行政院國家科學委員會專題研究計畫　成果報告

二相流在破裂介質中的宏觀模式(2/2)

計畫類別：個別型計畫
計畫編號：NSC94-2115-M-009-008-
執行期間：94 年 08 月 01 日至 95 年 07 月 31 日
執行單位：國立交通大學應用數學系(所)

計畫主持人：葉立明

報告類型：完整報告
報告附件：出席國際會議研究心得報告及發表論文
處理方式：本計畫可公開查詢

中　華　民　國 95 年 11 月 3 日

中文摘要

這是一個二年期的計劃。我們建議探討在破裂介質中的二相流的宏觀模式。一個已知的事實是二相流在獨立的區塊中的流動雖然很緩慢，但它對流體在整體上的表現卻佔了十分重要的角色，它具有所謂的調節功能。表現在外的一種形式是在不同區塊會有不同的 time scale 的現象。此現象在宏觀模式下的單孔模式(single-porosity model)中無法看出，但期望在宏觀模式下的雙孔模式(dual-porosity model)可看出來。在之前的研究中，我們考慮了區塊為小尺寸及中尺寸的情形，並證明了它們會收斂到某種的雙孔介質模式。且這些模式在區塊與整體之間確有不同的 time scale 的現象。我們希望繼續先前的工作討論柱形區塊及混和型區塊所構成的破裂介質的情形。目的就是想找出各種不同區塊的微觀模式所對應的宏觀模式。

關鍵詞：單孔模式、雙孔模式。

英文摘要

This is a two-year project. We propose to investigate macroscopic model for two-phase flows in fractured media. It is known that although flow moves slow in matrix blocks, it has profound influence on global flow behaviors. Usually, different time scales in different regions can be observed. This phenomenon does not show in single-porosity models, but it is expected to be captured in dual-porosity models. In previous study, we consider small-sized block case and medium-sized block case, and prove they converge to some dual-porosity models. They do show different time-scales in matrix blocks and fracture system. We plan to continue this effort to study column-block type fracture media and mixed-block fracture media. The purpose is to find the relations between microscopic models and macroscopic models for two-phase flows in fractured media.

Key words：single-porosity model、dual-porosity model.

背景及目的

在很多的應用問題上須要了解流體在破裂介質中的運動。譬如地下污染源(污水或核廢料)的問題及多孔介質中多相流的問題。事實上，組成半導體晶片的 transistor(如 MOSFET [37])本身也是一種破裂介質，因此電子流在 transistor 中的運動也對應到微觀下的流體在破裂介質中的運動。不過我們有興趣的是多孔介質中二相流的問題。流體在破裂介質中的運動有一特殊的現

象，就是在不同的區塊中流體的變化會有不同的 time-scale 的特性[30,31,33]。由於二相流在破裂介質中的運動和介質的 porosity、permeability、geometry、及流體的性質等有關。因此描述破裂介質中二相流變化的微觀模式十分複雜且很難做分析。即使想用計算的方式去找出微觀模式的近似解也因為計算量太過龐大而不可行。目前對二相流在破裂介質中的問題的研究都是研究它所對應的較簡化的宏觀模式。在一般不均勻的介質中(非破裂介質)，流體的運動的宏觀模式可用 single-porosity model 描述。在已知的文獻中，大部份的地下污染源的問題及多孔介質中多相流的問題都是屬於這類模式。然而 single-porosity model 並不適用於破裂介質中。一種粗淺的說法是在微觀下描述二相流在破裂介質中運動的一些係數，會有急劇振動的情形，此急劇的變化在 single-porosity model 是無法看出。因此，就有工程的學者提出 dual-porosity model 的概念。經過多次實驗測試的結果，dual-porosity model 較 single-porosity model 更能準確描述流體在破裂介質中的變化。不過相對付出的是前者比後者的方程式更複雜。目前並沒有太多探討這類問題的數學結果。此外在 dual-porosity model 中正確的 interface condition 也是一個問題。目前常用的 interface condition 是由物理直覺得到的並無理論基礎。我們則希望借由數學工具建立起一套有系統的二相流在微觀與在宏觀下的模式之間的對應關係及其理論基礎。

二相流在破裂介質中的微觀數學模式是由不連續係數的退化型拋物線與橢圓方程式所組成[46,48]。此問題在工程上有很多的文獻討論，但他們著重在 modelling，數值計算，或資料統計的部份，而沒有數學理論。在數學界，不連續係數的方程早在 60 年代已有人開始討論，不過他們只著重在輕微的不連續的問題上[35]。若有急劇變化的情形則會忽略係數較小的部份不計，因此可以大大的簡化問題的困難度。在我們的情形以上兩種假設都不可行。所以產生微觀模式下二相流問題的解的存在性、穩定性等的証明的困難。這方面的參考資料目前並不多。Poisson equation 及 heat equation 的情形可參考[32,36]。在一般不均勻介質下的宏觀模式(即 single-porosity model)的部份，已知的結果較多。解的存在性、唯一性、穩定性、數值計算方法、及物理性質可參考[6,7,10,12,20,21,25,26,27,39,40]及其內的參考資料。解的 regularity 的部份可參考[4,5,13,28,34]。至於破裂介質的宏觀模式大多是由物理性質及不嚴謹的方式推導出一些 dual-porosity model [8,9,17,18,30,31,33]，但並無數學的證明。少數一些 dual-porosity model 的數學結果(但屬於較簡單的例子)可參考[11,15]。

在過去的研究中，我們討論了二相流在小尺寸與中尺寸的破裂介質中微觀模式與宏觀模式的關聯[46,48]。我們希望在接下來的三年計劃中能沿續先前的工作，討論在不同尺寸及混合尺寸下二相流的微觀模式及其對應的宏觀模式。除了希望建立起一套有系統的二相流在微觀與在宏觀下的模式之間的對應關係及其理論基礎外，也希望能將此結果應用到晶片中的 transistor 的電子流的問

題上。研究的方向是先考慮柱形區塊所構成的破裂介質的微觀模式的解的 well-posedness 問題，再考慮其對應的宏觀模式，接著再結合之前的結果(即小尺寸及中尺寸的模式)討論混合尺寸下的問題。Triple-porosity model(即含有三種不同 porosity 尺寸的模式)也是我們討論的範圍。底下我們列出想考慮的一個 model problem。

We consider a porous medium $\Omega \in \Re^3$, which is a two-connected domain with a periodic structure. Let $Y := [0,1]^3$ be a cell consisting of a matrix block domain Y_m completely surrounded by a connected fracture domain Y_f, and we denote by Γ the matrix-fracture interface in the cell Y. Let $\chi(y)$ be the characteristic function of Y_m extended Y-periodically to all of $\Re^2$. The medium Ω contains two subdomains, $\Omega^\varepsilon{}_f$ and $\Omega^\varepsilon{}_m$, representing the system of fracture plans and matrix blocks respectively, and satisfying

$$\Omega^\varepsilon{}_m \subset \{x \in \Omega \mid \chi(x/\varepsilon) = 1\}, \quad \Omega^\varepsilon{}_f = \Omega \setminus \overline{\Omega^\varepsilon{}_m}.$$

Let $\Gamma^\varepsilon := \partial\Omega^\varepsilon{}_f \bigcap \partial\Omega^\varepsilon{}_m \bigcap \Omega$ be that part of the interface of $\partial\Omega^\varepsilon{}_m$ with $\partial\Omega^\varepsilon{}_f$ that is interior to Ω.

For the fracture subdomain $\Omega^\varepsilon{}_f$, we denote porosity by Φ^ε, absolute permeability by K^ε, saturation of oil phase by $S^\varepsilon \in [0,1]$, capillary pressure by $\gamma(S^\varepsilon)$, the relative permeability by $\Lambda_\alpha(S^\varepsilon)$, phase pressure by $P_\alpha{}^\varepsilon$, and a function depending on gravity by $G_\alpha{}^\varepsilon$ for $\alpha = w, o$. We use $\phi^\varepsilon, \kappa^\varepsilon, s^\varepsilon, \nu(s^\varepsilon), \lambda_\alpha(s^\varepsilon), p_\alpha{}^\varepsilon, g_\alpha{}^\varepsilon$ for $\alpha = w, o$, in subdomain $\Omega^\varepsilon{}_m$ to represent same quantities as those denoted by upper case symbol in fracture subdomain. Let $\varpi(>0)$ be a constant. The conservation of mass in each phase, with the Darcy's law, can be written as, in $\Omega^\varepsilon{}_f$, $t > 0$,

$$-\Phi^\varepsilon(x)\partial_t S^\varepsilon - \nabla\cdot(K^\varepsilon(x)\Lambda_w(S^\varepsilon)\nabla(P_w^\varepsilon - G_w^\varepsilon)) = 0,$$

$$\Phi^\varepsilon(x)\partial_t S^\varepsilon - \nabla\cdot(K^\varepsilon(x)\Lambda_o(S^\varepsilon)\nabla(P_o^\varepsilon - G_o^\varepsilon)) = 0,$$

$$\gamma(S^\varepsilon) = P_0^\varepsilon - P_w^\varepsilon,$$

in $\Omega^\varepsilon{}_m$, $t > 0$,

$$-\phi^\varepsilon(x)\partial_t s^\varepsilon - \varepsilon^{2\varpi}\nabla_{x,y}\cdot(\kappa^\varepsilon(x)\lambda_w(s^\varepsilon)\nabla_{x,y}p_w^\varepsilon) - \nabla_z\cdot(\kappa^\varepsilon(x)\lambda_w(s^\varepsilon)\nabla_z(p_w^\varepsilon - g_w^\varepsilon)) = 0$$

$$\phi^\varepsilon(x)\partial_t s^\varepsilon - \varepsilon^{2\varpi}\nabla_{x,y}\cdot(\kappa^\varepsilon(x)\lambda_o(s^\varepsilon)\nabla_{x,y}p_o^\varepsilon) - \nabla_z\cdot(\kappa^\varepsilon(x)\lambda_o(s^\varepsilon)\nabla_z(p_o^\varepsilon - g_o^\varepsilon)) = 0,$$

$$\nu(s^\varepsilon) = p_0^\varepsilon - p_w^\varepsilon,$$

Phase fluxes and pressures are required to be continuous on interface Γ^ε, $t > 0$, $\alpha = w,o$,

$$K^\varepsilon(x)\Lambda_\alpha(S^\varepsilon)\nabla_{x,y}P_\alpha^\varepsilon\vec{\nu} = \varepsilon^{2\varpi}k^\varepsilon(x)\lambda_\alpha(s^\varepsilon)\nabla_{x,y}p_\alpha^\varepsilon\vec{\nu},$$

$$P_\alpha^\varepsilon = p_\alpha^\varepsilon,$$

where $\vec{\nu}$ is the unit vector outer normal to Γ^ε. Boundary $\partial\Omega$ of Ω includes Γ_1, Γ_2, which satisfying $\Gamma_1 \bigcap \Gamma_2 = \phi$, $\partial\Omega = \overline{\Gamma_1}\bigcup\overline{\Gamma_2}$. Boundary conditions are given by, for $\alpha = w,o$,

$$K^\varepsilon(x)\Lambda_\alpha(S^\varepsilon)\nabla(P_\alpha^\varepsilon - G_\alpha^\varepsilon)\vec{n} = 0, \quad \text{on } \Gamma_1,$$

$$P_\alpha^\varepsilon = P_{b,\alpha}, \quad \text{on } \Gamma_2,$$

where $\vec{n}$ is the unit vector outer normal to Γ_1. Initial conditions are

$$S^\varepsilon(x,0) = S_0^\varepsilon(x), \quad \text{in } \Omega_f^\varepsilon,$$

$$s^\varepsilon(x,0) = s_0^\varepsilon(x), \quad \text{in } \Omega_m^\varepsilon,$$

我們想問當ε很小時，以上的微觀模式所對應的宏觀模式為何?

研究方法、進行步驟及執行進度

第一步是討論微觀模式的解的 well-posedness 的問題。在這部份，須要仔細研究不連續係數的退化型拋物線與橢圓方程式[16,29]。古典的拋物線與橢圓方程式理論不適用於這問題。為了估計不連續截面的變化，也須要用到 psudodifferential operator [41]及 boundary integral method[14]。在函數空間方面 Holder space 或 Sobolev space 並不是正確的空間，要嘗試 Besov space 或其他函數空間才可[2,38,42]。第二步則是找出二相流的微觀模式與宏觀模式的關係，這將要借助 homogenization 或 multiple-scale convergence 的技巧[1,3]。

結果與討論

這兩年我們完成了以下的工作:第一年找出了二相流在柱形區塊所構成的破裂介質的微觀模式與宏觀模式的關係。此宏觀模式基本上是由退化型拋物線與橢圓方程式所組成。它和二相流在小尺寸與中尺寸的破裂介質中的運動模式最大的不同是重力在此方程式中所佔的比重。由於二相流是在柱形區塊所構成的破裂介質中運動重力成了引導流體運動的重要因子。很自然的，方程式也變得十分複雜。第二年我們找出了兩種可混合流體在中尺寸區塊所構成的破裂介質的宏觀模式。此模式是由均勻拋物線與橢圓方程所組成。不過此拋物線方程需用到橢圓方程式的解的導數，是個高度非線性的拋物線與橢圓方程組。底下附上的兩篇論文就是我們這兩年的研究結果。

計畫成果自評部份

這次的研究內容與原計畫相符合程、也達成預期目標情況、研究成果將發表於國際學術期刊上。

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Tall Block Models for Two-Phase flows in Fractured Media

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Two-phase, incompressible, immiscible flow in fractured media with tall matrix blocks is concerned. Suppose ϵ denotes horizontal size ratio of matrix blocks to whole medium, and suppose the horizontal widths of the fracture planes and matrix blocks are in same order. As ϵ goes to 0, microscopic model for the two-phase flow problem converges to 1) a dual-porosity model if permeability ratio of matrix blocks to fracture planes is of order ϵ^2; 2) a single-porosity model for fracture flow if the ratio is smaller than order ϵ^2; 3) another type of single-porosity model if the ratio is greater than order ϵ^2.

