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DIFFERENTIAL EQUATIONS SATISFIED BY BI-MODULAR FORMS AND K3 SURFACES

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Abstract. We study differential equations satisfied by bi-modular forms associated to genus zero subgroups of $SL_2(\mathbb{R})$ of the form $\Gamma_0(N)$ or $\Gamma_0(N)^*$. In some examples, these differential equations are realized as the Picard–Fuchs differential equations of families of $K3$ surfaces with large Picard numbers, e.g., $19, 18, 17, 16$. Our method rediscovers some of the Lian–Yau examples of "modular relations" involving power series solutions to second and third order differential equations of Fuchsian type in [10, 11].

1. Introduction

Lian and Yau [10, 11] studied arithmetic properties of mirror maps of pencils of certain $K3$ surfaces, and further, they considered mirror maps of certain families of Calabi–Yau threefolds [12]. Lian and Yau observed in a number of explicit examples a mysterious relationship (now the so-called *mirror moonshine phenomenon*) between mirror maps and the McKay–Thompson series (Hauptmoduls of one variable associated to a genus zero congruence subgroup of $SL_2(\mathbb{R})$) arising from the Monster. Inspired by the work of Lian and Yau, Verrill–Yui [16] further computed more examples of mirror maps of one-parameter families of lattice polarized $K3$ surfaces with Picard number 19. The outcome of Verrill–Yui's calculations suggested that the mirror maps themselves are not always Hauptmoduls, but they are commensurable with Hauptmoduls (referred as the modularity of mirror maps). This fact was indeed established by Doran [6] for M_n-lattice polarized $K3$ surfaces of Picard number 19 (where $M_n = U \perp (-E_8)^2 \perp \langle -2n \rangle$). The mirror maps were calculated via the Picard–Fuchs differential equations of the $K3$ families in question. Therefore, the determination of the Picard–Fuchs differential equations played the central role in their investigations.

In this paper, we will address the inverse problem of a kind. That is, instead of starting with families of $K3$ surfaces or families of Calabi–Yau threefolds, we start with modular forms and functions of more than one variable.

More specifically, the main focus our discussions in this paper are on modular forms and functions of two variables. Here is the precise definition.

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Definition 1.1. Let $\mathbb{H}$ denote the upper half-plane $\{\tau : \Im\tau > 0\}$, and let $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$. Let Γ_1 and Γ_2 be two subgroups of $SL_2(\mathbb{R})$ commensurable with $SL_2(\mathbb{Z})$. We call a function $F : \mathbb{H}^* \times \mathbb{H}^* \longmapsto \mathbb{C}$ of two variables a *bi-modular form of weight* (k_1, k_2) *on* $\Gamma_1 \times \Gamma_2$ *with character* χ if F is meromorphic on $\mathbb{H}^* \times \mathbb{H}^*$ such that

$$F(\gamma_1\tau_1, \gamma_2\tau_2) = \chi(\gamma_1, \gamma_2)(c_1\tau_1 + d_1)^{k_1}(c_2\tau_2 + d_2)^{k_2} F(\tau_1, \tau_2)$$

for all

$$\gamma_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \in \Gamma_1, \qquad \gamma_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \in \Gamma_2.$$

If F is a bi-modular form of weight $(0, 0)$ with trivial character, then we also call F a *bi-modular function on* $\Gamma_1 \times \Gamma_2$.

Notation. We let $q_1 = e^{2\pi i\tau_1}$ and $q_2 = e^{2\pi i\tau_2}$. For a variable t we let D_t denote the the differential operator $t\frac{\partial}{\partial t}$.

Remark 1.1. Stienstra and Zagier [15] have a notion of bi-modular forms. Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$, and let τ, $\tau^* \in \mathbb{H}$. Let k_1, k_2 be integers. A two-variable meromorphic function $F : \mathbb{H} \times \mathbb{H} \to \mathbb{C}$ is called a *bi-modular* form of weight (k_1, k_2) on Γ if for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, it satisfies the transformation formula:

$$F(\gamma\tau, \gamma\tau^*) = (c\tau + d)^{k_1}(c\tau^* + d)^{k_2} F(\tau, \tau^*).$$

For instance,

$$F(\tau, \tau^*) = \tau - \tau^*$$

is a bi-modular form for $\mathrm{SL}_2(\mathbb{Z})$ of weight $(-1, -1)$. Another typical example is

$$F(\tau, \tau^*) = E_2(\tau) - \frac{1}{\tau - \tau^*}$$

is a bi-modular form of weight $(2, 0)$ for $\mathrm{SL}_2(\mathbb{Z})$.

Our definition of bi-modular forms coincides with that of Stienstra and Zagier, if we take $\Gamma_1 = \Gamma_2$ and $\gamma_1 = \gamma_2$.

The problems that we will consider here are formulated as follows : *Given a bi-modular form F, determine a differential equation it satisfies, and construct a family of $K3$ surfaces (or degenerations of a family of Calabi–Yau threefolds at some limit points) having the determined differential equation as its Picard–Fuchs differential equation.*

In fact, a similar problem was already raised by Lian and Yau in their papers [10, 11]. They discussed the so-called "modular relations" involving power series solutions to second and third order differential equations of Fuchsian type (e.g., hypergeometric differential equations ${}_2F_1$, ${}_3F_2$) and modular forms of weight 4 using mirror symmetry.

In this paper, we will focus our discussion on bi-modular forms of weight $(1, 1)$. We will determine the differential equations satisfied by bi-modular forms of weight $(1, 1)$ associated to genus zero subgroups of $SL_2(\mathbb{R})$, e.g., $\Gamma_0(N)$ and $\Gamma_0(N)^*$. Then the existence and the construction of particular bi-modular forms of weight $(1, 1)$ are discussed, using solutions of some hypergeometric differential equations. Moreover, we determine the differential equations they satisfy. Further, several examples of bi-modular forms and their differential equations are discussed aiming to realize these differential equations as the Picard–Fuchs differential equations of some families of $K3$ surfaces (or degenerations of families of Calabi–Yau threefolds) with large Picard numbers $19, 18, 17$ and 16.

It should be pointed out that our paper and our results have non-empty intersections with the results of Lian and Yau [10, 11]. Indeed, our approach rediscovers some of the examples of Lian and Yau.

2. Differential equations satisfied by bi-modular forms

We will now determine differential equations satisfied by bi-modular forms of weight $(1,1)$.

Theorem 2.1. *Let $F(\tau_1,\tau_2)$ be a bi-modular form of weight $(1,1)$, and let $x(\tau_1,\tau_2)$ and $y(\tau_1,\tau_2)$ be non-constant bi-modular functions on $\Gamma_1\times\Gamma_2$. Then F, as a function of x and y, satisfy a system of partial differential equations*

$$\begin{aligned} D_x^2F+a_0D_xD_yF+a_1D_xF+a_2D_yF+a_3F=0,\\ D_y^2F+b_0D_xD_yF+b_1D_xF+b_2D_yF+b_3F=0, \end{aligned} \tag{2.1}$$

where a_i and b_i are algebraic functions of x and y, and can be expressed explicitly as follows. Suppose that, for each function t among F, x, and y, we let

$$G_{t,1}=\frac{D_{q_1}t}{t}=\frac{1}{2\pi i}\frac{dt}{d\tau_1},\qquad G_{t,2}=\frac{D_{q_2}t}{t}=\frac{1}{2\pi i}\frac{dt}{d\tau_2}.$$

Then we have

$$a_0=\frac{2G_{y,1}G_{y,2}}{G_{x,1}G_{y,2}+G_{y,1}G_{x,2}},\qquad b_0=\frac{2G_{x,1}G_{x,2}}{G_{x,1}G_{y,2}+G_{y,1}G_{x,2}},$$

$$a_1=\frac{G_{y,2}^2(D_{q_1}G_{x,1}-2G_{F,1}G_{x,1})-G_{y,1}^2(D_{q_2}G_{x,2}-2G_{F,2}G_{x,2})}{G_{x,1}^2G_{y,2}^2-G_{y,1}^2G_{x,2}^2},$$

$$b_1=\frac{-G_{x,2}^2(D_{q_1}G_{x,1}-2G_{F,1}G_{x,1})+G_{x,1}^2(D_{q_2}G_{x,2}-2G_{F,2}G_{x,2})}{G_{x,1}^2G_{y,2}^2-G_{y,1}^2G_{x,2}^2},$$

$$a_2=\frac{G_{y,2}^2(D_{q_1}G_{y,1}-2G_{F,1}G_{y,1})-G_{y,1}^2(D_{q_2}G_{y,2}-2G_{F,2}G_{y,2})}{G_{x,1}^2G_{y,2}^2-G_{y,1}^2G_{x,2}^2},$$

$$b_2=\frac{-G_{x,2}^2(D_{q_1}G_{y,1}-2G_{F,1}G_{y,1})+G_{x,1}^2(D_{q_2}G_{y,2}-2G_{F,2}G_{y,2})}{G_{x,1}^2G_{y,2}^2-G_{y,1}^2G_{x,2}^2},$$

$$a_3=-\frac{G_{y,2}^2(D_{q_1}G_{F,1}-G_{F,1}^2)-G_{y,1}^2(D_{q_2}G_{F,2}-G_{F,2}^2)}{G_{x,1}^2G_{y,2}^2-G_{y,1}^2G_{x,2}^2},$$

and

$$b_3=-\frac{-G_{x,2}^2(D_{q_1}G_{F,1}-G_{F,1}^2)+G_{x,1}^2(D_{q_2}G_{F,2}-G_{F,2}^2)}{G_{x,1}^2G_{y,2}^2-G_{y,1}^2G_{x,2}^2}.$$

In order to prove Theorem 2.1, we first need the following lemma, which is an analogue of the classical Ramanujan's differential equations

$$D_qE_2=\frac{E_2^2-E_4}{12}=-24\sum_{n\in\mathbb{N}}\frac{n^2q^n}{(1-q^n)^2},$$

$$D_qE_4=\frac{E_2E_4-E_6}{3}=240\sum_{n\in\mathbb{N}}\frac{n^4q^n}{(1-q^n)^2},$$

$$D_qE_6 = \frac{E_2E_6 - E_4^2}{2} = \sum_{n\in\mathbb{N}} \frac{n^6q^n}{(1-q^n)^2}$$

where

$$E_k = 1 - \frac{2k}{B_k}\sum_{n\in\mathbb{N}} \frac{n^{k-1}q^n}{1-q^n} \tag{2.2}$$

are the Eisenstein series of weight k on $SL_2(\mathbb{Z})$, where B_k denotes the k-th Bernoulli number, e.g., $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$ and $B_6 = \frac{1}{42}$.

