行政院國家科學委員會專題研究計畫　期中進度報告

半單對稱空間的幾何性質(1/3)

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期中報告

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中文摘要：

我們要用 Dynkin 表來代表實單對稱空間。給定實單李代數 g 及其卡當對合 (involution) θ，它導至 g = k + p 分解。第二個對合與 θ 交換，因此對 k 和 p 各別作用。所以，我們可將它看成是 k 的對合，並可擴展至 p。針對內類的 g，我們已成功瞭解 θ 於 Dynkin 表的性質，並利用它得知 k。這對本計劃很有幫助，因為我們將得以描述第二個對合於 k，並研究它是否可擴展至 g。

Abstract

We want to represent real simple symmetric spaces by Dynkin diagrams. Consider a real simple Lie algebra g with a Cartan involution θ. It leads to the decomposition $g = k+p$. Now a second involution commutes with θ, so it acts on k and p separately. So we can think of it as an involution on k which is extendable over p. For g of inner type, we have been successful to understand θ on a Dynkin diagram in such a way that k is revealed. This is helpful in our project, because we can later express the second involution on k and then consider its extension to g.

Progress Report:

Geometry of Semisimple Symmetric Spaces

Meng-Kiat Chuah

This project seeks to represent real simple symmetric spaces by Dynkin diagrams. We start by introducing the following algebraic theory as follows. A real simple symmetric space is equivalent to a pair of commuting involutions θ, σ on a complex simple Lie algebra L. If we regard θ as the first involution, then there exists a real form g of L such that θ is a Cartan involution on g. Assume that g is noncompact, for otherwise θ is trivial. It leads to the decomposition $g = k + p$, where k and p are respectively the 1 and -1 eigenspaces of θ. Now since σ commutes with θ, it follows that σ acts on k and p separately. So we can think of σ as an involution on k which is extendable over p. For g of inner type, we have been successful in the first step of our project, which is to understand θ on a Dynkin diagram in such a way that k is revealed. We describe this in better details as follows. Note that this is helpful in our project, because we can later express σ on k and then consider its extension to g.

The fixed points $k \subset g$ of θ form a maximally compact subalgebra of g. Consider the case where g is of inner type, so that k and g have the same rank. There are two possibilities on the reductive Lie algebra k: Either k has a one dimensional center, or k is semisimple. Kac describes a way to reveal the Dynkin diagram of k [4]: Choose a simple system Π in such a way that only one simple root α whose root space lies in p. This is always possible, by [2]. If k has a one dimensional center, then the number of simple roots for k is one fewer than the number of simple roots for g. In this case the Dynkin diagram of k is provided by $\Pi \backslash \{\alpha\}$. If k is semisimple, then k and g have the same number of simple roots, and in this case the Dynkin diagram of k is provided by $(\Pi \cup \{\varphi\}) \backslash \{\alpha\}$, where φ is the lowest root with respect to Π. Note that $\Pi \cup \{\varphi\}$ is just the extended Dynkin diagram of Π, so k can be found conveniently from there.

An example is $g = su(m, n)$. The Dynkin diagram is of type A_{m+n-1}, where the m-th vertex represents the unique simple root whose root space lies in p. Here

$k = s(u(m) \oplus u(n))$. It has a one dimensional center, and its Dynkin diagram is of type $A_{m-1} \times A_{n-1}$, which is the Dynkin diagram of g with its m-th vertex removed.

Besides Kac's method, I have also found way to obtain the Dynkin diagram of k. Namely, its Dynkin diagram has to contain the subdiagram of k-simple roots with respect to the Vogan diagram of any choice of θ-stable simple system. Therefore, as I vary the choices of simple system, I obtain several subdiagrams of k, and together these subdiagrams reveal the Dynkin diagram of k. The collection of Vogan diagrams of g (under different choices of simple systems) have already been computed by me and my student C. C. Hu [3]. These families of Vogan diagrams provide a good machinery to calculate the Dynkin diagram of k, as well as the subsequent study of the second involution σ. The behavior of σ on k shall be our next step in this project.

The final goal of this project is to obtain an independent characterization of simple symmetric spaces, which parallels Berger's classification of these symmetric spaces [1].

References

[1] M. Berger, Les espaces symetriques non compacts, *Ann. Sci. Ecole Norm. Sup. (3)* **74** (1957), 81-177.

[2] A. Borel and J. de Siebenthal, Les sous-groupes fermes de rang maximum des groupes de Lie clos, *Comment. Math. Helv.* **23** (1949), 200-221.

[3] M. K. Chuah and C. C. Hu, Equivalence classes of Vogan diagrams, *J. Algebra* **279** (2004), 22-37.

[4] V. Kac, Automorphisms of finite order of semisimple Lie algebras, *Funkcional Anal. i Prilozen.* **3** (1969), 94-96.