Keywords: dual-porosity model, fractured media

1. Introduction

Homogenization for two-phase, incompressible, immiscible flow in fractured media with tall matrix blocks is concerned. Within a fractured medium there is an interconnected system of fracture planes dividing the porous rock into a collection of matrix blocks. The fracture planes, while very thin, form paths of high permeability. Most of the fluids reside in matrix blocks, where they move very slow. Let ϵ be the horizontal size ratio of tall matrix blocks to the whole medium, and let the horizontal widths of the fracture planes and matrix blocks be in same order. In case permeability ratio of matrix blocks to fracture planes is of order ϵ^2, microscopic models for the two-phase flow problem converge to a dual-porosity model as ϵ tends to 0. For the macroscopic model, a fractured medium is regarded as a porous medium consisting of two superimposed continua, a continuous fracture system and a discontinuous system of matrix blocks. Matrix blocks play the role of a global source distributed over the entire medium. The immiscible two-phase flow is formulated by conservation of mass principles for each continuum plus sources from tall matrix blocks. This problem was also considered by formal asymptotic expansion in [8]. If the ratio is smaller than order ϵ^2, the microscopic models approach a single-porosity model for fracture flow. If the ratio is greater than order ϵ^2, then microscopic models tend to another type of single-porosity model. Our intention is to prove the convergence of the microscopic models.

Rest of the paper is organized as follows: In next section §2, we state microscopic model for two-phase flow in fractured media. Notation and assumption will be given in §3. Then in §4, we present our main results. Some known results needed for our main results will be recalled in §5. Proof of main result is in §6. In §6, we need to use the convergence of oil saturation in matrix blocks. The proof is lengthy and tedious, so we present it in last section §7.

2. Microscopic Model for Tall Matrix Blocks

Let $Y \equiv [0,1]^2$ be a cell consisting of a matrix block domain Y_m completely surrounded by a connected fracture domain Y_f. $\mathcal{X}_m(y)$ is the characteristic function of Y_m, extended Y-periodically to all of $\Re^2$. $\tilde{\Omega} \subset \Re^2$ contains two subdomains, $\tilde{\Omega}_f^\epsilon$ and $\tilde{\Omega}_m^\epsilon$. $\tilde{\Omega}_m^\epsilon \subset \{\tilde{x} \in \tilde{\Omega} | \mathcal{X}_m(\tilde{x}/\epsilon) = 1\}$, $\tilde{\Omega}_f^\epsilon = \tilde{\Omega} \setminus \tilde{\Omega}_m^\epsilon$. Let $\tilde{\Gamma}_\epsilon \equiv \partial\tilde{\Omega}_f^\epsilon \cap \partial\tilde{\Omega}_m^\epsilon \cap \tilde{\Omega}$. Boundary of $\tilde{\Omega}$ includes two parts $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ satisfying $\tilde{\Gamma}_1 \cup \tilde{\Gamma}_2 = \partial\tilde{\Omega}$ and $\tilde{\Gamma}_1^\circ \cap \tilde{\Gamma}_2^\circ = \emptyset$. Porous medium considered is a cylindrical aquifer $\Omega \equiv \tilde{\Omega} \times [0,H] \subset \Re^3$ and is assumed to be a two-connected domain with a periodic structure. It contains two subdomains, $\Omega_f^\epsilon \equiv \tilde{\Omega}_f^\epsilon \times [0,H]$ and $\Omega_m^\epsilon \equiv \tilde{\Omega}_m^\epsilon \times [0,H]$, representing the system of fracture planes and matrix blocks respectively. Let $\Gamma_\epsilon \equiv \tilde{\Gamma}_\epsilon \times [0,H]$ be that part of the interface of Ω_m^ϵ with Ω_f^ϵ that is interior to Ω. Both $\Gamma_1 \equiv \tilde{\Gamma}_1 \times [0,H]$ and $\Gamma_2 \equiv \tilde{\Gamma}_2 \times [0,H]$ are part of lateral boundary of Ω.

In fracture subdomain Ω_f^ϵ, porosity is denoted by Φ^ϵ, absolute permeability by K^ϵ, saturation of oil phase by S^ϵ, capillary pressure by $\Upsilon(S^\epsilon)$, relative permeability by $\Lambda_\alpha(S^\epsilon)$, phase pressure by P_α^ϵ, and a density-gravity term by G_α^ϵ for $\alpha = w, o$. $\phi^\epsilon, k^\epsilon, s^\epsilon, \upsilon(s^\epsilon), \lambda_\alpha(s^\epsilon), p_\alpha^\epsilon, g_\alpha^\epsilon$ for $\alpha = w, o$, in subdomain Ω_m^ϵ represent same quantities as those denoted by upper case symbol in fracture subdomain. Conservation of mass in each phase are written as, in Ω_f^ϵ, $t > 0$,

$$-\Phi^\epsilon \partial_t S^\epsilon - \nabla \cdot (K^\epsilon \Lambda_w(S^\epsilon) \nabla (P_w^\epsilon - G_w^\epsilon)) = 0, \tag{2.1}$$

$$\Phi^\epsilon \partial_t S^\epsilon - \nabla \cdot (K^\epsilon \Lambda_o(S^\epsilon) \nabla (P_o^\epsilon - G_o^\epsilon)) = 0, \tag{2.2}$$

$$\Upsilon(S^\epsilon) = P_o^\epsilon - P_w^\epsilon, \tag{2.3}$$

in Ω_m^ϵ, $t > 0$,

$$-\phi^\epsilon \partial_t s^\epsilon - \nabla \cdot \left(k^\epsilon \mathcal{I}_\epsilon^{2\varpi} \lambda_w(s^\epsilon) \nabla (p_w^\epsilon - G_w^\epsilon)\right) = 0, \tag{2.4}$$

$$\phi^\epsilon \partial_t s^\epsilon - \nabla \cdot \left(k^\epsilon \mathcal{I}_\epsilon^{2\varpi} \lambda_o(s^\epsilon) \nabla (p_o^\epsilon - G_o^\epsilon)\right) = 0, \tag{2.5}$$

$$\upsilon(s^\epsilon) = p_o^\epsilon - p_w^\epsilon, \tag{2.6}$$

where $\mathcal{I}_\epsilon^{\mathbf{d}}$ is a diagonal matrix defined by $\mathcal{I}_\epsilon^{\mathbf{d}} \equiv \begin{pmatrix} \epsilon^{\mathbf{d}} & 0 & 0 \\ 0 & \epsilon^{\mathbf{d}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Phase fluxes and pressures are required to be continuous on interface Γ_ϵ, $t > 0$, $\alpha = w, o$,

$$K^\epsilon \Lambda_\alpha(S^\epsilon) \nabla (P_\alpha^\epsilon - G_\alpha^\epsilon) \cdot \vec{\nu^\epsilon} = k^\epsilon \mathcal{I}_\epsilon^{2\varpi} \lambda_\alpha(s^\epsilon) \nabla (p_\alpha^\epsilon - G_\alpha^\epsilon) \cdot \vec{\nu^\epsilon}, \tag{2.7}$$

$$P_\alpha^\epsilon = p_\alpha^\epsilon, \tag{2.8}$$

where $\vec{\nu^\epsilon}$ is the unit vector normal to Γ_ϵ. Boundary conditions are, for $\alpha = w, o$,

$$K^\epsilon \Lambda_\alpha(S^\epsilon)\nabla(P^\epsilon_\alpha - G^\epsilon_\alpha)\cdot \vec{n} = 0 \qquad \text{on } \Gamma_1, \tag{2.9}$$

$$K^\epsilon \Lambda_\alpha(S^\epsilon)\partial_{x_3}(P^\epsilon_\alpha - G^\epsilon_\alpha)|_{x_3=0,H} = k^\epsilon \lambda_\alpha(s^\epsilon)\partial_{x_3}(p^\epsilon_\alpha - G^\epsilon_\alpha)|_{x_3=0,H} = 0, \tag{2.10}$$

$$P^\epsilon_\alpha = P_{b,\alpha} \qquad \text{on } \Gamma_2, \tag{2.11}$$

where $\vec{n}$ is the unit vector outer normal to Γ_1. Initial conditions are

$$S^\epsilon(0,x) = S^\epsilon_0(x) \quad \text{in } \Omega^\epsilon_f, \tag{2.12}$$

$$s^\epsilon(0,x) = s^\epsilon_0(x) \quad \text{in } \Omega^\epsilon_m. \tag{2.13}$$

3. Notation and Assumption

For any $x \in \Re^3$, $x = (\tilde{x}, x_3)$ where $\tilde{x} \in \Re^2$. $\tilde{\Omega}(2\epsilon) \equiv \{\tilde{x} \in \tilde{\Omega} : dist(\tilde{x}, \partial\tilde{\Omega}) > 2\epsilon\}$, $\tilde{\Omega}^\epsilon_m \equiv \{\tilde{x} : \tilde{x} \in \epsilon(Y_m + j) \subset \tilde{\Omega}(2\epsilon) \text{ for } j \in Z^2\}$, $\tilde{\Omega}^\epsilon_f \equiv \tilde{\Omega} \setminus \tilde{\Omega}^\epsilon_m$, and $\tilde{\Omega}^\epsilon \equiv \{z : z \in \epsilon(Y + j), \epsilon(Y_m + j) \subset \tilde{\Omega}(2\epsilon) \text{ for } j \in Z^2\}$. $\Omega^\epsilon \equiv \tilde{\Omega}^\epsilon \times [0,H]$, $\Omega^\epsilon_i \equiv \tilde{\Omega}^\epsilon_i \times [0,H]$, $Y^H_m \equiv Y_m \times [0,H]$, $\mathcal{Q} \equiv \Omega \times Y$, $\mathcal{Q}^\epsilon_m \equiv \Omega^\epsilon \times Y_m$, $\mathcal{Q}_i \equiv \Omega \times Y_i$, $i = f, m$. $\mathcal{B}^t \equiv (0,t)\times\mathcal{B}$ for $\mathcal{B} = Y^H_m, \Gamma_\epsilon, \mathcal{Q}, \mathcal{Q}^\epsilon_m, \Omega, \Omega^\epsilon_i, \mathcal{Q}_i$ $i = f, m$.

$\Re^+_0 \equiv \Re^+ \cup \{0\}$. Time difference is defined to be $\partial^h\psi(t) \equiv \frac{\psi(t+h)-\psi(t)}{h}$. For a set $\mathcal{B}$, $\mathcal{X}_\mathcal{B}$ is a characteristic function of $\mathcal{B}$. $\psi(t,x,y) \in L^r(\Omega^T; L^r_{per}(Y)), 1 < r < \infty$, coincides with a function in $L^r(\mathcal{Q}^T)$ extended by Y-periodicity in y to the whole of $\Re^2$. For $\mathcal{B} = Y_f, Y_m$, we define $L^r(\Omega^T; L^r_{per}(\mathcal{B})) \equiv \{\psi \in L^r(\Omega^T; L^r_{per}(Y)) : \psi(t,x,y) = 0 \text{ if } y \in Y \setminus \mathcal{B}\}$. $\mathcal{W}^{i,r}_0(\Omega) \equiv \{\psi \in W^{i,r}(\Omega) : \psi|_{\Gamma_2} = 0\}$ if $i \in \mathbf{N}$ and $r > 1$, $\mathcal{U} \equiv \mathcal{W}^{1,2}_0(\Omega)$, $\mathcal{U}_2 \equiv \mathcal{U} \times \mathcal{U}$, *dual* $X \equiv$ dual space of X, s_l (resp. $1 - s_r$) is residual matrix oil (resp. water) saturation. $L^{q,r}(\Omega^T) \equiv L^r(0,T; L^q(\Omega))$.

If $\Upsilon : [0,1) \to \Re^+_0$ (resp. $\upsilon : [s_l, s_r) \to \Re^+_0$) is onto and strictly increasing, Υ^{-1} (resp. υ^{-1}) denotes the inverse function of Υ (resp. υ). Then we define $\mathcal{J} : [s_l, s_r) \to [0,1)$ by $\mathcal{J}(z) \equiv \Upsilon^{-1}(\upsilon(z))$, and denote by $\mathcal{J}^{-1}$ the inverse function of $\mathcal{J}$.

$P_{b,c} \equiv P_{b,o} - P_{b,w}$, $S_b \equiv \Upsilon^{-1}(P_{b,c})$, $\Lambda \equiv \Lambda_w + \Lambda_o$, $\lambda \equiv \lambda_w + \lambda_o$,

$$\begin{cases} \mathcal{R}(z) \equiv \int_0^z \frac{\Lambda_w\Lambda_o}{\Lambda}\frac{d\Upsilon}{dS}(\xi)d\xi & \text{for } z \in [0,1), \\ \mathcal{A}(z) \equiv \int_0^z \sqrt{\frac{\Lambda_w\Lambda_o}{\Lambda}(\Upsilon^{-1}(\xi))}d\xi & \text{for } z \in [0,\infty), \\ \mathcal{M}(z) \equiv \int_{s_l}^z \frac{\lambda_w\lambda_o}{\lambda}\frac{d\upsilon}{ds}(\xi)d\xi & \text{for } z \in [s_l, s_r). \end{cases} \tag{3.1}$$

$\vartheta \in (0, 1/8)$ is a number such that $\mathcal{R}'$ is increasing (resp. decreasing) in $(0,\vartheta)$ (resp. $(1-\vartheta, 1)$).