Lemma 2.2. *We retain the notations of Theorem 2.1. Then*

(a) $G_{x,1}$ and $G_{y,1}$ are bi-modular forms of weight $(2,0)$,

(b) $G_{x,2}$ and $G_{y,2}$ are bi-modular forms of weight $(0,2)$,

(c) $D_{q_1}G_{x,1} - 2G_{F,1}G_{x,1}$, $D_{q_1}G_{y,1} - 2G_{F,1}G_{y,1}$ and $D_{q_1}G_{F,1} - G_{F,1}^2$ are bi-modular forms of weight $(4,0)$, and

(d) $D_{q_2}G_{x,2} - 2G_{F,2}G_{x,2}$, $D_{q_2}G_{y,2} - 2G_{F,2}G_{y,2}$ and $D_{q_2}G_{F,2} - G_{F,2}^2$ are bi-modular forms of weight $(0,4)$.

Proof. We shall prove (a) and (c); the proof of (b) and (d) is similar.

By assumption, x is a bi-modular function on $\Gamma_1 \times \Gamma_2$. That is, for all $\gamma_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \in \Gamma_1$ and all $\gamma_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \in \Gamma_2$, one has

$$x(\gamma_1\tau_1, \gamma_2\tau_2) = x(\tau_1, \tau_2)$$

Taking the logarithmic derivatives of the above equation with respect to τ_1, we obtain

$$\frac{1}{(c_1\tau_1 + d_1)^2}\frac{\dot{x}}{x}(\gamma_1\tau_1, \tau_2) = \frac{\dot{x}}{x}(\tau_1, \tau_2),$$

or

$$G_{x,1}(\gamma_1\tau_1, \gamma_2\tau_2) = (c_1\tau_1 + d_1)^2 G_{x,1}(\tau_1, \tau_2), \tag{2.3}$$

where we let $\dot{x}$ denote the derivative of the two-variable function x with respect to the first variable. This shows that $G_{x,1}$ is a bi-modular form of weight $(2,0)$ on $\Gamma_1 \times \Gamma_2$ with the trivial character. The proof for the case $G_{y,1}$ is similar.

Likewise, taking the logarithmetic derivatives of the equation

$$F(\gamma_1\tau_1, \gamma_2\tau_2) = \chi(\gamma_1, \gamma_2)(c_1\tau_1 + d_1)(c_2\tau_2 + d_2)F(\tau_1, \tau_2)$$

with respect to τ_1, we obtain

$$\frac{1}{(c_1\tau_1 + d_1)^2}\frac{\dot{F}}{F}(\gamma_1\tau_1, \gamma_2\tau_2) = \frac{c_1}{(c_1\tau_1 + d_1)} + \frac{\dot{F}}{F}(\tau_1, \tau_2),$$

or, equivalently

$$G_{F,1}(\gamma_1\tau_1, \gamma_2\tau_2) = \frac{c_1(c_1\tau_1 + d_1)}{2\pi i} + (c_1\tau_1 + d_1)^2 G_{F,1}(\tau_1, \tau_2). \tag{2.4}$$

Now, differentiating (2.3) with respect to τ_1 again, we obtain

$$\frac{\dot{G}_{x,1}}{(c_1\tau_1 + d_1)^2}(\gamma_1\tau_1, \gamma_2\tau_2) = 2c_1(c_1\tau_1 + d_1)G_{x,1}(\tau_1, \tau_2) + (c_1\tau_1 + d_1)^2\dot{G}_{x,1}(\tau_1, \tau_2),$$

or

$$D_{q_1}G_{x,1}(\gamma_1\tau_1, \gamma_2\tau_2) = \frac{c_1(c_1\tau_1 + d_1)^3}{\pi i}G_{x,1}(\tau_1, \tau_2) + (c_1\tau_1 + d_1)^4 D_{q_1}G_{x,1}(\tau_1, \tau_2).$$

On the other hand, we also have, by (2.3) and (2.4),

$$G_{F,1}G_{x,1}(\gamma_1\tau_1,\gamma_2\tau_2)=\frac{c_1(c_1\tau_1+d_1)^3}{2\pi i}G_{x,1}(\tau_1,\tau_2)+(c_1\tau_1+d_1)^4G_{F,1}G_{x,1}(\tau_1,\tau_2).$$

From these two equations we see that $D_{q_1}G_{x,1}-2G_{F,1}G_{x,1}$ is a bi-modular form of weight $(4,0)$ with the trivial character.

Finally, differentiating (2.4) with respect to τ_1 and multiplying by $(c_1\tau_1+d_1)^2$ we have

$$\begin{aligned}D_{q_1}G_{F,1}(\gamma_1\tau_1,\gamma_2\tau_2)&=\frac{c_1^2(c_1\tau_1+d_1)^2}{(2\pi i)^2}+\frac{c_1(c_1\tau_1+d_1)^3}{\pi i}G_{F,1}(\tau_1,\tau_2)\\&\qquad+(c_1\tau_1+d_1)^4D_{q_1}G_{F,1}(\tau_1,\tau_2).\end{aligned}$$

Combining this with the square of (2.4) we see that $D_{q_1}G_{F,1}-G_{F,1}^2$ is a bi-modular form of weight $(4,0)$ on $\Gamma_1\times\Gamma_2$. This completes the proof of the lemma. □

Proof of Theorem 2.1. In light of Lemma 2.2, the functions a_k, b_k are all bi-modular functions on $\Gamma_1\times\Gamma_2$, and thus can be expressed as algebraic functions of x and y. Therefore, it suffices to verify (2.1) as formal identities. By the chain rule we have

$$\begin{pmatrix}D_{q_1}F\\D_{q_2}F\end{pmatrix}=\begin{pmatrix}x^{-1}D_{q_1}x & y^{-1}D_{q_1}y\\x^{-1}D_{q_2}x & y^{-1}D_{q_2}y\end{pmatrix}\begin{pmatrix}D_xF\\D_yF\end{pmatrix}.$$

It follows that

$$\begin{pmatrix}D_xF\\D_yF\end{pmatrix}=\frac{F}{G_{x,1}G_{y,2}-G_{x,2}G_{y,1}}\begin{pmatrix}G_{y,2} & -G_{y,1}\\-G_{x,2} & G_{x,1}\end{pmatrix}\begin{pmatrix}G_{F,1}\\G_{F,2}\end{pmatrix}$$

Writing

$$\Delta=G_{x,1}G_{y,2}-G_{x,2}G_{y,1},$$

and

$$\Delta_x=G_{F,1}G_{y,2}-G_{F,2}G_{y,1},\qquad \Delta_y=-G_{x,2}G_{F,1}+G_{x,1}G_{F,2},$$

we have

$$D_xF=F\frac{\Delta_x}{\Delta},\qquad D_yF=F\frac{\Delta_y}{\Delta}. \tag{2.5}$$

Applying the same procedure on D_xF again, we obtain

$$\begin{aligned}\begin{pmatrix}D_x^2F\\D_yD_xF\end{pmatrix}&=\frac{1}{\Delta}\begin{pmatrix}G_{y,2} & -G_{y,1}\\-G_{x,2} & G_{x,1}\end{pmatrix}\begin{pmatrix}D_{q_1}(F\Delta_x/\Delta)\\D_{q_2}(F\Delta_x/\Delta)\end{pmatrix}\\&=\frac{F}{\Delta}\begin{pmatrix}G_{y,2} & -G_{y,1}\\-G_{x,2} & G_{x,1}\end{pmatrix}\left\{\frac{\Delta_x}{\Delta}\begin{pmatrix}G_{F,1}\\G_{F,2}\end{pmatrix}+\begin{pmatrix}D_{q_1}(\Delta_x/\Delta)\\D_{q_2}(\Delta_x/\Delta)\end{pmatrix}\right\}.\end{aligned}$$

That is,

$$D_x^2F=F\frac{\Delta_x^2}{\Delta^2}+\frac{F}{\Delta}\left(G_{y,2}D_{q_1}\frac{\Delta_x}{\Delta}-G_{y,1}D_{q_2}\frac{\Delta_x}{\Delta}\right) \tag{2.6}$$

and

$$D_yD_xF=F\frac{\Delta_x\Delta_y}{\Delta^2}+\frac{F}{\Delta}\left(-G_{x,2}D_{q_1}\frac{\Delta_x}{\Delta}+G_{x,1}D_{q_2}\frac{\Delta_x}{\Delta}\right). \tag{2.7}$$

We then substitute (2.5), (2.6), and (2.7) into (2.1) and find that (2.1) indeed holds. (The details are tedious, but straightforward calculations. We omit the details here.) □

3. Bi-modular forms associated to solutions of hypergeometric differential equations

Here we will construct bi-modular forms of weight $(1,1)$ using solutions of some hypergeometric differential equations. Our main result of this section is the following theorem.

Theorem 3.1. *Let $0<a<1$ be a positive real number. Let $f(t)={}_2F_1(a,a;1;t)$ be a solution of the hypergeometric differential equation*

$$t(1-t)f''+[1-(1+2a)t]f'-a^2f=0. \tag{3.1}$$

Let

$$F(t_1,t_2)=f(t_1)f(t_2)(1-t_1)^a(1-t_2)^a,$$

$$x=\frac{t_1+t_2}{(t_1-1)(t_2-1)},\quad y=\frac{t_1t_2}{(t_1+t_2)^2}.$$

Then F is a bi-modular form of weight $(1,1)$ for $\Gamma_1\times\Gamma_2$. Furthermore, F, as a function of x and y is a solution of the partial differential equations

$$D_x(D_x-2D_y)F+x(D_x+a)(D_x+1-a)F=0, \tag{3.2}$$

and

$$D_y^2F-y(2D_y-D_x+1)(2D_y-D_x)F=0, \tag{3.3}$$

where $D_x=\partial/\partial x$ and $D_y=\partial/\partial y$.

