. Notice that $\boldsymbol{P} = \begin{bmatrix} \boldsymbol{P}_{11} & \boldsymbol{P}_{12} \\ \boldsymbol{P}_{12}^T & \boldsymbol{P}_{22} \end{bmatrix} > 0$, $\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{Q}_{11} & \boldsymbol{Q}_{12} \\ \boldsymbol{Q}_{12}^T & \boldsymbol{Q}_{22} \end{bmatrix} > 0$, and $\rho_i > 0$, $i = 1, 2, \cdots, 9$. If $\begin{bmatrix} \Xi_{11} & \Xi_{12} \\ \Xi_{12}^T & \Xi_{22} \end{bmatrix} < 0$, then $H[\boldsymbol{d}] < 0$ and the property of disturbance attenuation (9) is also satisfied. Through Shur decomposition, if there exists $\boldsymbol{Q}_{11} - \left(\rho_7^{-1} + \rho_8^{-1} + \rho_9^{-1} + \rho_{10}^{-1}\right)\boldsymbol{H}_d^T\boldsymbol{H}_d > 0$, inequality $\begin{bmatrix} \Xi_{11} & \Xi_{12} \\ \Xi_{12}^T & \Xi_{22} \end{bmatrix} < 0$ is equivalent to the LMI (20). If there exist γ, $\boldsymbol{P}_{11}$, $\boldsymbol{P}_{12}$, $\boldsymbol{P}_{22}$, $\boldsymbol{Q}_{11}$, $\boldsymbol{Q}_{12}$, and $\boldsymbol{Q}_{22}$ satisfying the LMI, it implies to $\sup_{\boldsymbol{d}\in L_2} H[\boldsymbol{d}] < 0$, and to guarantee the robust disturbance attenuation. The proof of this theorem is completed.

Huan-Chan Ting received the B.S. degree in electrical engineering from National Taipei University of Technology, Taipei, Taiwan in 2003. He is currently a candidate of Ph.D. degree in electrical and control engineering from National Chiao-Tung University, Hsinchu, Taiwan. His major fields of research are sliding mode control theorem, robust control, and power electronics.

Jeang-Lin Chang received his B.S., M.S., and Ph.D. degrees in electrical and control engineering from National Chiao-Tung University, Hsinchu, Taiwan. He is a professor in the Department of Electrical Engineering, Orient Institute of Technology. His researches include variable structure control, robust control, and discrete signal processing.

Yon-Ping Chen received his B.S. degree in electrical engineering from National Taiwan University in 1981, and M.S. and Ph.D. degrees in electrical engineering from University of Texas at Arlington, U.S.A. He is a professor in the Department of Electrical and Control Engineering, National Chiao Tung University. His researches include variable structure control, intelligent control, and image processing.