Next let us assume the following conditions: For $\alpha = w, o$,

A1. $\Gamma_2 \neq \emptyset$, $Y_m \subset \Re^2$ is a bounded smooth domain, and $\Omega \subset \Re^3$ is open, bounded, and connected with Lipschitz boundary,

A2. $K^\epsilon, G^\epsilon_\alpha(x_3) \in W^{1,\infty}(\Omega)$, $\partial_t P_{b,\alpha} \in L^2(0,T; H^1(\Omega))$, $P_{b,\alpha} \in C(0,T; C^{1,\mathbf{d}_1}(\Omega))$, $S^\epsilon_0, s^\epsilon_0 \in H^1(\Omega) \cap C^{0,\mathbf{d}_2}(\overline{\Omega})$ for $\mathbf{d}_1, \mathbf{d}_2 \in (0,1)$, A3. $K^\epsilon, k^\epsilon, \Lambda, \lambda \in [\mathbf{d}_3, \mathbf{d}_4]$, $S_b, S_0^\epsilon, \mathcal{J}(s_0^\epsilon) \in (\mathbf{d}_5, 1-\mathbf{d}_5)$ and $\mathbf{d}_5 \in (0,1)$,

A4. $\phi^\epsilon = \phi(\frac{x}{\epsilon})$, $k^\epsilon = k(\frac{x}{\epsilon})$, where ϕ, k are smooth Y-periodic functions,

A5. Λ_w, λ_w (resp. Λ_o, λ_o) : $[0,1] \to [0,1]$ are continuous and decreasing (resp. increasing), $\Lambda_w(1-z) \propto z^{\mathbf{d}_6}, \Lambda_o(z) \propto z^{\mathbf{d}_7}$ for $z \in (0, \vartheta)$, $\frac{\Lambda_\alpha}{\Lambda}(\mathcal{J}(z)) = \frac{\lambda_\alpha}{\lambda}(z)$,

A6. $\Upsilon : [0,1) \to \Re_0^+$ ($\upsilon : [s_l, s_r) \to \Re_0^+$) is onto, increasing, and a locally Lipschitz continuous function, and $\inf\limits_{z\in[0,1)} \frac{d\Upsilon}{dS}(z) > 0$, $\frac{\frac{d\Upsilon}{dS}(\mathcal{J}(z))}{\frac{dv}{ds}(z)}, \Phi^\epsilon, \phi^\epsilon \in [\mathbf{d}_8, \mathbf{d}_9]$ for $z \in [s_l, s_r]$, $\frac{\mathbf{d}_9}{\mathbf{d}_8} \sim 1$,

A7. $\Lambda_o^{3/2}(z) \le \int_z^{2z} (\mathcal{A}(\Upsilon(2z)) - \mathcal{A}(\Upsilon(\xi)))d\xi$ for $z \in (0, \vartheta)$ and
$\Lambda_w^{3/2}(1-z) \le \int_{1-2z}^{1-z} (\mathcal{A}(\Upsilon(\xi)) - \mathcal{A}(\Upsilon(1-2z)))d\xi$ for $z \in (0, \vartheta)$,

A8. $|\Lambda_\alpha(z_1) - \Lambda_\alpha(z_2)| \le \mathbf{d}_{10}\sqrt{(\mathcal{R}(z_1) - \mathcal{R}(z_2))(z_1 - z_2)}$ for any $z_1, z_2 \in [0,1]$,

A9. $\max\limits_{z\in[0,1]} |\Lambda(z) - 1| + \max\limits_{z\in[s_l,s_r]} |\lambda(z) - 1| \le \mathbf{d}_{11}$ ($\mathbf{d}_{11}$ only depends on $\Omega, K^\epsilon, k^\epsilon$),

A10. $\Lambda_o\Lambda_w(z) \le \mathbf{d}_{12} z|1-z|\sqrt{\mathcal{R}'(z)}$, $\mathcal{R}'(z) \propto z^{\mathbf{m}}|1-z|^{\mathbf{m}_1}$ for $z \in (0,\vartheta) \cup (1-\vartheta, 1)$ and $\mathbf{m}, \mathbf{m}_1 > 1$,

where $\mathbf{m}, \mathbf{m}_1$, $\mathbf{d}_i, i = 1, \cdots, 12$ are positive constants.

Remark 3.1 *From A1, Ω_f^ϵ is an open, bounded, and connected domain with Lipschitz boundary. In A2, the density-gravity terms $G_w^\epsilon, G_o^\epsilon$ are functions depending on x_3 variable. Initial and boundary saturations are away from two end points* 0 *and* 1 *(see A3). A5 implies that relative permeability Λ_w (resp. λ_w) in the neighbor of end point* 1 *has similar properties as Λ_o (resp. λ_o) in the neighbor of end point* 0*. Relative phase mobilities in fractures and matrix blocks behave similar. A6 requires that fracture capillary pressure increases as fast as capillary pressure of matrix blocks. Usually, derivative of capillary pressure $\Upsilon'(z)$ (resp. $\upsilon'(z)$) tends to infinity as $z \to 0$ or* 1 *(resp. s_l or s_r). A10 allows parabolic equations considered are degenerate at end points* 0 *and* 1*, a characteristic of a porous medium equation. Indeed, it also implies $\mathcal{R}' \in L^\infty(0,1)$. A7-8,10 are the restrictions on relative permeability and capillary pressure in fractures. Indeed, if $\mathbf{d}_6, \mathbf{d}_7$ (see A5) are large enough (depending on capillary pressure), A7-8,10 hold. One may also note that because of A5-10, Λ_o and $\mathcal{R}'$ at the end point* 0 *have similar properties as Λ_w and $\mathcal{R}'$ at the end point* 1*.*

4. Main Result

In this section, we present the limit models of (2.1–2.13) as $\epsilon \to 0$. Roughly speaking, the limit models are fracture flow equations plus interior sources from matrix blocks. The source terms depend on how fast the matrix permeability tends to 0 as $\epsilon \to 0$. For $0 < \varpi < 1$ case, matrix permeability tends to 0 very slow and saturation variation in fracture system and in matrix blocks is almost simultaneous.

So the limit model is a single-porosity model with sources from matrix blocks. For $\varpi = 1$ case, saturation variation in fracture system and in matrix blocks is not simultaneous and the limit model is a dual-porosity model. In this case, domain acts as a porous medium consisting of two superimposed continua, a continuous fracture system Ω and a discontinuous system of matrix blocks $\mathcal{Q}_m$. Primary flow occurs in fracture system Ω, and each point $x \in \Omega$ is associated with a matrix block Y_m. Flow in matrix blocks plays the role of a global source in the whole fracture system. The model includes two systems of equations, one for flow in fracture system and the other for flow in matrix block system. The two systems are coupled through nonlinear sources. For $1 < \varpi$ case, matrix permeability tends to 0 so fast that matrix blocks play no roles in the limit model. The limit model is a single-porosity model containing only fracture flow equations without matrix sources.

4.1. *For* $\varpi = 1$ *case*

Let $\Omega \subset \Re^3$ be a fractured medium. Equations for fracture flow are, for $x \in \Omega,\ t > 0$,

$$-\Phi\partial_t S - \nabla \cdot (K\Lambda_w(S)\nabla(P_w - G_w)) = q_w, \tag{4.1}$$

$$\Phi\partial_t S - \nabla \cdot (K\Lambda_o(S)\nabla(P_o - G_o)) = q_o, \tag{4.2}$$

$$\Upsilon(S) = P_o - P_w. \tag{4.3}$$

Φ is porosity, K is permeability field, S is oil saturation, $\Upsilon(S)$ is capillary pressure curve, Λ_α ($\alpha = w, o$) is relative permeability curve of α-phase, P_α denotes phase pressure, G_α is a function depending on density, gravity, and position, and q_α is the matrix-fracture source.

Above each point $x \in \Omega$ is suspended topologically a matrix block $Y_m \subset \Re^2$. Equations for flow in a matrix block are, for $x \in \Omega,\ y \in Y_m,\ t > 0$,

$$-\phi\partial_t s - \partial_{y,x_3} \cdot (k\lambda_w(s)\partial_{y,x_3}(p_w - G_w)) = 0, \tag{4.4}$$

$$\phi\partial_t s - \partial_{y,x_3} \cdot (k\lambda_o(s)\partial_{y,x_3}(p_o - G_o)) = 0, \tag{4.5}$$

$$\upsilon(s) = p_o - p_w. \tag{4.6}$$

Here functions s, p_w, p_0 are defined in space domain $\mathcal{Q}_m$ and $\partial_{y,x_3} = (\partial_{y_1}, \partial_{y_2}, \partial_{x_3})$. Each lower case symbol denotes the quantity on Y_m corresponding to that denoted by an upper case symbol in the fracture system equations.

The matrix-fracture sources are given by, for $x \in \Omega$, $t > 0$,

$$q_\alpha = \frac{-1}{|Y_m|}\int_{Y_m} (\sigma_\alpha \phi\partial_t s - \partial_{x_3}(k\lambda_\alpha(s)\partial_{x_3}(p_\alpha - G_\alpha)))\, dy, \tag{4.7}$$

where $\sigma_w = -1$, $\sigma_o = 1$, and $|Y_m|$ is the volume of Y_m. Boundary conditions are, for $t > 0$, $\alpha = w, o$,

$$K\Lambda_\alpha(S)\nabla(P_\alpha - G_\alpha) \cdot \vec{n} = 0 \qquad \text{for } x \in \Gamma_1, \tag{4.8}$$

$$K\Lambda_\alpha(S)\partial_{x_3}(P_\alpha - G_\alpha)|_{x_3=0,H} = k\lambda_\alpha(s)\partial_{x_3}(p_\alpha - G_\alpha)|_{x_3=0,H} = 0, \tag{4.9}$$

$$P_\alpha = P_{b,\alpha} \qquad \text{for } x \in \Gamma_2, \tag{4.10}$$

where $\vec{n}$ is the unit vector outward normal to Γ_1. On interface, pressures are continuous, that is, for $t > 0$, $x \in \Omega$, $y \in \partial Y_m$, $\alpha = w, o$,

$$p_\alpha(t, x, y) = P_\alpha(t, x). \tag{4.11}$$

Initial conditions are

$$S(0, x) = S_0(x) \qquad \text{for } x \in \Omega, \tag{4.12}$$

$$s(0, x, y) = s_0(x) \qquad \text{for } x \in \Omega, \quad y \in Y_m. \tag{4.13}$$

Theorem 4.1 *Under* $A1-10$*, a subsequence of solutions of the microscopic models (2.1–2.13) converges in two-scale sense to a solution of (4.1–4.13) (see next section for the definition of convergence in two-scale sense).*

4.2. *For* $0 < \varpi < 1$ *case*

Equations are, for $x \in \Omega,\ t > 0$,

$$-\Phi\partial_t S - \nabla \cdot (K\Lambda_w(S)\nabla(P_w - G_w)) = q_w, \tag{4.14}$$

$$\Phi\partial_t S - \nabla \cdot (K\Lambda_o(S)\nabla(P_o - G_o)) = q_o, \tag{4.15}$$

$$\Upsilon(S) = P_o - P_w = \upsilon(s). \tag{4.16}$$

Φ, K, S, $\Upsilon(S)$, $\upsilon(s)$, Λ_α, P_α, G_α, and q_α ($\alpha = w, o$) are the same quantities as those in $\varpi = 1$. The matrix-fracture sources are given by, for $x \in \Omega$, $t > 0$,

$$q_\alpha = \frac{-1}{|Y_m|} \int_{Y_m} \left(\sigma_\alpha \phi \partial_t s - \partial_{x_3}\left(k\lambda_\alpha(s)\partial_{x_3}(P_\alpha - G_\alpha)\right)\right) dy, \tag{4.17}$$

where $\sigma_w = -1$, $\sigma_o = 1$, and $|Y_m|$ is the volume of Y_m. Boundary conditions are, for $t > 0$, $\alpha = w, o$,

$$K\Lambda_\alpha(S)\nabla(P_\alpha - G_\alpha) \cdot \vec{n} = 0 \qquad \text{for } x \in \Gamma_1, \tag{4.18}$$

$$K\Lambda_\alpha(S)\partial_{x_3}(P_\alpha - G_\alpha)|_{x_3=0,H} = 0, \tag{4.19}$$

$$P_\alpha = P_{b,\alpha} \qquad \text{for } x \in \Gamma_2, \tag{4.20}$$

where $\vec{n}$ is the unit vector outward normal to Γ_1. Initial condition is

$$S(0, x) = S_0(x) \qquad \text{for } x \in \Omega. \tag{4.21}$$

Theorem 4.2 *Under* $A1-10$*, a subsequence of solutions of the microscopic models (2.1–2.13) converges in two-scale sense to a solution of (4.14–4.21) (see next section for the definition of convergence in two-scale sense).*

4.3. *For $\varpi > 1$ case*

Equations are, for $x \in \Omega,\ t > 0$,

$$-\Phi\partial_t S - \nabla \cdot (K\Lambda_w(S)\nabla(P_w - G_w)) = 0, \tag{4.22}$$

$$\Phi\partial_t S - \nabla \cdot (K\Lambda_o(S)\nabla(P_o - G_o)) = 0, \tag{4.23}$$

$$\Upsilon(S) = P_o - P_w. \tag{4.24}$$

Φ, K, S, $\Upsilon(S)$, $\upsilon(s)$, Λ_α, P_α, G_α, and q_α ($\alpha = w, o$) are the same quantities as those in $\varpi = 1$. Boundary conditions are, for $t > 0$, $\alpha = w, o$,

$$K\Lambda_\alpha(S)\nabla(P_\alpha - G_\alpha) \cdot \vec{n} = 0 \qquad \text{for } x \in \Gamma_1, \tag{4.25}$$

$$K\Lambda_\alpha(S)\partial_{x_3}(P_\alpha - G_\alpha)|_{x_3=0,H} = 0, \tag{4.26}$$

$$P_\alpha = P_{b,\alpha} \qquad \text{for } x \in \Gamma_2, \tag{4.27}$$

where $\vec{n}$ is the unit vector outward normal to Γ_1. Initial condition is

$$S(0, x) = S_0(x) \qquad \text{for } x \in \Omega. \tag{4.28}$$

Theorem 4.3 *Under $A1-10$, a subsequence of solutions of the microscopic models (2.1–2.13) converges in two-scale sense to a solution of (4.22–4.28) (see next section for the definition of convergence in two-scale sense).*

5. Some Known Results

Lemma 5.1 [1] *Let $1 \le r < \infty$ and A1 hold. There is a constant $\mathbf{d}_{13}(Y_f, r)$ and a linear continuous extension operator $\Pi_\epsilon : W^{1,r}(\Omega_f^\epsilon) \cap L^\infty(\Omega_f^\epsilon) \to W^{1,r}(\Omega) \cap L^\infty(\Omega)$ such that if $\varphi \in W^{1,r}(\Omega_f^\epsilon) \cap L^\infty(\Omega_f^\epsilon)$ and $\mathbf{d}_{14} \le \varphi \le \mathbf{d}_{15}$, then*

$$\begin{cases} \Pi_\epsilon\varphi = \varphi \quad \textit{in } \Omega_f^\epsilon \textit{ almost everywhere}, \\ \|\Pi_\epsilon\varphi\|_{W^{1,r}(\Omega)} \le \mathbf{d}_{13}\|\varphi\|_{W^{1,r}(\Omega_f^\epsilon)}, \\ \mathbf{d}_{14} \le \Pi_\epsilon\varphi \le \mathbf{d}_{15}. \end{cases}$$

Definition 5.1 *For a given $\epsilon > 0$ and $1 \le r < \infty$, we define a dilation operator "–" mapping a measurable function $\varphi \in L^r(\Omega_m^{\epsilon,T})$ to a measurable function $\overline{\varphi} \in L^r(\mathcal{Q}_m^T)$ by, for $(t, \tilde{x}, x_3, y) \in \mathcal{Q}_m^T$,*

$$\overline{\varphi}(t, \tilde{x}, x_3, y) \equiv \begin{cases} \varphi(t, \ell^\epsilon(\tilde{x}) + \epsilon y, x_3) & \textit{if } (\ell^\epsilon(\tilde{x}) + \epsilon y, x_3) \in \Omega_m^\epsilon, \\ 0 & \textit{elsewhere}, \end{cases}$$

where $\ell^\epsilon(\tilde{x}) \equiv \epsilon j$ if $\tilde{x} \in \epsilon(Y + j)$, $j \in Z^2$, denoting the lattice translation point of ϵ-cell domain containing $\tilde{x}$.