Remark 3.1. Theorem 2.1 of Lian and Yau [11] is essentially the same as our Theorem 3.1, though the formulation and proof are different.

We will present our proof of Theorem 3.1 now. For this, we need one more ingredient, namely, the Schwarzian derivatives.

Lemma 3.2. *Let $f(t)$ and $f_1(t)$ be two linearly independent solutions of a differential equation*

$$f''+p_1f'+p_2f=0.$$

Set $\tau:=f_1(t)/f(t)$. Then the associated Schwarzian differential equation

$$2Q\left(\frac{dt}{d\tau}\right)^2+\{t,\tau\}=0,$$

where $\{t,\tau\}$ is the Schwarzian derivative

$$\{t,\tau\}=\frac{dt^3/d\tau^3}{dt/d\tau}-\frac{3}{2}\left(\frac{dt^2/d\tau^2}{dt/d\tau}\right)^2,$$

satisfies

$$Q=\frac{4p_2-2p_1'-p_1^2}{4}.$$

Proof. This is standard, and proof can be found, for instance, in Lian and Yau [12]. □

Proof of Theorem 3.1. Let f_1 be another solution of (3.1) linearly independent of f, and set $\tau = f_1/f$. Then a classical identity asserts that

$$f^2 = c\exp\left\{-\int^t \frac{1-(1+2a)u}{u(1-u)}\,du\right\}\frac{dt}{d\tau} = \frac{c\,dt/d\tau}{t(1-t)^{2a}},$$

where c is a constant depending on the choice of f_1. Thus, letting

$$q_1 = e^{2\pi i f_1(t_1)/f(t_1)} \quad\text{and}\quad q_2 = e^{2\pi i f_1(t_2)/f(t_2)},$$

the function F, with a suitable choice of f_1, is in fact

$$F(t_1,t_2) = \left(\frac{D_{q_1}t_1\cdot D_{q_2}t_2}{t_1t_2}\right)^{1/2}.$$

We now apply the differential identities in (2.1), which hold for arbitrary F, x, and y. We have

$$G_{x,1} := \frac{D_{q_1}x}{x} = \frac{(1+t_2)D_{q_1}t_1}{(t_1+t_2)(1-t_1)}, \qquad G_{x,2} := \frac{D_{q_2}x}{x} = \frac{(1+t_1)D_{q_2}t_2}{(t_1+t_2)(1-t_2)},$$

$$G_{y,1} := \frac{D_{q_1}y}{y} = \frac{(t_2-t_1)D_{q_1}t_1}{t_1(t_1+t_2)}, \qquad G_{y,2} := \frac{D_{q_2}y}{y} = \frac{(t_1-t_2)D_{q_2}t_2}{t_2(t_1+t_2)},$$

$$G_{F,1} := \frac{D_{q_1}F}{F} = \frac{t_1D_{q_1}^2t_1-(D_{q_1}t_1)^2}{2t_1D_{q_1}t_1}, \qquad G_{F,2} := \frac{D_{q_2}F}{F} = \frac{t_2D_{q_2}^2t_2-(D_{q_2}t_2)^2}{2t_2D_{q_2}t_2}.$$

It follows that

$$a_0 := \frac{2G_{y,1}G_{y,2}}{G_{x,1}G_{y,2}+G_{y,1}G_{x,2}} = -\frac{2(t_1-1)(t_2-1)}{t_1t_2+1} = -\frac{2}{1+x},$$

$$b_0 := \frac{2G_{x,1}G_{x,2}}{G_{x,1}G_{y,2}+G_{y,1}G_{x,2}} = \frac{2t_1t_2(t_1+1)(t_2+1)}{(t_1-t_2)^2(t_1t_2+1)} = \frac{2y(1+2x)}{(1+x)(1-4y)},$$

$$\begin{aligned} a_1 :&= \frac{G_{y,2}^2(D_{q_1}G_{x,1}-2G_{F,1}G_{x,1})-G_{y,1}^2(D_{q_2}G_{x,2}-2G_{F,2}G_{x,2})}{G_{x,1}^2G_{y,2}^2-G_{y,1}^2G_{x,2}^2} \\ &= \frac{t_1+t_2}{t_1t_2+1} = \frac{x}{1+x}, \end{aligned}$$

$$\begin{aligned} b_1 :&= \frac{-G_{x,2}^2(D_{q_1}G_{x,1}-2G_{F,1}G_{x,1})+G_{x,1}^2(D_{q_2}G_{x,2}-2G_{F,2}G_{x,2})}{G_{x,1}^2G_{y,2}^2-G_{y,1}^2G_{x,2}^2} \\ &= \frac{t_1t_2(t_1+1)(t_2+1)}{(t_1-t_2)^2(t_1t_2+1)} = \frac{y(1+2x)}{(1+x)(1-4y)}, \end{aligned}$$

$$a_2 := \frac{G_{y,2}^2(D_{q_1}G_{y,1}-2G_{F,1}G_{y,1})-G_{y,1}^2(D_{q_2}G_{y,2}-2G_{F,2}G_{y,2})}{G_{x,1}^2G_{y,2}^2-G_{y,1}^2G_{x,2}^2} = 0,$$

$$\begin{aligned} b_2 :&= \frac{-G_{x,2}^2(D_{q_1}G_{y,1}-2G_{F,1}G_{y,1})+G_{x,1}^2(D_{q_2}G_{y,2}-2G_{F,2}G_{y,2})}{G_{x,1}^2G_{y,2}^2-G_{y,1}^2G_{x,2}^2} \\ &= -\frac{2t_1t_2}{(t_1-t_2)^2} = -\frac{2y}{1-4y}. \end{aligned}$$

Moreover, we have

$$\begin{aligned} a_3 :&= -\frac{G_{y,2}^2(D_{q_1}G_{F,1} - G_{F,1}^2) - G_{y,1}^2(D_{q_2}G_{F,2} - G_{F,2}^2)}{G_{x,1}^2 G_{y,2}^2 - G_{y,1}^2 G_{x,2}^2} \\ &= \frac{(t_1-1)(t_2-1)(t_1+t_2)\left\{t_1^2\dot t_2^4(2\dot t_1\dddot t_1 - 3\ddot t_1^2) - t_2^2\dot t_1^4(2\dot t_2\dddot t_2 - 3\ddot t_2^2)\right\}}{4(t_1-t_2)(t_1^2t_2^2-1)\dot t_1^4\dot t_2^4}, \end{aligned}$$

where, for brevity, we let $\dot t_j$, $\ddot t_j$, $\dddot t_j$ denote the derivatives $D_{q_j}t_j$, $D_{q_j}^2 t_j$, and $D_{q_j}^3 t_j$, respectively. To express a_3 in terms of x and y, we note that, by Lemma 3.2,

$$\begin{aligned} 2\dot t_j\dddot t_j - 3\ddot t_j^2 &= -\dot t_j^4\left(\frac{-4a^2}{t_j(1-t_j)} - 2\frac{d}{dt_j}\frac{1-(1+2a)t_j}{t_j(1-t_j)} - \frac{(1-(1+2a)t_j)^2}{t_j^2(1-t_j)^2}\right) \\ &= -\frac{(t_j-1)^2 + 4a(1-a)t_j}{t_j^2(t_j-1)^2}\dot t_j^4. \end{aligned}$$

It follows that

$$a_3 = a(1-a)\frac{t_1+t_2}{t_1t_2+1} = \frac{a(1-a)x}{1+x}.$$

Likewise, we have

$$\begin{aligned} b_3 :&= -\frac{-G_{x,2}^2(D_{q_1}G_{F,1} - G_{F,1}^2) + G_{x,1}^2(D_{q_2}G_{F,2} - G_{F,2}^2)}{G_{x,1}^2 G_{y,2}^2 - G_{y,1}^2 G_{x,2}^2} \\ &= a(1-a)\frac{t_1t_2(t_1+t_2)}{(t_1-t_2)^2(t_1t_2+1)} = \frac{a(1-a)xy}{(1+x)(1-4y)}. \end{aligned}$$

Then, by (2.1), the function F, as a function of x and y, satisfies

$$D_x^2F - \frac{2}{1+x}D_xD_yF + \frac{x}{1+x}D_xF + \frac{a(1-a)x}{1+x}F = 0 \tag{3.4}$$

and

$$\begin{aligned} D_y^2F &+ \frac{2y(1+2x)}{(1+x)(1-4y)}D_xD_yF + \frac{y(1+2x)}{(1+x)(1-4y)}D_xF \\ &- \frac{2y}{1-4y}D_yF + \frac{a(1-a)xy}{(1+x)(1-4y)}F = 0. \end{aligned} \tag{3.5}$$

Finally, we can deduce the claimed differential equations by taking (3.4) times $(1+x)$ and (3.5) times $(1-4y)$ minus (3.4) times y, respectively. □

4. Examples

Example 4.1. Let j be the elliptic modular j-function, and let $E_4(q)$ be the Eisenstein series of weight 4 on $SL_2(\mathbb{Z})$. Set

$$x = 2\frac{1/j(q_1) + 1/j(q_2) - 1728/(j(q_1)j(q_2))}{1+\sqrt{(1-1728/j(q_1))(1-1728/j(q_2))}}, \qquad y = \frac{1}{j(q_1)j(q_2)x^2},$$

and

$$F = (E_4(q_1)E_4(q_2))^{1/4}.$$

Then F is a bi-modular form of weight $(1,1)$ for $SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z})$, and it satisfies the system of partial differential equations:

$$\begin{aligned} (1-432x)D_x^2F - 2D_xD_yF - 432xD_x - 60xF &= 0, \\ (1-4y)D_y^2F + 4yD_xD_yF - yD_x^2F - yD_xF - 2yD_yF &= 0. \end{aligned}$$

We have noticed that this system of differential equation belongs to a general class of partial differential equations which involve solutions of hypergeometric hypergeometric differential equations discussed in Theorem 3.1.

Here we will prove the assertion of Example 4.1 using Theorem 3.1.