Definition 5.2 *A sequence of functions φ^ϵ in $L^r(\Omega^T), 1 < r < \infty$, is said to two-scale converge to φ in $L^r(\Omega^T; L^r_{per}(Y))$ if, for any function $\psi \in C_0^\infty(\Omega^T; C^\infty_{per}(Y))$, we have*

$$\lim_{\epsilon \to 0} \int_{\Omega^T} \varphi^\epsilon(t, x)\psi(t, x, \tilde{x}/\epsilon)dxdt = \int_{\mathcal{Q}^T} \varphi(t, x, y)\psi(t, x, y)dydxdt,$$

denoted by $\varphi^\epsilon \xrightarrow{2} \varphi \in L^r(\Omega^T; L^r_{per}(Y))$. *Besides* $\lim_{\epsilon\to 0} \|\varphi^\epsilon\|_{L^r(\Omega^T)} = \|\varphi\|_{L^r(\mathcal{Q}^T)}$, φ^ϵ *is said to two-scale converge to* φ *in* $L^r(\Omega^T; L^r_{per}(Y))$ *strongly, and denoted by* $\varphi^\epsilon \xrightarrow{2} \varphi \in L^r(\Omega^T; L^r_{per}(Y))$ *strongly.*

6. Proof of Main Result

A1-10 are assumed from now on. Let us derive a weak formulation of (2.1–2.6). Multiplying (2.1) and (2.4) by η as well as (2.2) and (2.5) by ζ, integrating over Ω^T, and employing boundary conditions (2.7) and (2.9), we obtain

$$-\int_{\Omega_f^{\epsilon,T}} \Phi^\epsilon \partial_t S^\epsilon \eta + \int_{\Omega_f^{\epsilon,T}} K^\epsilon \Lambda_w(S^\epsilon)\nabla(P_w^\epsilon - G_w^\epsilon)\nabla\eta - \int_{\Omega_m^{\epsilon,T}} \phi^\epsilon \partial_t s^\epsilon \eta + \int_{\Omega_m^{\epsilon,T}} k^\epsilon \mathcal{I}_\epsilon^{2\varpi} \lambda_w(s^\epsilon)\nabla(p_w^\epsilon - G_w^\epsilon)\nabla\eta = 0, \tag{6.1}$$

$$\int_{\Omega_f^{\epsilon,T}} \Phi^\epsilon \partial_t S^\epsilon \zeta + \int_{\Omega_f^{\epsilon,T}} K^\epsilon \Lambda_o(S^\epsilon)\nabla(P_o^\epsilon - G_o^\epsilon)\nabla\zeta + \int_{\Omega_m^{\epsilon,T}} \phi^\epsilon \partial_t s^\epsilon \zeta + \int_{\Omega_m^{\epsilon,T}} k^\epsilon \mathcal{I}_\epsilon^{2\varpi} \lambda_o(s^\epsilon)\nabla(p_o^\epsilon - G_o^\epsilon)\nabla\zeta = 0, \tag{6.2}$$

for smooth functions $\eta, \zeta \in L^2(0,T;\mathcal{U})$. Next we define global pressure [11] as

$$\begin{cases} P^\epsilon \equiv \frac{1}{2}\left(P_o^\epsilon + P_w^\epsilon + \int_0^{\Upsilon(S^\epsilon)} \left(\frac{\Lambda_o}{\Lambda}(\Upsilon^{-1}(\xi)) - \frac{\Lambda_w}{\Lambda}(\Upsilon^{-1}(\xi))\right) d\xi\right), \\ p^\epsilon \equiv \frac{1}{2}\left(p_o^\epsilon + p_w^\epsilon + \int_0^{\upsilon(s^\epsilon)} \left(\frac{\lambda_o}{\lambda}(\upsilon^{-1}(\xi)) - \frac{\lambda_w}{\lambda}(\upsilon^{-1}(\xi))\right) d\xi\right), \end{cases} \tag{6.3}$$

P_b is defined as P^ϵ in $(6.3)_1$ except replacing $P_o^\epsilon, P_w^\epsilon, \Upsilon(S^\epsilon)$ by $P_{b,o}^\epsilon, P_{b,w}^\epsilon, P_{b,c}$ respectively. Then $\nabla P^\epsilon = \frac{\Lambda_w}{\Lambda}(S^\epsilon)\nabla P_w^\epsilon + \frac{\Lambda_o}{\Lambda}(S^\epsilon)\nabla P_o^\epsilon$ and $\nabla p^\epsilon = \frac{\lambda_w}{\lambda}(s^\epsilon)\nabla p_w^\epsilon + \frac{\lambda_o}{\lambda}(s^\epsilon)\nabla p_o^\epsilon$ by (2.3) and (2.6). (6.2) can be rewritten as

$$\int_{\Omega_f^{\epsilon,T}} \Phi^\epsilon \partial_t S^\epsilon \zeta + \int_{\Omega_f^{\epsilon,T}} K^\epsilon \big(\Lambda_o(S^\epsilon)\nabla(P^\epsilon - G_o^\epsilon) + \nabla\mathcal{R}(S^\epsilon)\big)\nabla\zeta + \int_{\Omega_m^{\epsilon,T}} \phi^\epsilon \partial_t s^\epsilon \zeta + \int_{\Omega_m^{\epsilon,T}} k^\epsilon \mathcal{I}_\epsilon^{2\varpi} \big(\lambda_o(s^\epsilon)\nabla(p^\epsilon - G_o^\epsilon) + \nabla\mathcal{M}(s^\epsilon)\big)\nabla\zeta = 0. \tag{6.4}$$

See §3 for $\mathcal{R}, \mathcal{M}$. Summing (6.1) and (6.2), we obtain, for $\eta \in L^2(0,T;\mathcal{U})$,

$$\int_{\Omega_f^{\epsilon,T}} K^\epsilon \big(\Lambda(S^\epsilon)\nabla(P^\epsilon - G_o^\epsilon) - \Lambda_w(S^\epsilon)\nabla(G_w^\epsilon - G_o^\epsilon)\big)\nabla\eta + \int_{\Omega_m^{\epsilon,T}} k^\epsilon \mathcal{I}_\epsilon^{2\varpi} \big(\lambda(s^\epsilon)\nabla(p^\epsilon - G_o^\epsilon) - \lambda_w(s^\epsilon)\nabla(G_w^\epsilon - G_o^\epsilon)\big)\nabla\eta = 0. \tag{6.5}$$

For $\zeta \in L^2(0,T;\mathcal{U}) \cap H^1(\Omega^T), \zeta(T) = 0$,

$$\int_{\Omega_f^{\epsilon,T}} \Phi^\epsilon \partial_t S^\epsilon \zeta + \Phi^\epsilon (S^\epsilon - S_0^\epsilon)\partial_t\zeta = -\int_{\Omega_m^{\epsilon,T}} \phi^\epsilon \partial_t s^\epsilon \zeta + \phi^\epsilon (s^\epsilon - s_0^\epsilon)\partial_t\zeta. \tag{6.6}$$

(6.1–6.6), (2.3), (2.6), (2.8), (2.11) form a weak formulation of (2.1–2.13).

Next we consider a regularized problem. Let $\mathbf{v}$ be a small number satisfying $0 < \mathbf{v} < \frac{\mathbf{d}_5}{4}$. Extend Λ_α ($\alpha = w, o$) constantly and continuously to $\Re$ and define $\Lambda_{\alpha,\mathbf{v}}, \Lambda_{\mathbf{v}}, \lambda_{\alpha,\mathbf{v}}, \lambda_{\mathbf{v}}$ as

$$\begin{cases} \Lambda_{\alpha,\mathbf{v}}(z) \equiv \Lambda_\alpha\big(0.5(\frac{z-\mathbf{v}}{0.5-\mathbf{v}})\big), & \Lambda_{\mathbf{v}} \equiv \Lambda_{w,\mathbf{v}} + \Lambda_{o,\mathbf{v}}, \\ \lambda_{\mathbf{v}}(z) \equiv \Lambda_{\mathbf{v}}(\mathcal{J}(z)), & \lambda_{\alpha,\mathbf{v}}(z) = \Lambda_{\alpha,\mathbf{v}}(\mathcal{J}(z)). \end{cases} \tag{6.7}$$

By A2-3, there exist smooth functions $S^\epsilon_{0,\mathbf{v}}, S_{b,\mathbf{v}}, s^\epsilon_{0,\mathbf{v}}$ such that

$$S^\epsilon_{0,\mathbf{v}}, S_{b,\mathbf{v}}, \mathcal{J}(s^\epsilon_{0,\mathbf{v}}) \in (\mathbf{d}_5/2, 1-\mathbf{d}_5/2), \quad S^\epsilon_{0,\mathbf{v}}|_{\Gamma_2} = S_{b,\mathbf{v}}|_{\Gamma_2}(t=0), \tag{6.8}$$

$$\begin{cases} S^\epsilon_{0,\mathbf{v}}, S_{b,\mathbf{v}}, s^\epsilon_{0,\mathbf{v}} \to S^\epsilon_0, S_b, s^\epsilon_0 & \text{in } L^2(0,T;H^1(\Omega)), \\ \partial_t \Upsilon(S_{b,\mathbf{v}}) \to \partial_t(P_{b,o} - P_{b,w}) & \text{in } L^1(\Omega^T), \end{cases} \quad \text{as } \mathbf{v} \to 0. \tag{6.9}$$

The regularized problem is: Find $\{S^\epsilon_{\mathbf{v}}\mathcal{X}_{\Omega^\epsilon_f} + s^\epsilon_{\mathbf{v}}\mathcal{X}_{\Omega^\epsilon_m}, P^\epsilon_{\mathbf{v}}\mathcal{X}_{\Omega^\epsilon_f} + p^\epsilon_{\mathbf{v}}\mathcal{X}_{\Omega^\epsilon_m}\}$ satisfying

$$\Phi^\epsilon\partial_t S^\epsilon_{\mathbf{v}}\mathcal{X}_{\Omega^\epsilon_f} + \phi^\epsilon\partial_t s^\epsilon_{\mathbf{v}}\mathcal{X}_{\Omega^\epsilon_m} \in dual\ L^2(0,T;\mathcal{U}), \tag{6.10}$$

$$\mathbf{v} \le S^\epsilon_{\mathbf{v}}\mathcal{X}_{\Omega^\epsilon_f} + \mathcal{J}(s^\epsilon_{\mathbf{v}})\mathcal{X}_{\Omega^\epsilon_m} \le 1-\mathbf{v}, \tag{6.11}$$

$$\mathcal{R}(S^\epsilon_{\mathbf{v}}\mathcal{X}_{\Omega^\epsilon_f} + \mathcal{J}(s^\epsilon_{\mathbf{v}})\mathcal{X}_{\Omega^\epsilon_m}) - \mathcal{R}(S^\epsilon_b),\ P^\epsilon_{\mathbf{v}}\mathcal{X}_{\Omega^\epsilon_f} + p^\epsilon_{\mathbf{v}}\mathcal{X}_{\Omega^\epsilon_m} - P_b \in L^2(0,T;\mathcal{U}), \tag{6.12}$$

$$\begin{aligned} &\int_{\Omega^{\epsilon,T}_f} \Phi^\epsilon\partial_t S^\epsilon_{\mathbf{v}}\zeta + \int_{\Omega^{\epsilon,T}_f} K^\epsilon\big(\Lambda_{o,\mathbf{v}}(S^\epsilon_{\mathbf{v}})\nabla(P^\epsilon_{\mathbf{v}} - G^\epsilon_o) + \nabla\mathcal{R}(S^\epsilon_{\mathbf{v}})\big)\nabla\zeta \\ &\quad + \int_{\Omega^{\epsilon,T}_m} \phi^\epsilon\partial_t s^\epsilon_{\mathbf{v}}\zeta + \int_{\Omega^{\epsilon,T}_m} k^\epsilon\mathcal{I}^{2\varpi}_\epsilon\big(\lambda_{o,\mathbf{v}}(s^\epsilon_{\mathbf{v}})\nabla(p^\epsilon_{\mathbf{v}} - G^\epsilon_o) + \nabla\mathcal{M}(s^\epsilon_{\mathbf{v}})\big)\nabla\zeta = 0, \end{aligned} \tag{6.13}$$

$$\begin{aligned} &\int_{\Omega^{\epsilon,T}_f} K^\epsilon\big(\Lambda_{\mathbf{v}}(S^\epsilon_{\mathbf{v}})\nabla(P^\epsilon_{\mathbf{v}} - G^\epsilon_o) - \Lambda_{w,\mathbf{v}}(S^\epsilon_{\mathbf{v}})\nabla(G^\epsilon_w - G^\epsilon_o)\big)\nabla\eta \\ &\quad + \int_{\Omega^{\epsilon,T}_m} k^\epsilon\mathcal{I}^{2\varpi}_\epsilon\big(\lambda_{\mathbf{v}}(s^\epsilon_{\mathbf{v}})\nabla(p^\epsilon_{\mathbf{v}} - G^\epsilon_o) - \lambda_{w,\mathbf{v}}(s^\epsilon_{\mathbf{v}})\nabla(G^\epsilon_w - G^\epsilon_o)\big)\nabla\eta = 0, \end{aligned} \tag{6.14}$$

$$S^\epsilon_{\mathbf{v}}\mathcal{X}_{\Omega^\epsilon_f}(0,x) + s^\epsilon_{\mathbf{v}}\mathcal{X}_{\Omega^\epsilon_m}(0,x) = S^\epsilon_{0,\mathbf{v}}\mathcal{X}_{\Omega^\epsilon_f} + s^\epsilon_{0,\mathbf{v}}\mathcal{X}_{\Omega^\epsilon_m}, \tag{6.15}$$

for any $\zeta, \eta \in L^2(0,T;\mathcal{U})$. It is easy to see that (6.13) is a nondegenerate (depending on $\mathbf{v}$) parabolic equation, and (6.13–6.14) imply, if $S^\epsilon_{w,\mathbf{v}} \equiv 1 - S^\epsilon_{\mathbf{v}}$,

$$\begin{aligned} 0 = &\int_{\Omega^{\epsilon,T}_f} \Phi^\epsilon\partial_t S^\epsilon_{w,\mathbf{v}}\zeta + K^\epsilon\big(\Lambda_{w,\mathbf{v}}(1-S^\epsilon_{w,\mathbf{v}})\nabla(P^\epsilon_{\mathbf{v}} - G^\epsilon_w) - \nabla\mathcal{R}(1-S^\epsilon_{w,\mathbf{v}})\big)\nabla\zeta \\ + &\int_{\Omega^{\epsilon,T}_m} \phi^\epsilon\partial_t s^\epsilon_{w,\mathbf{v}}\zeta + k^\epsilon\mathcal{I}^{2\varpi}_\epsilon\left(\lambda_{w,\mathbf{v}}(1-s^\epsilon_{w,\mathbf{v}})\nabla(p^\epsilon_{\mathbf{v}} - G^\epsilon_w) - \nabla\mathcal{M}(1-s^\epsilon_{w,\mathbf{v}})\right)\nabla\zeta. \end{aligned} \tag{6.16}$$

By [4, 5, 6, 9, 12, 20, 22, 29], it is known

Lemma 6.1 *Under (6.8–6.9), there exist functions $S^\epsilon_{\mathbf{v}}, P^\epsilon_{\mathbf{v}}$ in Ω^ϵ_f and $s^\epsilon_{\mathbf{v}}, p^\epsilon_{\mathbf{v}}$ in Ω^ϵ_m satisfying (6.10–6.15) for each $\mathbf{v}, \epsilon$ as well as there exist functions $S^\epsilon, P^\epsilon, P^\epsilon_\alpha$ in Ω^ϵ_f and $s^\epsilon, p^\epsilon, p^\epsilon_\alpha$ in Ω^ϵ_m for $\alpha = w, o$ satisfying (6.1–6.6), (2.3), and (2.6–2.11). $\check{\mathcal{S}}^\epsilon_{\mathbf{v}} \equiv S^\epsilon_{\mathbf{v}}\mathcal{X}_{\Omega^\epsilon_f} + \mathcal{J}(s^\epsilon_{\mathbf{v}})\mathcal{X}_{\Omega^\epsilon_m}$ is in $L^2(0,T;H^1(\Omega))$ and is Hölder continuous in $\overline{\Omega}^T$,* *and, as* $\mathbf{v} \to 0$,

$$\begin{cases} \check{\mathcal{S}}^\epsilon_{\mathbf{v}} \to \check{\mathcal{S}}^\epsilon \equiv S^\epsilon \mathcal{X}_{\Omega^\epsilon_f} + \mathcal{J}(s^\epsilon)\mathcal{X}_{\Omega^\epsilon_m} & \textit{pointwise}, \\ \mathcal{R}(\check{\mathcal{S}}^\epsilon_{\mathbf{v}}), P^\epsilon_{\mathbf{v}}\mathcal{X}_{\Omega^\epsilon_f} + p^\epsilon_{\mathbf{v}}\mathcal{X}_{\Omega^\epsilon_m} \to \mathcal{R}(\check{\mathcal{S}}^\epsilon), P^\epsilon\mathcal{X}_{\Omega^\epsilon_f} + p^\epsilon\mathcal{X}_{\Omega^\epsilon_m} & \textit{in } L^2(0,T;H^1(\Omega)). \end{cases}$$

Moreover, $0 < S^\epsilon < 1$, $s_l < s^\epsilon < s_r$, *and*

$$\begin{aligned} \sum_{\alpha=w,o} \Big(\|\sqrt{\Lambda_\alpha(S^\epsilon)}\nabla P^\epsilon_\alpha\|_{L^2(\Omega^{\epsilon,T}_f)} + \|\mathcal{I}^\varpi_\epsilon \sqrt{\lambda_\alpha(s^\epsilon)}\nabla p^\epsilon_\alpha\|_{L^2(\Omega^{\epsilon,T}_m)}\Big) \\ + \| \, |\nabla P^\epsilon| + |\nabla \mathcal{R}(S^\epsilon)| + |\nabla \mathcal{A}^\epsilon| \, \|_{L^2(\Omega^{\epsilon,T}_f)} \\ + \| \, |\mathcal{I}^\varpi_\epsilon \nabla p^\epsilon| + |\mathcal{I}^\varpi_\epsilon \nabla \mathcal{M}(s^\epsilon)| + |\mathcal{I}^\varpi_\epsilon \nabla \mathcal{A}^\epsilon| \, \|_{L^2(\Omega^{\epsilon,T}_m)} \le c, \end{aligned}$$

where $\mathcal{A}^\epsilon \equiv \begin{cases} \mathcal{A}(\Upsilon(S^\epsilon)) & \textit{if } x \in \Omega^\epsilon_f, \\ \mathcal{A}(\upsilon(s^\epsilon)) & \textit{if } x \in \Omega^\epsilon_m. \end{cases}$ *and c is a constant independent of* ϵ.

Lemma 6.2 *For any β, τ satisfying $2 \le \beta_0 \le \beta - 2 \in \mathbf{N}$, $\frac{\mathbf{d}_5}{\beta_0} \le \vartheta$, and $\tau \le T$, the following inequality holds:*

$$\sup_{t \le \tau} \big|\{x \in \Omega : \check{\mathcal{S}}^\epsilon(t) \le \mu \textit{ or } 1-\mu \le \check{\mathcal{S}}^\epsilon(t)\}\big| \le \frac{c_0|c_0\tau|^{\beta-\beta_0}}{(\beta-\beta_0)^{(\beta-\beta_0)\mathbf{f}_\beta}}, \tag{6.17}$$

where $\mu \equiv \frac{\mathbf{d}_5}{2^\beta}$, $\lim_{\beta\to\infty} \mathbf{f}_\beta = 1$, *and c_0 is a constant independent of* $\tau, \beta, \epsilon, \mu$.

Proof: Let us define $\mathcal{L}_\mu, \mathcal{K}_\mu, \widehat{\mathcal{K}_\mu}$ as

$$\begin{cases} \mathcal{L}_\mu(z) \equiv \begin{cases} 1 & \text{if } \mu \le z \le 2\mu, \\ 0 & \text{elsewhere}, \end{cases} \\ \mathcal{K}_\mu(z) \equiv \int_{\mathcal{A}(\Upsilon(2\mu))}^z \mathcal{L}_\mu(\Upsilon^{-1}(\mathcal{A}^{-1}(\xi)))d\xi & \text{for } z \in [0, \mathcal{A}(\infty)), \\ \widehat{\mathcal{K}_\mu}(z) \equiv \int_{\mathcal{A}(\Upsilon(2\mu))}^z (\mathcal{L}_\mu \frac{\Lambda_o}{\Lambda}) \circ (\Upsilon^{-1}(\mathcal{A}^{-1}(\xi)))d\xi & \text{for } z \in [0, \mathcal{A}(\infty)). \end{cases}$$

By $2\mu \le \frac{\mathbf{d}_5}{2}$ and A2-3,5, we take $\zeta = \mathcal{K}_\mu(\mathcal{A}^\epsilon) \in L^2(0,T;\mathcal{U})$ in (6.4) and $\eta = \widehat{\mathcal{K}_\mu}(\mathcal{A}^\epsilon) \in L^2(0,T;\mathcal{U})$ in (6.5) to obtain

$$\begin{aligned} \int_{\Omega^{\epsilon,\tau}_f} \Phi^\epsilon \mathcal{K}_\mu(\mathcal{A}^\epsilon)\partial_t S^\epsilon + \int_{\Omega^{\epsilon,\tau}_f} K^\epsilon \Lambda_o(S^\epsilon)\mathcal{L}_\mu(S^\epsilon)\nabla\Upsilon(S^\epsilon)\nabla\mathcal{A}^\epsilon \\ + \int_{\Omega^{\epsilon,\tau}_m} \phi^\epsilon \mathcal{K}_\mu(\mathcal{A}^\epsilon)\partial_t s^\epsilon + \int_{\Omega^{\epsilon,\tau}_m} k^\epsilon \mathcal{I}^{2\varpi}_\epsilon \Lambda_o(\mathbf{u}^\epsilon)\mathcal{L}_\mu(\mathbf{u}^\epsilon)\nabla\upsilon(s^\epsilon)\nabla\mathcal{A}^\epsilon \\ \le c_1\Big(\int_{\Omega^{\epsilon,\tau}_f} K^\epsilon\Lambda_o(S^\epsilon)\mathcal{L}_\mu(S^\epsilon)|\partial_{x_3}\mathcal{A}^\epsilon| + \int_{\Omega^{\epsilon,\tau}_m} k^\epsilon\Lambda_o(\mathbf{u}^\epsilon)\mathcal{L}_\mu(\mathbf{u}^\epsilon)|\partial_{x_3}\mathcal{A}^\epsilon|\Big), \end{aligned} \tag{6.18}$$

where $\mathbf{u}^\epsilon \equiv \mathcal{J}(s^\epsilon)$ and constant c_1 is independent of ϵ, μ. Suppose

$$\int \mathcal{K}_\mu(\mathcal{A}^\epsilon)\big(\Phi^\epsilon\partial_t S^\epsilon \mathcal{X}_{\Omega^{\epsilon,\tau}_f} + \phi^\epsilon\partial_t s^\epsilon\mathcal{X}_{\Omega^{\epsilon,\tau}_m}\big) \ge 0, \tag{6.19}$$

(6.18–6.19) imply

$$\int_{\Omega_f^{\epsilon,\tau}} K^\epsilon \Lambda_o(S^\epsilon)\mathcal{L}_\mu(S^\epsilon)|\partial_{x_3}\mathcal{A}^\epsilon| + \int_{\Omega_m^{\epsilon,\tau}} k^\epsilon \Lambda_o(\mathbf{u}^\epsilon)\mathcal{L}_\mu(\mathbf{u}^\epsilon)|\partial_{x_3}\mathcal{A}^\epsilon|$$

$$\leq c_2\Big(\int_{\Omega_f^{\epsilon,\tau}} K^\epsilon \Lambda_o^{3/2}\mathcal{L}_\mu(S^\epsilon)\Big)^{\frac{1}{2}}\Big(\int_{\Omega_f^{\epsilon,\tau}} K^\epsilon \Lambda_o(S^\epsilon)\mathcal{L}_\mu(S^\epsilon)\partial_{x_3}\Upsilon(S^\epsilon)\partial_{x_3}\mathcal{A}^\epsilon\Big)^{\frac{1}{2}}$$

$$+c_2\Big(\int_{\Omega_m^{\epsilon,\tau}} k^\epsilon \Lambda_o^{3/2}\mathcal{L}_\mu(\mathbf{u}^\epsilon)\Big)^{\frac{1}{2}}\Big(\int_{\Omega_m^{\epsilon,\tau}} k^\epsilon \Lambda_o(\mathbf{u}^\epsilon)\mathcal{L}_\mu(\mathbf{u}^\epsilon)\partial_{x_3}\upsilon(s^\epsilon)\partial_{x_3}\mathcal{A}^\epsilon\Big)^{\frac{1}{2}}, \quad (6.20)$$

where constant c_2 is independent of ϵ, μ. A3 and (6.18–6.20) imply

$$\int \mathcal{K}_\mu(\mathcal{A}^\epsilon)\big(\Phi^\epsilon\partial_t S^\epsilon \mathcal{X}_{\Omega_f^{\epsilon,\tau}} + \phi^\epsilon \partial_t s^\epsilon \mathcal{X}_{\Omega_m^{\epsilon,\tau}}\big) \leq c_3 \int_{\Omega^\tau} \Lambda_o^{3/2}\mathcal{L}_\mu(\check{\mathcal{S}}^\epsilon). \quad (6.21)$$

Let us define

$$\mathcal{Z}(S^\epsilon, s^\epsilon, \mu) \equiv \begin{cases} \Phi^\epsilon \int_{2\mu}^{S^\epsilon} \mathcal{K}_\mu(\mathcal{A}(\Upsilon(\xi)))d\xi & \text{in } \Omega_f^\epsilon, \\ \phi^\epsilon \int_{\mathcal{J}^{-1}(2\mu)}^{s^\epsilon} \mathcal{K}_\mu(\mathcal{A}(\upsilon(\xi)))d\xi & \text{in } \Omega_m^\epsilon. \end{cases}$$

(6.21) implies

$$\int_{\Omega^\tau} \partial_t \mathcal{Z}(S^\epsilon, s^\epsilon, \mu) \leq c_4 \int_{\Omega^\tau} \Lambda_o^{3/2}\mathcal{L}_\mu(\check{\mathcal{S}}^\epsilon). \quad (6.22)$$

(6.22) and A6-7 yield that, if $0 \leq t_1 \leq t_2 \leq T$,

$$\int_{t_1}^{t_2}\int_\Omega \partial_t \mathcal{Z}(S^\epsilon, s^\epsilon, \mu) \leq c_4 \int_{t_1}^{t_2}\int_\Omega \mathcal{Z}(S^\epsilon, s^\epsilon, 2\mu), \quad (6.23)$$

where c_4 is independent of t_1, t_2, μ, ϵ. Define

$$\mathcal{F}^\epsilon(\tau,\mu) \equiv \frac{1}{\Lambda_o(\mu)^{3/2}} \sup_{t\leq\tau} \int_\Omega \mathcal{Z}(S^\epsilon, s^\epsilon, \mu).$$

A5 and (6.23) imply that, for $0 \leq t_1 \leq t_2 \leq T$,

$$\mathcal{F}^\epsilon(t_2,\mu) - \mathcal{F}^\epsilon(t_1,\mu) \leq c_5(t_2-t_1)\mathcal{F}^\epsilon(t_2, 2\mu),$$

where c_5 is independent of t_1, t_2, μ, ϵ. By induction and A3, one obtains, for $j \in \mathbf{N}$, $jh \leq T$,

$$\mathcal{F}^\epsilon(jh, \frac{\mathbf{d}_5}{2^\beta}) \leq (\beta-\beta_0+1)^{j-1}|c_5 h|^{\beta-\beta_0}\mathcal{F}^\epsilon(jh, \frac{\mathbf{d}_5}{2^{\beta_0}}). \quad (6.24)$$