Proof of Example 4.1. We first make a change of variable $x \mapsto -\bar{x}/432$. For convenience, we shall denote the new variable $\bar{x}$ by x again. Thus, we are required to show that the functions

$$x = -864\frac{1/j(q_1)+1/j(q_2)-1728/(j(q_1)j(q_2))}{1+\sqrt{(1-1728/j(q_1))(1-1728/j(q_2))}}, \qquad y = \frac{432^2}{j(q_1)j(q_2)x^2},$$

and $F = (E_4(q_1)E_4(q_2))^{1/4}$ satisfy

$$(1+x)D_x^2F - 2D_xD_yF + xD_x + \frac{5}{36}xF = 0,$$

and

$$(1-4y)D_y^2F + 4yD_xD_yF - yD_x^2F - yD_xF - 2yD_yF = 0.$$

For brevity, we let j_1 denote $j(q_1)$ and j_2 denote $j(q_2)$. We now observe that the function x can be alternatively expressed as

$$\begin{aligned} x &= -864\frac{1/j_1+1/j_2-1728/(j_1j_2)}{1-(1-1728/j_1)(1-1728/j_2)}\left(1-\sqrt{(1-1728/j_1)(1-1728/j_2)}\right) \\ &= \frac{1}{2}\left(\sqrt{(1-1728/j_1)(1-1728/j_2)}-1\right). \end{aligned}$$

Setting

$$t_1 = \frac{\sqrt{1-1728/j_1}-1}{\sqrt{1-1728/j_1}+1}, \qquad t_2 = \frac{\sqrt{1-1728/j_2}-1}{\sqrt{1-1728/j_2}+1},$$

we have

$$x = \frac{t_1+t_2}{(t_1-1)(t_2-1)}.$$

Moreover, the functions j_k, written in terms of t_k, are $j_k = 432(t_k-1)^2/t_k$ for $k = 1, 2$. It follows that

$$y = \frac{432^2}{j_1j_2x^2} = \frac{t_1t_2}{(t_1+t_2)^2}.$$

In view of Theorem 3.1, setting

$$t = \frac{\sqrt{1-1728/j(q)}-1}{\sqrt{1-1728/j(q)}+1}$$

it remains to show that the function $f(t) = E_4(q)^{1/4}(1-t)^{-1/6}$ is a solution of the hypergeometric differential equation

$$t(1-t)f'' + (1+4t/3)f' - \frac{1}{36}f = 0,$$

or equivalently, that

$$\frac{E_4(q)^{1/4}}{(1-t)^{1/6}} = {}_2F_1(1/6, 1/6; 1; t).$$

This, however, follows from the classical identity

$$E_4(q)^{1/4} = {}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; \frac{1728}{j(q)}\right)$$

and Kummer's transformation formula

$$\left(\frac{1+\sqrt{1-z}}{2}\right)^{2a} {}_2F_1\left(a,b;a+b+\frac{1}{2};z\right) = {}_2F_1\left(2a,a-b+\frac{1}{2};a+b+\frac{1}{2};\frac{\sqrt{1-z}-1}{\sqrt{1-z}+1}\right).$$

This completes the proof of Example 4.1. □

Remark 4.1. The functions x and y in Example 4.1 (up to constant multiple) have also appeared in the paper of Lian and Yau [12], Corollary 1.2, as the mirror map of the family of $K3$ surfaces defined by degree 12 hypersurfaces in the weighted projective space $\mathbb{P}^3[1,1,4,6]$. Further, this $K3$ family is derived from the square of a family of elliptic curves in the weighted projective space $\mathbb{P}^2[1,2,3]$. (The geometry behind this phenomenon is the so-called Shoida–Inose structures, which has been studied in detail by Long [13] for one-parameter families of $K3$ surfaces, and their Picard–Fuchs differential equations.) Lian and Yau [12] proved that the mirror map of the $K3$ family can be given in terms of the elliptic j-function, and indeed, by the functions x and y (up to constant multiple). We will discuss more examples of families of $K3$ surfaces, their Picard–Fuchs differential equations and mirror maps in the section 6.

Along the same vein, we obtain more examples of bi-modular forms of weight $(1,1)$ and bi-modular functions on $\Gamma_0(N)\times\Gamma_0(N)$ for $N=2,3,4$.

Theorem 4.1. *We retain the notations of Theorem 3.1. Then the solutions of the differential equations (3.2) and (3.3) for the cases $a=1/2,1/3,1/4,1/6$ can be expressed in terms of bi-modular forms and bi-modular functions.*

(a) For $a=1/2$, they are given by

$$F(q_1,q_2)=\theta_4(q_1)^2\theta_4(q_2)^2, \qquad t=\theta_2(q)^4/\theta_3(q)^4,$$

which are modular on $\Gamma_0(4)\times\Gamma_0(4)$.

(b) For $a=1/3$, they are

$$F(q_1,q_2)=\frac{1}{2}(3E_2(q_1^3)-E_2(q_1))^{1/2}(3E_2(q_2^3)-E_2(q_2))^{1/2}, \quad t=-27\frac{\eta(3\tau)^{12}}{\eta(\tau)^{12}},$$

which are modular on $\Gamma_0(3)\times\Gamma_0(3)$.

(c) For $a=1/4$, they are

$$F(q_1,q_2)=(2E_2(q_1^2)-E_2(q_1))^{1/2}(2E_2(q_2^2)-E_2(q_2))^{1/2}, \quad t=-64\frac{\eta(2\tau)^{24}}{\eta(\tau)^{24}},$$

which are modular are $\Gamma_0(2)\times\Gamma_0(2)$.

(d) For $a=1/6$, they are given as in Example 4.1.

Here

$$\eta(\tau)=q^{1/24}\prod_{n\in\mathbb{N}}(1-q^n)$$

is the Dedekind eta-function, and

$$\theta_2(q)=q^{1/4}\sum_{n\in\mathbb{Z}}q^{n(n+1)}, \quad \theta_3(q)=\sum_{n\in\mathbb{Z}}q^{n^2}, \quad \theta_4(q)=\sum_{n\in\mathbb{Z}}(-1)^nq^{n^2}$$

are theta-series.

Lemma 4.2. *Let Γ be a subgroup of $SL_2(\mathbb{R})$ commensurable with $SL_2(\mathbb{Z})$. Let $f(\tau)$ be a modular form of weight 1, and $t(\tau)$ be a non-constant modular function on Γ. Then, setting*

$$G_t = \frac{D_q t}{t}, \qquad G_f = \frac{D_q f}{f},$$

we have

$$D_t^2 f + \frac{D_q G_t - 2G_f G_t}{G_t^2} D_t f - \frac{D_q G_f - G_f^2}{G_t^2} f = 0.$$

Proof of Theorem 4.1. To prove part (a) we use the well-known identities

$$\theta_3^2 = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{\theta_2^4}{\theta_3^4}\right)$$

(see [17] for a proof using Lemma 4.2) and

$$\theta_3^4 = \theta_2^4 + \theta_4^4.$$

Applying Theorem 3.1 and observing that

$$\theta_3^2 \left(1 - \frac{\theta_2^4}{\theta_3^4}\right)^{1/2} = \theta_3^2 \frac{\theta_4^2}{\theta_3^2} = \theta_4^2,$$

we thus obtain the claimed differential equation.

For parts (b), we need to show that the function

$$f(\tau) = \frac{(3E_2(q^3) - E_2(q))^{1/2}}{(1-t)^{1/3}}$$

satisfies

$$t(1-t)\frac{d^2}{dt^2} f + (1 - 5t/3)\frac{d}{dt} f - \frac{1}{9} f = 0,$$

or, equivalently,

$$(1-t)D_t^2 f - \frac{2}{3} t D_t f - \frac{1}{9} t f = 0. \tag{4.1}$$

Let G_t and G_f be defined as in Lemma 4.2. For convenience we also let $g = (3E_2(q^3) - E_2(q))/2$. We have

$$G_t = \frac{1}{2}(3E_2(q^3) - E_2(q)) = g$$

and

$$G_f = \frac{D_q g}{2g} - \frac{1}{3(1-t)} D_q t = \frac{D_q g}{2g} + \frac{t}{3(1-t)} g.$$

It follows that

$$\frac{D_q G_t - 2G_f G_t}{G_t^2} = g^{-2}\left(D_q g - 2\left(\frac{D_q g}{2g} + \frac{t}{3(1-t)} g\right) g\right) = -\frac{2t}{3(1-t)}.$$

Moreover, we can show that $(D_q G_f - G_f^2)/G_t^2$ is equal to $-t/(9(1-t))$ by comparing enough Fourier coefficients. This establishes (4.1) and hence part (b).

The proof of part (c) is similar, and we shall skip the details here. □

5. More examples

We may also consider groups like $\Gamma_0(N)^* \times \Gamma_0(N)^*$ where $\Gamma_0(N)^*$ denotes the group generated by $\Gamma_0(N)$ and the Atkin–Lehner involution $w_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$ for some N. (Note that $\Gamma_0(N)^*$ is contained in the normalizer of $\Gamma_0(N)$ in $SL_2(\mathbb{R})$.) Also the entire list of N giving rise to genus zero groups $\Gamma_0(N)^*$ is known (cf.[4]), and we will be interested in some of those genuz zero groups. We can determine differential equations satisfied by bi-modular forms of weight $(1,1)$ on $\Gamma_0(N)^* \times \Gamma_0(N)^*$ for some N (giving rise to genus zero subgroups $\Gamma_0(N)^*$).

We first prove a generalization of Theorem 3.1.