If $j = \frac{\beta-\beta_0}{\log(\beta-\beta_0)}$ and $\tau = jh$ in (6.24), then

$$\mathcal{F}^\epsilon(\tau, \frac{\mathbf{d}_5}{2^\beta}) \leq \frac{|c_5\tau|^{\beta-\beta_0}}{(\beta-\beta_0)^{(\beta-\beta_0)\mathbf{f}_\beta}}\mathcal{F}^\epsilon(\tau, \frac{\mathbf{d}_5}{2^{\beta_0}}), \quad (6.25)$$

where $\mathbf{f}_\beta \to 1$ as $\beta \to \infty$. Define $\mathcal{B}(t) \equiv \{x \in \Omega : \check{\mathcal{S}}^\epsilon(t,x) \le \frac{\mathbf{d}_5}{2^\beta}\}$. (6.25) implies

$$\sup_{t\le\tau} \int \mathcal{X}_{\mathcal{B}(t)} \le c_6 \mathcal{F}^\epsilon(\tau, \frac{\mathbf{d}_5}{2^\beta}) \le \frac{c_6 |c_5 \tau|^{\beta-\beta_0}}{(\beta-\beta_0)^{(\beta-\beta_0)\mathbf{f}_\beta}} \mathcal{F}^\epsilon(\tau, \frac{\mathbf{d}_5}{2^{\beta_0}}),$$

where constant c_6 is independent of $\tau, \beta, \epsilon, \mu$. So proof of first part of (6.17) is completed. The other part can be proved in a similar way, so we skip it. ■

Lemma 6.3 *If $r \in (1,2)$, $\|P_\alpha^\epsilon\|_{L^r(0,T;W^{1,r}(\Omega_f^\epsilon))} + \|\mathcal{I}_\epsilon^\varpi \nabla p_\alpha^\epsilon\|_{L^r(\Omega_m^{\epsilon,T})} \le c$, where $\alpha = w, o$ and c is a constant independent of ϵ. Moreover, if $\varpi \le 1$, then $\|p_\alpha^\epsilon\|_{L^r(\Omega_m^{\epsilon,T})} \le c$.*

Proof: We define, for $2 \le \beta_0 \in \mathbf{N}$,

$$\begin{cases} \mathcal{B}_{1+\beta_0} \equiv \{(t,x) \in \Omega_f^{\epsilon,T} : \ \frac{\mathbf{d}_5}{2^{2+\beta_0}} \le S^\epsilon\}, \\ \mathcal{B}_\beta \equiv \{(t,x) \in \Omega_f^{\epsilon,T} : \ \frac{\mathbf{d}_5}{2^{\beta+1}} \le S^\epsilon < \frac{\mathbf{d}_5}{2^\beta}\} \quad \text{if } 2+\beta_0 \le \beta \in \mathbf{N}. \end{cases}$$

A5, Lemmas 6.1-6.2, and Hölder inequality imply

$$\begin{aligned} \|\nabla P_o^\epsilon\|_{L^r(\Omega_f^{\epsilon,T})}^r &\le \|\sqrt{\Lambda_o(S^\epsilon)}\ \nabla P_o^\epsilon\|_{L^2(\Omega_f^{\epsilon,T})}^r \|\Lambda_o^{-1}(S^\epsilon)\|_{L^{r/(2-r)}(\Omega_f^{\epsilon,T})}^{r/2} \\ &\le c_1 \Big(\int_{\Omega_f^{\epsilon,T}} |\Lambda_o(S^\epsilon)|^{\frac{-r}{2-r}} \sum_{\beta=1+\beta_0}^{\infty} \mathcal{X}_{\mathcal{B}_\beta} \Big)^{\frac{2-r}{2}} \le c_2 \text{ (indep. of } \epsilon). \end{aligned} \tag{6.26}$$

Similar argument will give $\|\nabla P_w^\epsilon\|_{L^r(\Omega_f^{\epsilon,T})} + \sum_{\alpha=w,o} \|\mathcal{I}_\epsilon^\varpi \nabla p_\alpha^\epsilon\|_{L^r(\Omega_m^{\epsilon,T})} \le c$. By boundary condition A2, $\|P_\alpha^\epsilon\|_{L^r(\Omega_f^{\epsilon,T})} \le c$, $\alpha = w, o$. By Lemma 5.1, (2.8), and $\varpi \le 1$, $\|p_\alpha^\epsilon - \Pi_\epsilon P_\alpha^\epsilon\|_{L^r(\Omega_m^{\epsilon,T})} \le \|\epsilon \partial_{x_1}(p_\alpha^\epsilon - \Pi_\epsilon P_\alpha^\epsilon)\|_{L^r(\Omega_m^{\epsilon,T})} \le c$. So $\|p_\alpha^\epsilon\|_{L^r(\Omega_m^{\epsilon,T})}$ is bounded. ■

Lemma 6.4 *For $r \in [1,\infty)$ and sufficiently small δ,*

$$\|\delta^2 \partial^{-\delta} S^\epsilon\ \partial^{-\delta} \mathcal{A}^\epsilon\|_{L^r((\delta,T)\times\Omega_f^\epsilon)} + \|\delta^2 \partial^{-\delta} s^\epsilon\ \partial^{-\delta} \mathcal{A}^\epsilon\|_{L^r((\delta,T)\times\Omega_m^\epsilon)} \le c\delta^{1/r}, \tag{6.27}$$

where c is independent of ϵ, δ. See §4 for notation $\partial^{-\delta}$.

Proof: Note $\zeta(t,x) \equiv \int_{max(t,\delta)}^{min(t+\delta,T)} \delta\ \partial^{-\delta}\big(\mathcal{A}^\epsilon - \mathcal{A}(P_{b,c})\big)(\tau,x)d\tau \in L^2(0,T;\mathcal{U})$ by A2-3 and Lemma 6.1. Take ζ above in (6.2) to get, by Fubini's theorem, A2, and Lemma 6.1,

$$\begin{aligned} &\int_\delta^T \int_{\Omega_f^\epsilon} \Phi^\epsilon \delta^2 \partial^{-\delta} S^\epsilon\ \partial^{-\delta} \mathcal{A}^\epsilon(\tau,x) + \int_\delta^T \int_{\Omega_m^\epsilon} \phi^\epsilon \delta^2 \partial^{-\delta} s^\epsilon \partial^{-\delta} \mathcal{A}^\epsilon(\tau,x) \\ &\quad = \int_{\Omega_f^{\epsilon,T}} \Phi^\epsilon \partial_t S^\epsilon(t,x)\zeta + \int_{\Omega_m^{\epsilon,T}} \phi^\epsilon \partial_t s^\epsilon(t,x)\zeta \\ &\qquad + \int_\delta^T \int_{\Omega_f^\epsilon} \Phi^\epsilon \delta^2 \partial^{-\delta} S^\epsilon\ \partial^{-\delta} \mathcal{A}(P_{b,c}) + \int_\delta^T \int_{\Omega_m^\epsilon} \phi^\epsilon \delta^2 \partial^{-\delta} s^\epsilon \partial^{-\delta} \mathcal{A}(P_{b,c}) \le c\delta, \end{aligned}$$

where c is independent of ϵ, δ. So we prove (6.27) for $r = 1$ case. (6.27) for $r > 1$ case follows directly because $\mathcal{A}^\epsilon, \check{\mathcal{S}}^\epsilon$ are bounded and (6.27) for $r = 1$ holds. ■

Lemma 6.5 *A subsequence of* $\Pi_\epsilon(\mathcal{A}^\epsilon|_{\Omega_f^\epsilon})$ *converges to* $\mathcal{A}^*$ *in* $L^2(\Omega^T)$ *and pointwise.*

Proof: This is due to A6,10, Lemmas 5.1, 6.1-6.4, and compactness principle. ∎

Lemma 6.6 $\overline{s^\epsilon}, \overline{p^\epsilon}, \overline{p^\epsilon_\alpha}$ $(\alpha = w, o)$ *satisfy, for almost all* $x \in \tilde{\Omega}^\epsilon$,

$$\phi\partial_t\overline{s^\epsilon} - \partial_{y,x_3}\cdot\big(k\mathcal{I}_\epsilon^{2\varpi-2}\big(\partial_{y,x_3}\mathcal{M}(\overline{s^\epsilon}) + \lambda_o(\overline{s^\epsilon})\partial_{y,x_3}(\overline{p^\epsilon} - \overline{G^\epsilon_o})\big)\big) = 0, \tag{6.28}$$

$$\partial_{y,x_3}\cdot\Big(k\mathcal{I}_\epsilon^{2\varpi-2}\big(\lambda(\overline{s^\epsilon})\partial_{y,x_3}\overline{p^\epsilon} - \sum\lambda_\alpha(\overline{s^\epsilon})\partial_{y,x_3}\overline{G^\epsilon_\alpha}\big)\Big) = 0, \tag{6.29}$$

$$-\phi\partial_t\overline{s^\epsilon} - \partial_{y,x_3}\cdot\big(k\mathcal{I}_\epsilon^{2\varpi-2}\lambda_w(\overline{s^\epsilon})\partial_{y,x_3}(\overline{p^\epsilon_w} - \overline{G^\epsilon_w})\big) = 0, \tag{6.30}$$

$$\phi\partial_t\overline{s^\epsilon} - \partial_{y,x_3}\cdot\big(k\mathcal{I}_\epsilon^{2\varpi-2}\lambda_o(\overline{s^\epsilon})\partial_{y,x_3}(\overline{p^\epsilon_o} - \overline{G^\epsilon_o})\big) = 0, \tag{6.31}$$

in $L^2(0,T;H^{-1}(Y_m^H))$.

Proof: Let $\hat{\zeta} \in L^2(0,T;C_0^\infty(Y_m^H))$. For $x \in \Omega, y \in \Re^2$, we define

$$\check{\zeta}(t,x,y) \equiv \begin{cases} \hat{\zeta}(t, \frac{y-\ell^\epsilon(\tilde{x})}{\epsilon}, x_3) & \text{for } y \in \epsilon Y_m + \ell^\epsilon(\tilde{x}), \\ 0 & \text{elsewhere.} \end{cases}$$

Then we plug $\zeta(t,x) \equiv \mathcal{X}_{\epsilon(Y_m+j)}(\tilde{x})\check{\zeta}(t,x,\tilde{x})$ for $j \in Z^2$ into (6.4). Since *supp* $\zeta \subset (0,T)\times\epsilon(Y_m+j)\times[0,H]$,

$$\int_0^T\int_0^H\int_{\epsilon(Y_m+j)} \phi^\epsilon\partial_t s^\epsilon\zeta + k^\epsilon\mathcal{I}_\epsilon^{2\varpi}\big(\lambda_o(s^\epsilon)\nabla(p^\epsilon - G_o^\epsilon) + \nabla\mathcal{M}(s^\epsilon)\big)\nabla\zeta = 0.$$

Since $\tilde{x} \in \epsilon(Y_m+j)$, $\ell^\epsilon(\tilde{x}) = \epsilon j$. Changing variable $y = \frac{\tilde{x}-\ell^\epsilon(\tilde{x})}{\epsilon}$ gives

$$\int_0^T\int_{Y_m^H} \phi\partial_t\overline{s^\epsilon}\hat{\zeta} + k\mathcal{I}_\epsilon^{2\varpi-2}\big(\partial_{y,x_3}\mathcal{M}(\overline{s^\epsilon}) + \lambda_o(\overline{s^\epsilon})\partial_{y,x_3}(\overline{p^\epsilon} - \overline{G^\epsilon_o})\big)\partial_{y,x_3}\hat{\zeta} = 0, \tag{6.32}$$

for almost all $\tilde{x} \in \epsilon(Y_m+j), j \in Z^2$. Actually, by Definition 5.1, (6.32) holds for $\tilde{x} \in \tilde{\Omega}^\epsilon$, i.e., (6.28). (6.29–6.31) can be proved in a similar way. ∎

Remark 6.2 *By Lemmas 5.1, 6.5, if we define* $S^\epsilon \equiv \Upsilon^{-1}(\mathcal{A}^{-1}(\Pi_\epsilon(\mathcal{A}^\epsilon|_{\Omega_f^\epsilon})))$ *and* $S \equiv \begin{cases} \Upsilon^{-1}(\mathcal{A}^{-1}(\mathcal{A}^*)) & \text{if } \mathcal{A}^* < \mathcal{A}(\infty), \\ 1 & \text{if } \mathcal{A}^* = \mathcal{A}(\infty), \end{cases}$ *then* $0 \le S^\epsilon, S \le 1$.

Lemma 6.7 *There is a* $r \in (1,2)$ *and a subsequence of* $\{S^\epsilon, s^\epsilon, S_0^\epsilon, s_0^\epsilon, \phi^\epsilon, k^\epsilon, P_\alpha^\epsilon, p_\alpha^\epsilon, \alpha = w, o\}$ *such that, as* $\epsilon \to 0$,

$$\begin{cases} \mathcal{X}_{\Omega_f^\epsilon}P_\alpha^\epsilon \overset{2}{\rightharpoonup} \mathcal{X}_{Y_f}(y)P_\alpha(t,x) \text{ where } P_\alpha \in L^r(0,T;W^{1,r}(\Omega)), P_\alpha = P_{b,\alpha} \text{ in } \Gamma_2, \\ \mathcal{X}_{\Omega_f^\epsilon}\nabla P_\alpha^\epsilon \overset{2}{\rightharpoonup} \mathcal{X}_{Y_f}(y)(\nabla P_\alpha + \partial_y P_{\alpha,1}(t,x,y)) \text{ where } P_{\alpha,1} \in L^r(\Omega^T; L^r_{per}(Y_f)), \\ \mathcal{X}_{\Omega_f^\epsilon}S_0^\epsilon \overset{2}{\rightharpoonup} S_0 \in L^2(\Omega; L^2_{per}(Y_f)), \\ S^\epsilon \to S \text{ strongly in } L^2(\Omega^T) \text{ and pointwise}, \\ \mathcal{X}_{\Omega_f^\epsilon}S^\epsilon \overset{2}{\to} \mathcal{X}_{Y_f}(y)S(t,x) \text{ strongly}, \\ \mathcal{X}_{\Omega_m^\epsilon}s_0^\epsilon \overset{2}{\to} s_0 \in L^2(\Omega; L^2_{per}(Y_m)) \text{ strongly}, \\ \overline{p_\alpha^\epsilon} \rightharpoonup p_\alpha \quad \text{weakly in } L^r(\Omega^T; W^{1,r}(Y_m)). \end{cases}$$

Proof: By Lemma 5.1 and Lemma 6.3, $\Pi_\epsilon P_\alpha^\epsilon$ is bounded in $L^r(0,T;W^{1,r}(\Omega))$. So a subsequence of $\Pi_\epsilon P_\alpha^\epsilon$ converges weakly to limit $P_\alpha \in L^r(0,T;W^{1,r}(\Omega))$. Since $\Pi_\epsilon P_\alpha^\epsilon = P_{b,\alpha}$ in Γ_2, $P_\alpha = P_{b,\alpha}$ in Γ_2. Rest of proof are due to A2-4,6,10, Lemmas 6.1, 6.3, 6.5, and [3]. ■

Lemma 6.8 *$\overline{s^\epsilon}$ converges to s in $L^2(\mathcal{Q}_m^T)$ if $0 < \varpi \leq 1$.*

Proof of this lemma is lengthy, and will be postponed untill the last five sections.