Theorem 5.1. *Let $0 < a, b < 1$ be positive real numbers. Let $f(t) = {}_2F_1(a,b;1;t)$ be a solution of the hypergeometric differential equation*

$$t(1-t)f'' + [1-(1+a+b)t]f' - abf = 0. \tag{5.1}$$

Set

$$F(t_1,t_2) = f(t_1)f(t_2)(1-t_1)^{(a+b)/2}(1-t_2)^{(a+b)/2},$$
$$x = t_1 + t_2 - 2, \qquad y = (1-t_1)(1-t_2).$$

Then F, as a function of x and y, satisfies

$$D_x^2 F + 2D_xD_yF - \frac{1}{x+y+1}D_xF + \frac{x}{x+y+1}D_yF + \frac{(2ab-a-b)x}{2(x+y+1)}F = 0 \tag{5.2}$$

and

(5.3)
$$D_y^2F + \frac{2y}{x^2}D_xD_yF + \frac{y^2}{x^2(x+y+1)}D_xF + \frac{y-x-x^2}{x(x+y+1)}D_yF - \frac{(a+b)(a+b-2)(x^2+x)+(a-b)^2xy-(4ab-2a-2b)y}{4x(x+y+1)}F = 0.$$

Proof. The proof is very similar to that of Theorem 3.1. Let f_1 be another solution of the hypergeometric differential equation (5.1), and set $\tau := f_1/f$. We find

$$f^2 = c\exp\left\{-\int^t \frac{1-(1+a+b)u}{u(1-u)}\,du\right\}\frac{dt}{d\tau} = \frac{c\,dt/d\tau}{t(1-t)^{a+b}}$$

for some constant c depending on the choice of f_1. Thus, setting

$$q_1 = e^{2\pi i f_1(t_1)/f(t_1)} \quad \text{and} \quad q_2 = e^{2\pi i f_1(t_2)/f(t_2)},$$

we have

$$F(t_1,t_2) = c'\left(\frac{D_{q_1}t_1 \cdot D_{q_2}t_2}{t_1t_2}\right)^{1/2}$$

for some constant c'. We now apply the differential identities (2.1). We have, for $j = 1,2$,

$$G_{x,j} := \frac{D_{q_j}x}{x} = \frac{D_{q_j}t_j}{t_1+t_2-2}, \qquad G_{y,j} := \frac{D_{q_j}y}{y} = -\frac{D_{q_j}t_j}{1-t_j},$$

and

$$G_{F,j} := \frac{D_{q_j}F}{F} = \frac{t_jD_{q_j}^2t_j - (D_{q_j}t_j)^2}{2t_jD_{q_j}t_j}.$$

It follows that the coefficients in (2.1) are

$$a_0 = 2, \qquad b_0 = \frac{2(1-t_1)(1-t_2)}{(t_1+t_2-2)^2} = \frac{2y}{x^2},$$

$$a_1 = -\frac{1}{t_1t_2} = -\frac{1}{x+y+1}, \qquad b_1 = \frac{(1-t_1)^2(1-t_2)^2}{t_1t_2(t_1+t_2-2)^2} = \frac{y^2}{x^2(x+y+1)},$$

$$a_2 = \frac{t_1+t_2-2}{t_1t_2} = \frac{x}{x+y+1},$$

$$b_2 = -\frac{t_1^2+t_1t_2+t_2^2-2t_1-2t_2+1}{t_1t_2(t_1+t_2-2)} = \frac{y-x-x^2}{x(x+y+1)}.$$

Moreover, we have

$$a_3 = \left\{-\frac{(1-t_1)^2(2\dot{t}_1\dddot{t}_1-3\ddot{t}_1^2)}{4(t_1-t_2)\dot{t}_1^4} + \frac{(1-t_2)^2(2\dot{t}_2\dddot{t}_2-3\ddot{t}_2^2)}{4(t_1-t_2)\dot{t}_2^4} - \frac{2t_1t_2-t_1-t_2}{4t_1^2t_2^2}\right\} \times (t_1+t_2-2),$$

where we, as before, employ the notations $\dot{t}_j$, $\ddot{t}_j$, $\dddot{t}_j$ for the derivatives $D_{q_j}t_j$, $D_{q_j}^2 t_j$, and $D_{q_j}^3 t_j$, respectively. Now, by Lemma 3.2, we have

$$2\dot{t}_j\dddot{t}_j - 3\ddot{t}_j^2 = \dot{t}_j^4\frac{(a-b)^2t_j^2-(1-t_j)^2+(4ab-2a-2b)t_j}{t_j^2(1-t_j)^2}.$$

It follows that

$$a_3 = \frac{(2ab-a-b)(t_1+t_2-2)}{2t_1t_2} = \frac{(2ab-a-b)x}{x+y+1}.$$

A similar calculation shows that

$$b_3 = -\frac{(a+b)(a+b-2)(x^2+x)+(a-b)^2xy-(4ab-2a-2b)y}{4x(x+y+1)}.$$

This proves the claimed result. □

Remark 5.1. *It should be pointed out that the first identity in our proof of Theorem 5.1 is equivalent to the formula in Proposition 4.4 of Lian and Yau* [10].

We now obtain new examples of bi-modular forms of weight $(1,1)$ on $\Gamma_0(N)^* \times \Gamma_0(N)^*$ for some N.

Theorem 5.2. *When the pairs of numbers (a,b) in Theorem 5.1 are given by $(1/12,5/12)$, $(1/12,7/12)$, $(1/8,3/8)$, $(1/8,5/8)$, $(1/6,1/3)$, $(1/6,2/3)$, $(1/4,1/4)$ and $(1/4,3/4)$, the solutions $F(t_1,t_2)$ of the differential equations* (5.2) *and* (5.3) *are bi-modular forms of weight* $(1,1)$ *on* $\Gamma_0(N)^*\times\Gamma_0(N)^*$ *with* $N = 1,1,2,2,3,3,4,4$, *respectively.*

Proof. We shall prove only the cases $(a,b) = (1/6,1/3)$ and $(1/6,2/3)$; the other cases can be proved in the same manner.

Let

$$s(\tau) = -27\frac{\eta(3\tau)^{12}}{\eta(\tau)^{12}}, \qquad E_2(q) = 1 - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}.$$

From the proof of Part (b) of Theorem 4.1 we know that

$$f(\tau) = \frac{(3E_2(q^3)-E_2(q))^{1/2}}{(1-s)^{1/3}},$$

as a function of s, is equal to $\sqrt{2}\,{}_2F_1(1/3,1/3;1;s)$. Now, applying the quadratic transformation formula

$$ {}_2F_1(\alpha,\beta;\alpha-\beta+1;x)=(1-x)^{-\alpha}\,{}_2F_1\left(\frac{\alpha}{2},\frac{1+\alpha}{2}-\beta;\alpha-\beta+1;-\frac{4x}{(1-x)^2}\right) $$

for hypergeometric functions (see, for example [1, Theorem 3.1.1]) with $\alpha=\beta=1/3$, we obtain

$$ (3E_2(q^3)-E_2(q))^{1/2}=\sqrt{2}\,{}_2F_1\left(\frac16,\frac13;1;-\frac{4s}{(1-s)^2}\right). $$

Observing that the action of the Atkin-Lehner involution w_3 sends s to $1/s$, we find that the function $s/(1-s)^2$ is modular on $\Gamma_0(3)^*$. This proves that $F(t_1,t_2)$ is a bi-modular form of weight $(1,1)$ for $\Gamma_0(3)^*\times\Gamma_0(3)^*$ in the case $(a,b)=(1/6,1/3)$.

Furthermore, an application of another hypergeometric function identity

$$ {}_2F_1(\alpha,\beta;\gamma;x)=(1-x)^{-\alpha}\,{}_2F_1\left(\alpha,\gamma-\beta;\gamma;\frac{x}{x-1}\right) $$

yields

$$ (3E_2(q^3)-E_2(q))^{1/2}=\sqrt{2}\left(\frac{1-s}{1+s}\right)^{1/3}{}_2F_1\left(\frac16,\frac23;1;\frac{4s}{(1+s)^2}\right). $$

This corresponds to the case $(a,b)=(1/6,2/3)$. Again, the function $4s/(1+s)^2$ is modular on $\Gamma_0(3)^*$. This implies that $F(t_1,t_2)$ is a bi-modular form of weight $(1,1)$ for $\Gamma_0(3)^*\times\Gamma_0(3)^*$ for the case $(a,b)=(1/6,2/3)$. □

Remark 5.2. *For the remaining pairs (a,b) in Theorem 5.2, we simply list the exact expressions of $F(t_1,t_2)$ in terms of modular forms as proofs are similar.*

For $(a,b)=(1/12,5/12)$ and $(1/12,7/12)$, they are

$$ \left(\frac{E_6(q_1)E_6(q_2)}{E_4(q_1)E_4(q_2)}\right)^{1/2},\qquad \text{and}\qquad \left(\frac{E_8(q_1)E_8(q_2)}{E_6(q_1)E_6(q_2)}\right)^{1/2}, $$

respectively, where E_k are the Eisenstein series in (2.2).

For $(a,b)=(1/8,3/8)$ and $(1/8,5/8)$, they are

$$ \prod_{j=1}^2\left(\frac{1+s_j}{1-s_j}(2E_2(q_j^2)-E_2(q_j))\right)^{1/2},\quad \text{and}\quad \prod_{j=1}^2\left(\frac{1-s_j}{1+s_j}(2E_2(q_j^2)-E_2(q_j))\right)^{1/2}, $$

respectively, where $s_j=-64\eta(\tau_j)^{24}/\eta(\tau_j)^{24}$.

For $(a,b)=(1/6,1/3)$ and $(1/6,2/3)$, they are

$$ \prod_{j=1}^2\left(\frac{1+s_j}{1-s_j}(3E_2(q_j^3)-E_2(q_j))\right)^{1/2},\quad \text{and}\quad \prod_{j=1}^2\left(\frac{1-s_j}{1+s_j}(3E_2(q_j^3)-E_2(q_j))\right)^{1/2}, $$

respectively, where $s_j=-27\eta(3\tau_j)^{12}/\eta(\tau_j)$.

For $(a,b)=(1/4,1/4)$ and $(1/4,3/4)$, they are

$$ \prod_{j=1}^2\left(2E_2(q_j^2)-E_2(q_j)\right)^{1/2},\qquad \text{and}\qquad \prod_{j=1}^2\left(2E_2(q_j^2)-E_2(q_j)\right)^{1/2}\frac{1-s_j}{1+s_j}, $$

respectively, where $s_j=\theta_2(q_j)^4/\theta_3(q_j)^4$.

6. Picard–Fuchs differential equations of Familes of $K3$ surfaces : Part I

One of the motivations of our investigation is to understand the mirror maps of families of $K3$ surfaces with large Picard nubmers, e.g., $19, 18, 17$ or 16. Some examples of such families of $K3$ surfaces were discussed in Lian–Yau [11], Hosono–Lian–Yau [8] and also in Verrill-Yui [16]. Some of $K3$ families occured considering degenerations of Calabi–Yau families.

Our goal here is to construct families of $K3$ surfaces whose Picard–Fuchs differential equations are given by the differential equations satisfied by bi-modular forms we constructed in the earlier sections. In this section, we will look into the families of $K3$ surfaces appeared in Lian and Yau [10, 11].

Let S be a $K3$ surface. We recall some general theory about $K3$ surfaces which are relevant to our discussion. We know that

$$H^2(S,\mathbb{Z}) \simeq (-E_8)^2 \perp U^3$$

where U is the hyperbolic plane $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and E_8 is the even unimodular negative definite lattice of rank 8. The Picard group of S, $\mathrm{Pic}(S)$, is the group of linear equivalence classes of Cartier divisors on S. Then $\mathrm{Pic}(S)$ injects to $H^2(X,\mathbb{Z})$, and the image of $\mathrm{Pic}(S)$ is the algebraic cycles in $H^2(S,\mathbb{Z})$. As $Pic(S)$ is torsion-free, it may be regarded as a lattice in $H^2(S,\mathbb{Z})$, called the *Picard lattice*, and its rank is denoted by $\rho(S)$.

According to Arnold–Dolgachev [5], two $K3$ surfaces form a mirror pair $(S,\hat{S})$ if

$$\mathrm{Pic}(S)^{\perp}_{H^2(S,\mathbb{Z})} = \mathrm{Pic}(\hat{S}) \perp U \quad \text{as lattices}$$

In terms of ranks, a mirror pair $(S,\hat{S})$ is related by the identity:

$$22 - \rho(S) = \rho(\hat{S}) + 2 \Leftrightarrow \rho(S) + \rho(\hat{S}) = 20.$$

Example 6.1. We will be interested in mirror pairs of $K3$ surfaces $(S,\hat{S})$ whose Picard lattices are of the form

$$Pic(S) = U \quad and \quad Pic(\hat{S}) = U_2 \perp (-E_8)^2.$$

We go back to our Example 4.1, and discuss geometry behind that example. Associated to this example, there is a family of $K3$ surfaces in the weighted projective 3-space $\mathbb{P}^3[1,1,4,6]$ with weight $(q_1,q_2,q_3,q_4) = (1,1,4,6)$. There is a mirror pair of $K3$ surfaces $(S,\hat{S})$. Here we know (cf. Belcastro [3]) that

$$\mathrm{Pic}(S) = U \quad \text{so that } \rho(S) = 2,$$

and that S has a mirror partner $\hat{S}$ whose Picard lattice is given by

$$\mathrm{Pic}(\hat{S}) = U \perp (-E_8)^2 \quad \text{so that } \rho(\hat{S}) = 18.$$

The mirror $K3$ family can be defined by a hypersurface in the orbifold ambient space $\mathbb{P}^3[1,1,4,6]/G$ of degree 12. Here G is the discrete group of symmetry and can be given explicitly by $G = (\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}) = \langle g_1 \rangle \times \langle g_2 \rangle$ where g_1, g_2 are generatoers whose actions are given by:

$$\begin{array}{rcl} g_1 : (Y_1,Y_2,Y_3,Y_4) & \mapsto & (\zeta_3 Y_1, Y_2, \zeta_3^{-1} Y_3, Y_4) \\ g_2 : (Y_1,Y_2,Y_3,Y_4) & \mapsto & (Y_1, -Y_2, Y_3, -Y_4) \end{array}$$

(Here $\zeta_3 = e^{2\pi i/3}$.) The G-invariant monomials are

$$Y_1^{12},\ Y_2^{12},\ Y_3^3,\ Y_4^2,\ Y_1^6Y_2^6,\ Y_1Y_2Y_3Y_4.$$

The matrix of exponents is the following 6×5 matrix

$$\begin{pmatrix} 12 & 0 & 0 & 0 & 1 \\ 0 & 12 & 0 & 0 & 1 \\ 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 2 & 1 \\ 6 & 6 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

whose rank is 2. Therefore we may conclude that the typical G-invariant polynomials is in 2-parameters, and $\hat{S}$ can be defined by the following 2-parameter family of hypersurfaces of degree 12

$$Y_1^{12} + Y_2^{12} + Y_3^3 + Y_4^2 + \lambda Y_1Y_2Y_3Y_4 + \phi Y_1^6Y_2^6 = 0$$

in $\mathbb{P}^3[1,1,4,6]/G$ with parameters λ and ϕ.

How do we conmpute the Picard–Fuchs differential equation of this $K3$ family?

Several physics articles are devoted to this question. For instance, Klemm–Lerche–Mayr [9], Hosono–Klemm–Theisen–Yau [7], Lian and Yau [11] determined the Picard–Fuchs differential equation of the Calabi–Yau family using the GKZ hypergeometric system. Also it was noticed (cf. [9], [11]) that the Picard–Fuchs system of this family of $K3$ surfaces can be realized as the degeneration of the Picard–Fuchs systems of the Calabi–Yau family. The family of Calabi–Yau threefolds is a degree 24 hypersurfaces in $\mathbb{P}^4[1,1,2,8,12]$ with $h^{1,1} = 3$. The defining equation for this family is given by

$$Z_1^{24} + Z_2^{24} + Z_3^{12} + Z_4^3 + Z_5^2 - 12\psi_0 Z_1^6 Z_2^6 Z_3^6 - 6\psi_1(Z_1Z_2Z_3)^6 - \psi_2(Z_1Z_2)^{12} = 0.$$

Its Picard–Fuchs system is given by

$$\begin{aligned} L_1 &= \Theta_x(\Theta_x - 2\Theta_z) - 12\,x(6\Theta_x+5)(6\Theta_x+1) \\ L_2 &= \Theta_y^2 - y(2\Theta_y - \Theta_z + 1)(2\Theta_y - \Theta_z) \\ L_3 &= \Theta_z(\Theta_z - 2\Theta_y) - z(\Theta_z - \Theta_x + 1)(2\Theta_z - \Theta_x) \end{aligned}$$

where

$$x = -\frac{2\psi_1}{1728^2\psi_0^6},\ y = \frac{1}{\psi_2^2} \quad \text{and} \quad z = -\frac{\psi_2}{4\psi_1^2}$$

are deformation coordinates.

Now the intersection of this Calabi–Yau hypersurface with the hyperplane $Z_2 - t\,Z_1 = 0$ gives rise to a family of $K3$ surfaces

$$b_0Y_1Y_2Y_3Y_4 + b_1Y_1^{12} + b_2Y_2^{12} + b_3Y_3^3 + b_4Y_4^2 + b_5Y_1^6Y_2^6 = 0$$

in $\mathbb{P}^3[1,1,4,6]$ of degree 12. Taking $(b_0,b_1,b_2,b_3,b_4,b_5) = (\lambda,1,1,1,1,\phi)$ we obtain the 2-parameter family of $K3$ surfaces described above. The Picard–Fuchs system of this $K3$ family is obtained by taking the limit $y = 0$ in the Picard–Fuchs system for the Calabi–Yau family:

$$\begin{aligned} L_1 &= \Theta_x(\Theta_x - 2\Theta_z) - 12\,x(6\Theta_x+5)(6\Theta_x+1) \\ L_3 &= \Theta_z^2 - z(2\Theta_z - \Theta_x + 1)(2\Theta_z - \Theta_x) \end{aligned}$$

Further, if we intersect this $K3$ family with the hyperplane $Y_2 - s\,Y_1 = 0$, we obtain a family of elliptic curves:

$$c_0 W_1 W_2 W_3 + c_1 W_1^6 + c_2 W_2^3 + c_3 W_3^2 = 0$$

in $\mathbb{P}^2[1,2,3]$, whose Picard–Fuchs equation is given by

$$L = \Theta_x^2 - 12\,x(6\Theta_x + 5)(6\Theta_x + 1).$$

Here we describe a relation of the Picard–Fuchs system of the above family of $K3$ surfaces to the differential equation discussed in Example 4.1.

Remark 6.1. We note that, in view of our proof of Example 4.1, the process of setting $z = 0$ in the above Picard–Fuchs system $\{L_1, L_3\}$ is equivalent to setting $t_1 = 0$ or $t_2 = 0$ in x and y in Example 4.1. Our Theorem 3.1 then implies that $F(t) = (1-t)^{1/6}\,{}_2F_1(1/6, 1/6; 1; t)$ satisfies

$$(1+x)D_x^2 F + xD_x F + \frac{5}{36} xF = 0$$

with $x = t/(1-t)$, or equivalently, (making a change of variable $x \mapsto -x$)

$$x(1-x)F'' + (1-2x)F' - \frac{5}{36}F = 0$$

with $x = t/(t-1)$. That is,

$$(1-t)^{1/6}\,{}_2F_1\left(\frac{1}{6}, \frac{1}{6}; 1; t\right) = {}_2F_1\left(\frac{1}{6}, \frac{5}{6}; 1; \frac{t}{t-1}\right).$$

This is the special case of the hypergeometric series identity

$$(1-t)^a\,{}_2F_1(a, b; c; t) = {}_2F_1\left(a, c-b; c; \frac{t}{t-1}\right).$$

We will discuss more examples of Picard–Fuchs systems of Calabi–Yau threefolds and $K3$ surfaces, which have already been considered by several people. For instance, the articles [7], [8], and [9] obtained the Picard–Fuchs operators for Calabi–Yau hypersurfaces with $h^{1,1} \le 3$. The next two examples consider Calabi–Yau hypersurfaces with $h^{1,1} > 3$, and the paper of Lian and Yau [11] addressed the question of determining the Picard–Fuchs system of the families of $K3$ surfaces $\mathbb{P}^3[1,1,2,2]$ of degree 6 and $\mathbb{P}^3[1,1,2,4]$ of degree 8. Their results are that

(1) there is an elliptic fibration on these $K3$ surfaces, and the Picard–Fuchs systems of the $K3$ families can be derived from the Picard–Fuchs system of the elliptic pencils, and that

(2) the solutions of the Picard–Fuchs systems for the $K3$ families are given by "squares" of those for the elliptic families.