Lemma 6.9 *If $\varpi = 1$, then $p_o - p_w = \upsilon(s)$, $P_o - P_w = \Upsilon(S)$, and $p_\alpha(t,x,y) = P_\alpha(t,x)$ for $x \in \Omega, y \in \partial Y_m$, $\alpha = w,o$. If $\varpi < 1$, then $\upsilon(s) = P_o - P_w = \Upsilon(S)$ and $p_\alpha(t,x,y) = P_\alpha(t,x)$ for $x \in \Omega, y \in Y_m$, $\alpha = w,o$.*

Proof: First we consider $\varpi = 1$ case. Note $0 \leq S < 1$, $s_l \leq s < s_r$ by Egoroff's theorem [25] and Lemmas 6.1-6.2, 6.7-6.8. Since $\overline{p_o^\epsilon} - \overline{p_w^\epsilon} = \upsilon(\overline{s^\epsilon})$, we get $p_o - p_w = \upsilon(s)$ by Lemmas 6.7-6.8. Similarly, one can derive $P_o - P_w = \Upsilon(S)$. By Lemmas 5.1, 6.3 and (2.8), $\overline{(\Pi_\epsilon P_\alpha^\epsilon)|_{\Omega_m^\epsilon}} - \overline{p_\alpha^\epsilon} \in L^r(\Omega^T; W_0^{1,r}(Y_m))$ for $1 < r < 2$. So, a subsequence of $\overline{(\Pi_\epsilon P_\alpha^\epsilon)|_{\Omega_m^\epsilon}} - \overline{p_\alpha^\epsilon}$ converges weakly to $\mathcal{X}_{Y_m}(y)P_\alpha(t,x) - p_\alpha \in L^r(\Omega^T; W_0^{1,r}(Y_m))$ by Lemma 6.7. So, $p_\alpha(t,x,y) = P_\alpha(t,x)$ for $y \in \partial Y_m$. Results for $\varpi < 1$ case can be obtained by similar argument as above, so we skip it. ■

Now we consider the limit model of (2.1–2.13) as $\epsilon \to 0$. Plug into (6.1) and (6.6) a test function $\eta = \hat{\zeta}(t,x) + \epsilon\hat{\eta}(t,x,\frac{\tilde{x}}{\epsilon})$ where $\hat{\zeta} \in C_0^\infty(\Omega^T), \hat{\eta} \in C_0^\infty(\Omega^T; C_{per}^\infty(Y))$ to obtain

$$0 = \int_{\Omega_f^{\epsilon,T}} \Phi^\epsilon S^\epsilon(\partial_t\hat{\zeta} + \epsilon\partial_t\hat{\eta}) + K^\epsilon\Lambda_w(S^\epsilon)\nabla(P_w^\epsilon - G_w^\epsilon)(\nabla\hat{\zeta} + \epsilon\partial_x\hat{\eta} + \partial_y\hat{\eta})$$
$$+ \int_{\Omega_m^{\epsilon,T}} \phi^\epsilon s^\epsilon(\partial_t\hat{\zeta} + \epsilon\partial_t\hat{\eta}) + k^\epsilon\mathcal{I}_\epsilon^{2\varpi}\lambda_w(s^\epsilon)\nabla(p_w^\epsilon - G_w^\epsilon)(\nabla\hat{\zeta} + \epsilon\partial_x\hat{\eta} + \partial_y\hat{\eta})$$
$$+ \int_{\Omega_f^\epsilon} \Phi^\epsilon S_0^\epsilon(\hat{\zeta} + \epsilon\hat{\eta})(0) + \int_{\Omega_m^\epsilon} \phi^\epsilon s_0^\epsilon(\hat{\zeta} + \epsilon\hat{\eta})(0).$$

By A2 and Lemma 6.7, $K^\epsilon\Lambda_w(S^\epsilon)$ converges to $K^*\Lambda_w(S)$ in $L^r(\Omega^T), r < \infty$ strongly. Passing to two-scale limit, we get, by A2-4, Lemmas 6.3-6.9, Theorem 2.28 of [2], Theorem 1.8 of [3], and [8, 10],

$$\int_{\mathcal{Q}_f^T} \Phi^* S\partial_t\hat{\zeta} + K^*\Lambda_w(S)(\nabla P_w + \partial_y P_{w,1} - \nabla G_w)(\nabla\hat{\zeta} + \partial_y\hat{\eta})$$
$$= -\int_{\mathcal{Q}_m^T} \phi s\partial_t\hat{\zeta} + \mathcal{F}_w^*\partial_{x_3}\hat{\zeta} - \int_{\mathcal{Q}_f} \Phi^* S_0\hat{\zeta}(0) - \int_{\mathcal{Q}_m} \phi s_0\hat{\zeta}(0),$$

where

$$\mathcal{F}_w^* \equiv \begin{cases} k\lambda_w(s)\partial_{x_3}(P_w - G_w) & \text{if } 0 < \varpi < 1, \\ k\lambda_w(s)\partial_{x_3}(p_w - G_w) & \text{if } \varpi = 1, \\ \text{An } L^2 \text{ function} & \text{if } \varpi > 1. \end{cases} \tag{6.33}$$

Apply Green's theorem in t variable to get

$$-\int_{\mathcal{Q}_f^T} \Phi^*\partial_t S\ \hat{\zeta} - K^*\Lambda_w(S)(\nabla P_w + \partial_y P_{w,1} - \nabla G_w)(\nabla\hat{\zeta} + \partial_y\hat{\eta})$$

$$= \int_{\mathcal{Q}_m^T} \phi \partial_t s \ \hat{\zeta} - \mathcal{F}_w^* \partial_{x_3} \hat{\zeta} + \int_{\mathcal{Q}_f} \Phi^*(S(0) - S_0)\hat{\zeta}(0) + \int_{\mathcal{Q}_m} \phi(s(0) - s_0)\hat{\zeta}(0).$$

So we have, in Ω^T,

$$(S(0) - S_0) \int_{Y_f} \Phi^* dy + \int_{Y_m} \phi(s(0) - s_0) dy = 0, \tag{6.34}$$

and the choice of $\hat{\eta} = 0$ gives, in Ω^T,

$$\begin{aligned} \int_{Y_f} \Phi^* dy \partial_t S + \nabla \int_{Y_f} K^* \Lambda_w(S)(\nabla P_w + \partial_y P_{w,1} - \nabla G_w) \\ = - \int_{Y_m} (\phi \partial_t s + \partial_{x_3} \mathcal{F}_w^*) \, dy. \end{aligned} \tag{6.35}$$

The choice of $\hat{\zeta} = 0$ gives, by A2-3 and Lemma 6.7,

$$\begin{cases} \partial_y^2 P_{w,1} = 0 & \text{in } \mathcal{Q}_f, \\ (\partial_{\tilde{x}} P_w + \partial_y P_{w,1}) \cdot \vec{\nu_y} = 0 & \text{on } \partial Y_m, \end{cases} \tag{6.36}$$

where $\vec{\nu_y}$ is the unit vector outward normal to ∂Y_m. Let $\vec{e_j}$ be the unit vector in jth direction. We denote by Ξ the tensor whose (i, j) component is $\partial \varphi_j / \partial y_i$, where φ_j is a periodic solution in Y of the auxiliary problem

$$\begin{cases} \Delta_y \varphi_j = 0 & \text{in } Y_f, \\ \partial_y \varphi_j \cdot \vec{\nu_y} = -\vec{e_j} \cdot \vec{\nu_y} & \text{on } \partial Y_m. \end{cases}$$

$P_{w,1}$ of (6.36) is given by the product $P_{w,1} = \sum_j \varphi_j(y) \partial_{x_j} P_w$. So (6.35) becomes

$$\Phi \partial_t S + \nabla \cdot (K \Lambda_w(S) \nabla (P_w - G_w)) = \frac{-1}{|Y_m|} \int_{Y_m} (\phi \partial_t s + \partial_{x_3} \mathcal{F}_w^*) dy, \tag{6.37}$$

where $\Phi \equiv \frac{1}{|Y_m|} \int_{Y_f} \Phi^* dy$ and K is a diagonal matrix satisfying

$$K_{11} = K_{22} = \frac{K^*}{|Y_m|} \int_{Y_f} (I + \Xi(y)) dy, \quad K_{33} = \frac{|Y_f| K^*}{|Y_m|}.$$

Proceeding as the proof of (6.37), we obtain, by (6.2),

$$-\Phi \partial_t S + \nabla \cdot (K \Lambda_o(S) \nabla (P_o - G_o)) = \frac{1}{|Y_m|} \int_{Y_m} (\phi \partial_t s - \partial_{x_3} \mathcal{F}_o^*) dy, \tag{6.38}$$

where

$$\mathcal{F}_o^* \equiv \begin{cases} k \lambda_o(s) \partial_{x_3} (P_o - G_o) & \text{if } 0 < \varpi < 1, \\ k \lambda_o(s) \partial_{x_3} (p_o - G_o) & \text{if } \varpi = 1, \\ \text{An } L^2 \text{ function} & \text{if } \varpi > 1. \end{cases} \tag{6.39}$$

Matrix sources for $0 < \varpi < 1$ case is clear from (6.33), (6.39), and Lemma 6.9. Next we consider the matrix source terms for $\varpi \geq 1$ cases.

6.1. *For* $\varpi = 1$ *case*

By (6.30) of Lemma 6.6, we have, for any $\eta \in L^2(\Omega^T; H_0^1(Y_m))$,

$$\int_{\mathcal{Q}_m^{\epsilon,T}} \phi \partial_t \overline{s^\epsilon} \eta + \int_{\mathcal{Q}_m^{\epsilon,T}} k\lambda_o(\overline{s^\epsilon})\partial_{y,x_3}(\overline{p_o^\epsilon} - \overline{G_o^\epsilon})\partial_{y,x_3}\eta = 0.$$

As $\epsilon \to 0$, by Lemmas 6.7-6.8, one obtains

$$\int_{\mathcal{Q}_m^T} \phi \partial_t s\ \eta + \int_{\mathcal{Q}_m^T} k\lambda_o(s)\partial_{y,x_3}(p_o - G_o)\partial_{y,x_3}\eta = 0. \tag{6.40}$$

In a similar way, we obtain, by (6.31),

$$\int_{\mathcal{Q}_m^T} \phi \partial_t s\ \eta - \int_{\mathcal{Q}_m^T} k\lambda_w(s)\partial_{y,x_3}(p_w - G_w)\partial_{y,x_3}\eta = 0. \tag{6.41}$$

By (6.37–6.41) and Lemmas 6.7-6.9, it is easy to show Theorem 4.1.

6.2. *For* $\varpi > 1$ *case*

By (6.30) of Lemma 6.6, we have, for any $\eta \in L^2(\Omega^T; H_0^1(Y_m))$,

$$\int_{\mathcal{Q}_m^{\epsilon,T}} \phi \partial_t \overline{s^\epsilon} \eta + \int_{\mathcal{Q}_m^{\epsilon,T}} k\mathcal{I}_\epsilon^{2\varpi-2}\lambda_o(\overline{s^\epsilon})\partial_{y,x_3}(\overline{p_o^\epsilon} - \overline{G_o^\epsilon})\partial_{y,x_3}\eta = 0.$$

As $\epsilon \to 0$, by Lemmas 6.7-6.8, one obtains

$$\int_{\mathcal{Q}_m^T} \phi \partial_t s\ \eta + \int_{\mathcal{Q}_m^T} \mathcal{F}_o^* \partial_{x_3}\eta = 0.$$

So we get $\phi\partial_t s - \partial_{x_3}\mathcal{F}_o^* = 0$. In a similar way, we obtain $\phi\partial_t s + \partial_{x_3}\mathcal{F}_w^* = 0$. Therefore we prove Theorem 4.3.

Rest of this work is to prove Lemma 6.8.

7. Convergence of $\overline{s^\epsilon}$

Remark 7.3 *Define*

$$\begin{cases} \mathcal{G}^\epsilon \equiv \begin{cases} \upsilon^{-1}(\mathcal{A}^{-1}(\Pi_\epsilon(\mathcal{A}^\epsilon|_{\Omega_f^\epsilon}))) & \textit{if } \Pi_\epsilon\mathcal{A}^\epsilon < \mathcal{A}(\infty), \\ s_r & \textit{if } \Pi_\epsilon\mathcal{A}^\epsilon = \mathcal{A}(\infty), \end{cases} \\ \mathcal{G} \equiv \begin{cases} \upsilon^{-1}(\mathcal{A}^{-1}(\mathcal{A}^*)) & \textit{if } \mathcal{A}^* < \mathcal{A}(\infty), \\ s_r & \textit{if } \mathcal{A}^* = \mathcal{A}(\infty). \end{cases} \end{cases}$$

See Lemma 6.5 for $\mathcal{A}^*$*. By Lemma 6.7, A1,3, Theorem 2.28 of [2], and [3, 8, 10], it is easy to see that*

$$\begin{cases} \|\mathcal{M}(\mathcal{G}^\epsilon)\|_{L^2(0,T;H^1(\Omega))} \textit{ are bounded independently of } \epsilon, \\ \mathcal{M}(\overline{\mathcal{G}^\epsilon|_{\Omega_m^\epsilon}}) \to \mathcal{M}(\mathcal{G}) \quad \textit{strongly in } L^2(\mathcal{Q}_m^T), \\ \mathcal{M}(\overline{\mathcal{G}^\epsilon|_{\Omega_m^\epsilon}}) - \mathcal{M}(\overline{s^\epsilon}) \in L^2(\Omega^T; H_0^1(Y_m)). \end{cases} \tag{7.1}$$

Assume that $\overline{s^{\epsilon_i}}, \overline{p^{\epsilon_i}}, i = 1, 2$ are two solutions of (6.28–6.29), and ζ, η are smooth functions satisfying

$$\zeta(T) = 0,\ \zeta|_{\partial Y_m \times [0,H]} = \eta|_{\partial Y_m \times [0,H]} = \partial_{x_3}\zeta|_{x_3 \in \{0,H\}} = \partial_{x_3}\eta|_{x_3 \in \{0,H\}} = 0. \quad (7.2)$$