The system of partial differential equations considered by Lian and Yau [11] is

$$\begin{aligned} L_1 &= \Theta_x(\Theta_x - 2\Theta_z) - \lambda\,x(\Theta_x + \tfrac{1}{2} + \nu)(\Theta_x + \tfrac{1}{2} - \nu) \\ L_2 &= \Theta_z^2 - z(2\Theta_z - \Theta_x + 1)(2\Theta_z - \Theta_x) \end{aligned}$$

and an ordinary differential equations

$$L = \Theta_x^2 - \lambda\,x(\Theta_x + \frac{1}{2} + \nu)(\Theta_x + \frac{1}{2} - \nu)$$

where $\Theta_x = x\frac{\partial}{\partial x}$, etc.) and λ, ν are complex numbers.

Also they noted that the $K3$ families correspond, respecitvely, to the families of Calabi–Yau threefolds $\mathbb{P}^4[1,1,2,4,4]$ of degree 12 and $\mathbb{P}^4[1,1,2,4,8]$ of degree 16. However, the Picard–Fuchs systems for the Calabi–Yau families are not explicitly determined.

Example 6.2. We now consider a family of $K3$ surfaces $\mathbb{P}^3[1,1,2,4]$. of degree 8. This $K3$ family is realized as the degeneration of the family of Calabi–Yau hypersurfaces $\mathbb{P}^4[1,1,2,4,8]$ of degree 16 and $h^{1,1}=4$. The most generic defining equation for this family is given by

$$a_0Z_1Z_2Z_3Z_4Z_5+a_1Z_1^{16}+a_2Z_2^{16}+a_3Z_3^8+a_4Z_4^4+a_5Z_5^2+a_6Z_3^2Z_4Z_5+a_7Z_1^8Z_2^8=0$$

Again the intersection with the hyperplane $Z_2-t\,Z_1=0$ gives rise to a family of $K3$ surfaces $\mathbb{P}^3[1,1,2,4]$:

$$Y_1^8+Y_2^8+Y_3^4+Y_4^2+\lambda Y_1Y_2Y_3Y_4+\phi Y_1^4Y_2^4=0$$

Let S denote this family of $K3$ surfaces. Then

$$Pic(S)=M_{(1,1),(1,1),0}\quad\text{with } \rho(S)=3.$$

The mirror family $\hat{S}$ exists and its Picard lattice is

$$Pic(\hat{S})=E_8\perp D_7\perp U\quad\text{with } \rho(\hat{S})=17.$$

The Picard lattices are determined by Belcastro [3]. The intersection of this family of $K3$ surfaces with the hyperplane $Y_2-s\,Y_2=0$ gives rise to the pencil of elliptic curves

$$c_0W_1W_2W_3+c_1W_1^4+c_2W_2^4+c_3W_3^2=0$$

in $\mathbb{P}^2[1,1,2]$ of degree 4. This means that this family of $K3$ surfaces has the elliptic fibration with section.

Now translate this “inductive” structure to the Picard–Fuchs systems. The Picard–Fuchs system for the $K3$ family is given by

$$\begin{array}{rcl} L_1 &=& \Theta_x(\Theta_x-2\Theta_z)-64\,x(\Theta_x+\frac{1}{2}+\frac{1}{4})(\Theta_x+\frac{1}{2}-\frac{1}{4}) \\ L_2 &=& \Theta_z^2-z(2\Theta_z-\Theta_x+1)(2\Theta_z-\Theta_x) \end{array}$$

and the Picard–Fuchs defferential equation of the elliptic family is given by

$$L=\Theta_x^2-64\,x(\Theta_x+\frac{1}{2}+\frac{1}{4})(\Theta_x+\frac{1}{2}-\frac{1}{4})$$

The same remark as Remark 6.1 is valid for the Picard–Fuchs system $\{L_1,L_3\}$ which corresponds to Theorem 4.1 (b) with $a=1/3$.

Example 6.3. We consider a family of $K3$ surfaces $\mathbb{P}^3[1,1,2,2]$ of degree 6. This $K3$ family is realized as the degeneration of the family of Calabi–Yau hypersurfaces $\mathbb{P}^4[1,1,2,4,4]$ of degree 12 and $h^{1,1}=5$:

$$a_0Z_1Z_2Z_3Z_4Z_5+a_1Z_1^{12}+a_2Z_2^{12}+a_3Z_3^6+a+4Z_4^3+a_5Z_5^3+a_6Z_1^6Z_2^6=0.$$

The intersection of this Calabi–Yau hypersurface with the hyperplane $Z_2-t\,Z_1=0$ gives rise to the family of $K3$ hypersurfaces $\mathbb{P}^3[1,1,2,4]$:

$$Y_1^6+Y_2^6+Y_3^3+Y_4^3+\lambda Y_1Y_2Y_3Y_4+\phi Y_1^3Y_2^3=0.$$

Let S denote this family of $K3$ surfaces. Then

$$\mathrm{Pic}(S)=M_{(1,1,1),(1,1,1),0}\quad\text{with } \rho(S)=4.$$

There is a mirror family of $K3$ surfaces, $\hat{S}$ with

$$\mathrm{Pic}(\hat{S}) = E_8 \perp D_4 \perp A_2 \perp U \quad \text{with } \rho(\hat{S}) = 16.$$

The Picard lattices are determined by Belcastro [3].

The intersection of this $K3$ family with the hyperplane $Y_2 - s\,Y_1 = 0$ gives rise to the family of elliptic curves

$$c_0 W_1 W_2 W_3 + c_1 W_1^3 + c_2 W_2^3 + c_3 W_3^3 = 0$$

in $\mathbb{P}^2[1,1,1]$ of degree 3.

The Picard–Fuchs system of this $K3$ family is

$$\begin{array}{rcl} L_1 & = & \Theta_x(\Theta_x - 2\Theta_z) - 27\,x(\Theta_x + \frac{1}{2} + \frac{1}{6})(\Theta_x + \frac{1}{2} - \frac{1}{6}) \\ L_2 & = & \Theta_z^2 - z(2\Theta_z - \Theta_x + 1)(2\Theta_z - \Theta_x) \end{array}$$

and the Picard–Fuchs differential equation for the elliptic family is given by

$$L = \Theta_x^2 - 27\,x(\Theta_x + \frac{1}{2} + \frac{1}{6})(\Theta_x + \frac{1}{2} - \frac{1}{6})$$

We note that the same remark is valid for the Picard–Fuchs system $\{L_1, L_3\}$ corresponding to $a = 1/4$ in Theorem 4.1(c).

We will summarize the above discussions for the families of $K3$ surfaces in the following form.

Proposition 6.1. *The Picard–Fuchs systems of families of $K3$ surfaces obtained by Lian and Yau [11] can be reconstructed starting from the bi-modular forms and then finding the differential equations satisfied by them. In other words, the differential equations satisfied by the bi-modular forms are realized as the Picard–Fuchs differential equations of the families of $K3$ surfaces, establishing, in a sense, the "modularity" of the $K3$ families.*

7. Picard–Fuchs differential equations of families of $K3$ surfaces: Part II

The purpose of this section is to study (one-parameter) families of $K3$ surfaces (some of which are realized as degenerations of some families of Calabi–Yau threefolds), whose mirror maps are expressed in terms of Hauptmodules for genus zero subgroups of the form $\Gamma_0(N)^*$, aiming to identify their Picard–Fuchs systems with differential equations assocaited to some to bi-modular forms (e.g., in Theorem 5.1).

Dolgachev [5] has discussed several examples of families of M_N-polarized $K3$ surfaces corresponding to $\Gamma_0(N)^*$ for small values of N, e.g., $N = 1, 2$ and 3.

Lian and Yau [10] have given examples of families of $K3$ surfaces and their Picard–Fuchs differential equqtions of order 3. The modular groups are genus zero subgroups of the form $\Gamma_0(N)^*$ where N ranging from 1 to 30. Here we try to analyze their examples and their method in relation to our results in the section 5.

Example 7.1. We start with the hypergeometric equation:

$$t(1-t)f'' + [1 - (1+a+b)t]f' - abf = 0$$

in Theorem 5.1. Take $a = b = \frac{1}{4}$ and consider a one-parameter deformation of this equation of the form:

$$t(1-t)f'' + (1 - \frac{3}{2}t)f' - \frac{1}{16}(1 - 4\nu^2)f = 0$$

with a deformation parameter ν. This has a unique solution $f_0(t)$ near $t=0$ with $f_0(0)=1$, and a solution $f_1(t)$ with $f_1(t)=f_0(t)\log t+O(t)$. The inverse $t(q)$ of the power series $q=\exp(\frac{f_1(t)}{f_0(t)})=t+O(t^2)$ defines an invertible holomorphic function in a disc, and $t(q)$ is the so-called mirror map. Put

$$x(q)=\frac{1}{\lambda}t(\lambda q) \quad \text{for a given } \lambda.$$

One of the main results of Lian and Yau [10] is that for any complex numbers λ, ν with $\lambda\neq 0$, there is a power series identity:

$${}_3F_2(\frac{1}{2},\frac{1}{2}+\nu,\frac{1}{2}-\nu;1,1;\lambda\, x(q))^2=\frac{x'^2}{x^2(1-\lambda\, x)}$$

in the common domain of definitions of both sides. As before, $x'(q)=D_qx(q)$.

For instance, take $(\lambda,\nu)=(2^63^3,\frac{1}{3})$, $(2^8,\frac{1}{4})$, $(2^23^3,\frac{1}{6})$ and $(2^6,0)$, then these relations are given below. The mirror maps in these examples are expressed in terms of Hauptmodules of genus zero modular groups of the form $\Gamma_0(N)^*$ ($\Gamma_0(1)^*=\Gamma$).

Label		Modular Relation			Modular Group
I	:	$\left(\sum_{n=0}^{\infty}\frac{(6n)!}{(3n)!(n!)^3}\frac{1}{j(q)^n}\right)^2$	$=$	$E_4(q)$	Γ
II	:	$\left(\sum_{n=0}^{\infty}\frac{(4n)!}{(n!)^4}x_2(q)^n\right)^2$	$=$	$\frac{x_2'^2}{x^2(1-256x)}$	$\Gamma_0(2)^*$
III	:	$\left(\sum_{n=0}^{\infty}\frac{(2n)!(3n)!}{(n!)^5}x_3(q)^n\right)^2$	$=$	$\frac{x_3'^2}{x_3^2(1-108x_3)}$	$\Gamma_0(3)^*$
IV	:	$\left(\sum_{n=0}^{\infty}\frac{(2n)!^3}{(n!)^6}x_4(q)^n\right)^2$	$=$	$\frac{x_4'^2}{x_4^2(1-64x_4)}$	$\Gamma_0(4)^*$.

Here $j(q)$, $x_2(q), x_3(q)$ and $x_4(q)$ are Hauptmodules for the genus zero subgroups Γ, $\Gamma_0(2)^*$, $\Gamma_0(3)^*$ and $\Gamma_0(4)^*$, respectively. Observe that in each modular relation, the right hand side is a modular form of weight 4 on the corresponding genus zero subgroup.

We know that ${}_3F_2(\frac{1}{2},\frac{1}{2}+\nu,\frac{1}{2}-\nu;1,1;\lambda\, x)$ is a unique solution with the leading term $1+O(x)$ to the differential operator

$$L=\Theta_x^3-\lambda\, x(\Theta_x+\frac{1}{2})(\Theta_x+\frac{1}{2}+\nu)(\Theta_x+\frac{1}{2}-\nu).$$

In these examples, this differential operator is identified with the Picard–Fuchs differential operator for a one-parameter family of $K3$ surfaces, which are obtained by degenerating Calabi–Yau families. (Cf. Lian and Yau [10], Klemm, Lercher and Myer [9].)

	CY family	$K3$ family	PF Operator
I	$X(1,1,2,2,2)[8]$	$X(1,1,1,3)[6]$	$\Theta^3-8x(6\Theta+5)(6\Theta+3)(6\Theta+1)$
II	$X(1,1,2,2,6)[12]$	$X(1,1,1,1)[4]$	$\Theta^3-4x(4\Theta+3)(4\Theta+2)(4\Theta+1)$
III	$X(1,1,2,2,2,2)[6,4]$	$X(1,1,1,1,1)[3,2]$	$\Theta^3-6x(2\Theta+1)(3\Theta+2)(3\Theta+1)$
IV	$X(1,1,2,2,2,2,2)[4,4,4]$	$X(1,1,1,1,1,1)[2,2,2]$	$\Theta^3-8x(2\Theta+1)^3$

The $K3$ families I and II have already been discussed in Lian–Yau [11] (see also Verrill–Yui [16]) in relation to mirror maps. The Picard group of I (resp. II) is given by

$$(-E_8)^2 \oplus U_2 \oplus <-4> \quad (\text{resp. } (-E_8)^2 \oplus U_2 \oplus <-2>).$$

The Calabi–Yau family III can be realized as a complete intersection of the two hypersurfaces:

$$\begin{gathered} Y_1^6+Y_2^6+Y_3^3+Y_4^3+Y_5^3+Y_6^3=0 \\ Y_1^4+Y_2^4+Y_3^2+Y_4^2+Y_5^2+Y_6^2=0 \end{gathered}$$

This Calabi–Yau family has $h^{1,2}=68$ and $h^{1,1}=2$. The $K3$ family is realized as the fiber space by setting

$$Y_1=Z_1^{1/2}, \quad Y_2=\lambda Z_1^{1/2}, \quad \text{and} \quad Y_i=Z_i \quad \text{for } i=3,\cdots,6$$

where $\lambda\in\mathbb{P}^1$ is a parameter. That is, we obtain a family of complete intersection $K3$ surfaces:

$$\begin{gathered} (1+\lambda^6)Z_1^3+Z_3^3+Z_4^3+Z_5^3+Z_6^3=0 \\ (1+\lambda^4)Z_1^2+Z_3^2+Z_4^2+Z_5^2+Z_6^2=0 \end{gathered}$$

$X(1,1,1,1,1)[3,2]$.

Question: What is the Picard group of this K3 family?

In the similar manner, the Calabi–Yau family IV can be realized as a complete intersection of the three hypersurfaces:

$$\begin{gathered} Y_1^4+Y_2^4+Y_3^2+Y_4^2+Y_5^2+Y_6^2+Y_7^2=0 \\ Z_1^4+Z_2^4+Z_3^2+Z_4^2+Z_5^2+Z_6^2+Z_7^2=0 \\ W_1^4+W_2^4+W_3^2+W_4^2+W_5^2+W_6^2+W_7^2=0 \end{gathered}$$

The $K3$ family is realized as the fiber space by setting

$$Y_1=Y_1'^{\frac{1}{2}},\, Y_2=\lambda Y_1'^{\frac{1}{2}} \quad \text{and} \quad Y_i=Y_i' \quad \text{for } i=3,\cdots,7$$

and similarly for Z_1, Z_2 and W_1, W_2 where $\lambda\in\mathbb{P}^1$ is a parameter.

This gives rise to the $K3$ family

$$\begin{gathered} (1+\lambda^4)Y_1'^2+Y_3'^2+Y_4'^2+Y_5'^2+Y_6'^2+Y_7'=0 \\ (1+\lambda^4)Z_1'^2+Z_3'^2+Z_4'^2+Z_5'^2+Z_6'^2+Z_7'^2=0 \\ (1+\lambda^4)W_1'^2+W_3'^2+W_4'^2+W_5'^2+W_6'^2+W_7'^2=0 \end{gathered}$$

Question: What is the Picard group of this K3 family?

Here is the summary:

(1) One starts with a Hauptmodule $x(=x(q))$ for a genus zero subgroup $\Gamma_0(N)^*$;

(2) then there associate a modular form $\frac{x'^2}{x\,r(x)}$ of weight 4,

(3) and a power series solution $\omega_0(x)$ of an order three differential operator;

(4) this differential operator coincides with the Picard–Fuchs differential operator of a one-parameter family of $K3$ surfaces in weighted projective spaces.

Lian and Yau [10] further considered generalizations of the above phenomenon, constructing many more examples. Given a genus zero subgroup of the form $\Gamma_0(N)^*$ and a Hauptmodul $x(q)$, constract (by taking a Schwarzian derivative) a modular form E of weight 4 of the form $\frac{x'^2}{x\,r(x)}$ and a differential operator L whose monodromy has maximal unipotency at $x=0$, such that $L\,E^{1/2}=0$. Further, identify L as the Picard–Fuchs differential operator of a family of $K3$ surfaces. Let $\omega_0(x)$ denotes the fundamental period of this manifold. Then it should be subject to the modular relation

$$\omega_0(x)^2 = \frac{x'^2}{x\,r(x)}$$

How do we associate bi-modular forms of weight $(1,1)$ corresponding to the groups $\Gamma_0(N)^* \times \Gamma_0(N)^*$ in this situation?

Taking the square root of both sides of the modular relation, we obtain that $\omega_0(x)^{1/2}$ is a modular form of weight 1 for the group $\Gamma_0(N)^*$. Take $\omega_0(q_1)\omega_0(q_2)$. Then this is a bi-modular form for $\Gamma_0(N)^* \times \Gamma_0(N)^*$ of weight $(1,1)$. Then this bi-modular form satisfies a differential equation, which may be identified with the Picard–Fuchs differential equation of the K3 family considered above. We summarize the above discussion in the following proposition.

Proposition 7.1. *The examples I–IV above are related to our Theorem 5.2. Indeed, the connection is established by the identity*

$${}_2F_1\left(a,b;a+b+\frac{1}{2};z\right)^2 = {}_3F_2\left(2a,a+b,2b;a+b+\frac{1}{2},2a+2b;z\right).$$

More explicitly, the examples I–IV correspond to the cases $(1/12,5/12)$, $(1/8,3/8)$, $(1/6,1/3)$, *and* $(1/4,1/4)$, *respectively.*

Note that the generalized hypergeometric series ${}_3F_2(\alpha_1,\alpha_2,\alpha_3;1,1;z)$ satisfies the differential equation of the form:

$$[\Theta_z^3 - \lambda\, z(\Theta_z+\alpha_1)(\Theta_z+\alpha_2)(\Theta_z+\alpha_3)]f = 0$$

for some $\alpha_1,\alpha_2,\alpha_3 \in \mathbb{Q}$ and $\lambda \in \mathbb{Q}, \neq 0$.

A natural question we may ask now is: Is is possible to construct families of $K3$ surfaces corresponding to Theorem 5.2 from this observation?

When the order 3 differential equation of this form becomes the symmetric square of an order 2 differential equation, and if the order 2 differential equation is realized as the Picard–Fuchs differential equation of a family of elliptic curves, we may be able to construct a family of $K3$ surfaces using the method of Long [13], especially when the Picard number of the $K3$ family in question is 19 or 20. In fact, Rodriguez–Villegas [14] has discussed 4 families of $K3$ surfaces which fall into this class.

However, at the moment, we do not know if there are readily available methods for constructing $K3$ families starting from differential equations.

Remark 7.1. *If we consider the order* 4 *generalized hypergeomtric series, there are* 14 *families of Calabi–Yau threefolds whose Picard–Fuchs differential equations are of the form*

$$[\Theta_z^4 - \lambda\, z(\Theta_z+\alpha_1)(\Theta_z+\alpha_2)(\Theta_z+\alpha_3)(\Theta_z+\alpha_4)]f = 0$$

for some $\alpha_i \in \mathbb{Q}$ *and* $\lambda \in \mathbb{Q}, \neq 0$. *These* 14 *differential operators have been found in Almkvist–Zudilin* [2] *and Villegas* [14] *found the corresponding families of Calabi–Yau threefolds all in weighted projective spaces with* $h^{1,1} = 1$.

8. Generalizations and open problems

Problem 1. We have determined differential equations satisfied by bi-modular forms of weight $(1, 1)$. The arguments can be generalized to bi-modular forms of any weight (k_1, k_2), using the result of Yang [17]. However, differerntial equations satisfied by them are getting too big to display.

Problem 2. A natural generalization is to consider tri-modular forms $F(\tau_1, \tau_2, \tau_3)$ of weight (k_1, k_2, k_3) on $\Gamma_1 \times \Gamma_2 \times \Gamma_3$.

Examples of this kind should correspond to Picard–Fuchs differential equations of families of Calabi–Yau threefolds, or Picard–Fuchs differential equations of degenerate families of Calabi–Yau fourfolds.

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