Let $x \in \Omega^{\epsilon_1} \cap \Omega^{\epsilon_2}$. By subtracting one solution from the other and integration by parts, we obtain

$$\int_{Y_m^{H,T}} (\overline{s^{\epsilon_1}} - \overline{s^{\epsilon_2}})\big(\phi\partial_t\zeta + \mathcal{F}_1\partial_{y,x_3}(k\mathcal{I}_\epsilon^{2\varpi-2}\partial_{y,x_3}\zeta) - \mathcal{F}_2\partial_{y,x_3}\zeta - \mathcal{F}_3\partial_{y,x_3}\eta\big)$$
$$+ \int_{Y_m^{H,T}} (\overline{p^{\epsilon_1}} - \overline{p^{\epsilon_2}})\big(\partial_{y,x_3}k\mathcal{I}_\epsilon^{2\varpi-2}(\lambda(\overline{s^{\epsilon_1}})\partial_{y,x_3}\eta + \lambda_o(\overline{s^{\epsilon_1}})\partial_{y,x_3}\zeta)\big) = \mathcal{F}_4 + \mathcal{F}_5, \quad (7.3)$$

where

$$\mathcal{F}_1 \equiv \mu + \begin{cases} \frac{\mathcal{M}(\overline{s^{\epsilon_1}}) - \mathcal{M}(\overline{s^{\epsilon_2}})}{\overline{s^{\epsilon_1}} - \overline{s^{\epsilon_2}}} & \text{if } \overline{s^{\epsilon_1}} \neq \overline{s^{\epsilon_2}}, \\ 0 & \text{otherwise}, \end{cases} \quad (7.4)$$

$$\mathcal{F}_2 \equiv \begin{cases} \frac{k(\lambda_o(\overline{s^{\epsilon_1}}) - \lambda_o(\overline{s^{\epsilon_2}}))\mathcal{I}_\epsilon^{2\varpi-2}\partial_{y,x_3}(\overline{p^{\epsilon_2}} - \overline{G_o^{\epsilon_2}})}{\overline{s^{\epsilon_1}} - \overline{s^{\epsilon_2}}} & \text{if } \overline{s^{\epsilon_1}} \neq \overline{s^{\epsilon_2}}, \\ 0 & \text{otherwise}, \end{cases} \quad (7.5)$$

$$\mathcal{F}_3 \equiv \begin{cases} \sum_\alpha \frac{k(\lambda_\alpha(\overline{s^{\epsilon_1}}) - \lambda_\alpha(\overline{s^{\epsilon_2}}))\mathcal{I}_\epsilon^{2\varpi-2}\partial_{y,x_3}(\overline{p^{\epsilon_2}} - \overline{G_\alpha^{\epsilon_2}})}{\overline{s^{\epsilon_1}} - \overline{s^{\epsilon_2}}} & \text{if } \overline{s^{\epsilon_1}} \neq \overline{s^{\epsilon_2}}, \\ 0 & \text{otherwise}, \end{cases} \quad (7.6)$$

$$\mathcal{F}_4 \equiv \mu \int_{Y_m^{H,T}} (\overline{s^{\epsilon_1}} - \overline{s^{\epsilon_2}})\partial_{y,x_3}(k\mathcal{I}_\epsilon^{2\varpi-2}\partial_{y,x_3}\zeta), \quad (7.7)$$

$$\mathcal{F}_5 \equiv \int_{Y_m^{H,T}} \epsilon^{2\varpi-2}\partial_y \left(\left(\mathcal{M}(\overline{\mathcal{G}^{\epsilon_1}|_{\Omega_m^{\epsilon_1}}}) - \mathcal{M}(\overline{\mathcal{G}^{\epsilon_2}|_{\Omega_m^{\epsilon_2}}})\right) k\partial_y\zeta\right)$$
$$+ \int_{Y_m^{H,T}} \epsilon^{2\varpi-2}\partial_y \left((\overline{\Pi_{\epsilon_1}P^{\epsilon_1}|_{\Omega_m^{\epsilon_1}}} - \overline{\Pi_{\epsilon_2}P^{\epsilon_2}|_{\Omega_m^{\epsilon_2}}})\big(k\lambda(\overline{s^{\epsilon_1}})\partial_y\eta + k\lambda_o(\overline{s^{\epsilon_1}})\partial_y\zeta\big)\right)$$
$$- \sum_{\alpha \in \{w,o\}} \int_{Y_m^{H,T}} k\lambda_\alpha(\overline{s^{\epsilon_1}})\partial_{x_3}(\overline{G_\alpha^{\epsilon_1}} - \overline{G_\alpha^{\epsilon_2}})\partial_{x_3}\eta$$
$$- \int_{Y_m^{H,T}} k\lambda_o(\overline{s^{\epsilon_1}})\partial_{x_3}(\overline{G_o^{\epsilon_1}} - \overline{G_o^{\epsilon_2}})\partial_{x_3}\zeta - \int_{Y_m^H} (\overline{s_0^{\epsilon_1}} - \overline{s_0^{\epsilon_2}})\phi\zeta(0). \quad (7.8)$$

Define $\widetilde{\mathcal{U}}_1 \equiv \{\zeta : \zeta \in H^1(Y_m^{H,T}) \cap L^\infty(0,T; H^1(Y_m^H)), \zeta|_{\partial Y_m \times [0,H]} = \partial_{x_3}\zeta|_{x_3=0,H} = \zeta(0) = 0\}$. We consider the following auxiliary problem for fixed μ:

Lemma 7.1 *Let $\mathcal{F}_2, \mathcal{F}_3 \in L^\infty(Y_m^{H,T})$ and $0 < \mathbf{d}_{18} < \mathcal{F}_1 < \mathbf{d}_{19} < \infty$. For $(f_1, f_2) \in L^2(Y_m^{H,T}) \times L^2(Y_m^{H,T})$, there is a unique $(\zeta, \eta) \in \widetilde{\mathcal{U}}_1 \times L^2(0,T; H^1(Y_m^H))$ such that*

$$-\phi\partial_t\zeta + \mathcal{F}_1\partial_{y,x_3}(k\mathcal{I}_\epsilon^{2\varpi-2}\partial_{y,x_3}\zeta) - \mathcal{F}_2\partial_{y,x_3}\zeta - \mathcal{F}_3\partial_{y,x_3}\eta = f_1, \quad (7.9)$$
$$\partial_{y,x_3}\big(k\mathcal{I}_\epsilon^{2\varpi-2}(\lambda\partial_{y,x_3}\eta + \lambda_o\partial_{y,x_3}\zeta)\big) = f_2. \quad (7.10)$$

Moreover,

$$\sup_{\tau \le T} \|\mathcal{I}_\epsilon^{\varpi-1}\partial_{y,x_3}\zeta(\tau)\|_{L^2(Y_m^H)} + \||\mathcal{I}_\epsilon^{\varpi-1}\partial_{y,x_3}\eta| + \mathbf{d}_{18}^{1/2}|\partial_{y,x_3}(k\mathcal{I}_\epsilon^{2\varpi-2}\partial_{y,x_3}\zeta)|\|_{L^2(Y_m^{H,T})}$$
$$\le c\left(\mathbf{d}_{19}, \|(|\mathcal{F}_2| + |\mathcal{F}_3|)/\mathcal{F}_1^{1/2}\|_{L^\infty(Y_m^{H,T})}\right) \|\ |f_1|/\mathcal{F}_1^{1/2} + |f_2|\ \|_{L^2(Y_m^{H,T})}. \quad (7.11)$$

Proof: This is proved by following the argument of Lemma 5.1 [29]. ■

Finally we give the proof of Lemma 6.8.

Proof: For $x \in \Omega^{\epsilon_1} \cap \Omega^{\epsilon_2}$, we take $f_1 = \mathcal{M}(\overline{s^{\epsilon_1}}) - \mathcal{M}(\overline{s^{\epsilon_2}})$ in (7.9) and $f_2 = \overline{p^{\epsilon_1}} - \overline{p^{\epsilon_2}}$ in (7.10) to obtain solution (ζ^μ, η^μ) for each μ by (7.4–7.6), Remark 7.3, and Lemma 7.1. After substitution $t \to T - t$ for the solution (ζ^μ, η^μ), we plug it into (7.3) to obtain

$$\int_{Y_m^{H,T}} (\overline{s^{\epsilon_1}} - \overline{s^{\epsilon_2}})(\mathcal{M}(\overline{s^{\epsilon_1}}) - \mathcal{M}(\overline{s^{\epsilon_2}})) + \int_{Y_m^{H,T}} |\overline{p^{\epsilon_1}} - \overline{p^{\epsilon_2}}|^2 = \mathcal{F}_4 + \mathcal{F}_5. \qquad (7.12)$$

By Lemmas 6.1, 7.1 and [13, 15, 26, 27], we see 1) $\mathcal{F}_4$ is bounded by $c\sqrt{\mu}$, where c is a constant independent of $\mu, \epsilon_1, \epsilon_2$; and 2) For fixed μ, $\mathcal{F}_5$ converges to 0 as ϵ_1, ϵ_2 tend to 0. So it is not difficult to show that $\mathcal{M}(\overline{s^{\epsilon_2}})$ is a Cauchy sequence in $L^2(\mathcal{Q}_m^T)$, which implies $\overline{s^{\epsilon_2}}$ is a Cauchy sequence in $L^2(\mathcal{Q}_m^T)$ as well. ■

Acknowledgement

This research is supported by the grant number NSC 93-2115-M-009-09 from the research program of National Science Council of Taiwan.

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Two-Component Miscible Displacement in Fractured Media

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A Peaceman model, which describes the effects of turbulent mechanical mixing of non-stationary, incompressible, two-component, miscible displacement in fractured media is studied. In a fractured medium, there is an interconnected system of fracture planes dividing the porous rock into a collection of matrix blocks. The fracture planes, while very thin, form paths of high permeability. Most of the fluids reside in matrix blocks, where they move very slow. Let ϵ denote the size ratio of the matrix blocks to the whole medium and let permeability, gravity, and width ratios of matrix blocks to fracture planes be of the orders ϵ^2, ϵ, and ϵ^0 respectively. Microscopic models for the two-component, miscible displacement in fractured media converge to a dual-porosity model as ϵ tends to 0.

Keywords: dual-porosity model, fractured media, miscible displacement

1. Introduction

Homogenization of non-stationary, incompressible, two-component, miscible displacement in a fractured medium with moderate-sized matrix blocks is concerned. Within a fractured medium there is an interconnected system of fracture planes dividing the porous rock into a collection of matrix blocks. The system of fracture planes is on a much finer scale than the whole reservoir; hence, it can be viewed as a porous structure. The fractures form the void space while the matrix blocks play the role of the solid rock. Certainly, the solid part itself is permeable. The fracture planes, while very thin, form paths of high permeability. Most of the fluids reside in matrix blocks, where they move very slow. No flow is allowed between blocks, and fluids that reside in matrix blocks must enter the fracture planes to move great distance. The two-component, miscible displacement in the system of fracture planes and the system of matrix blocks can be described by Peaceman model [6, 23, 25]. On the interface of the two systems, fluxes, pressure, and concentration are continuous.

Let ϵ denote the size ratio of the matrix blocks to the whole medium and let permeability, gravity, and width ratios of matrix blocks to fracture planes be of the orders ϵ^2, ϵ, and ϵ^0 respectively. As ϵ tends to 0, microscopic models for the two-

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component, miscible displacement in fractured media converge to a dual-porosity model. In the macroscopic model, domain is regarded as a porous medium consisting of two superimposed continua, a continuous fracture system and a discontinuous system of matrix blocks. The two-component, miscible displacement is formulated by conservation of mass principle for each continuum. Matrix blocks play the role of a global source distributed over the entire medium. A source term is included in flow equations in the fracture system, which representing the exchange of fluid between matrix blocks and the fractures. Similar works were also studied by some other authors. For miscible displacement in non-fractured media case, the existence and uniqueness of the solution are well-known, see [14, 21] and references therein. For fractured media case, if dispersion tensors in fracture planes and matrix blocks are uniformly elliptic, then macroscopic equations are derived and its convergence proofs for some special cases are given in [4, 5, 22]. The problem considered here is for non-uniformly elliptic dispersion tensor case and our intention is to study the convergence of the microscopic models. To this end, we derive some kind of uniform estimates for elliptic equations and for parabolic equations in fractured media, and then use two-scale technique to show the convergence. The definition of two-scale method and its applications to a diffusion process in highly heterogeneous media can be found in [3, 8, 9, 11].

Rest of the paper is organized as follows: In §2, we state a microscopic model for two-component, miscible flows in fractured media. Notation and assumption will be given in §3. Then in §4, we present the main result. We shall recall some known results related to this problem in §5. Proof of the main result is given in §6. The proof in §6 requires the existence of smooth solutions, which is proved in §7.

2. A Microscopic Model with Moderate-Sized Matrix Blocks

We consider a porous medium Ω, which is a two-connected domain with a periodic structure. Let $Y \equiv [0,1]^3$ be a cell consisting of a matrix block domain Y_m completely surrounded by a connected fracture domain Y_f, and denote by Γ the matrix-fracture interface in the cell Y. Let $\mathcal{X}(y)$ be the characteristic function of Y_m, extended Y-periodically to $\Re^3$. The medium Ω contains two subdomains, Ω_f^ϵ and Ω_m^ϵ, representing the system of fracture planes and matrix blocks respectively, and $\Omega_m^\epsilon \subset \{x \in \Omega | \mathcal{X}(\frac{x}{\epsilon}) = 1\}$, $\Omega_f^\epsilon = \Omega \setminus \Omega_m^\epsilon$. Let $\Gamma^\epsilon \equiv \partial\Omega_f^\epsilon \cap \partial\Omega_m^\epsilon \cap \Omega$ be that part of the interface of Ω_m^ϵ with Ω_f^ϵ that is interior to Ω.

The effects of turbulent mechanical mixing of non-stationary, incompressible, two-component, miscible displacement in fracture subdomain Ω_f^ϵ can be described as [6, 23, 25, 26], in Ω_f^ϵ, $t > 0$,

$$\partial_t \Theta^\epsilon - \nabla \cdot \big(\mathbf{D}(\Theta^\epsilon, \mathbf{V}^\epsilon)\nabla\Theta^\epsilon - \mathbf{V}^\epsilon\Theta^\epsilon\big) - \mathbf{Q}^\epsilon\Theta^\epsilon = 0, \tag{2.1}$$

$$\mathbf{V}^\epsilon = \frac{-K^\epsilon}{\mu(\Theta^\epsilon)}(\nabla P^\epsilon - \rho(\Theta^\epsilon, x)), \tag{2.2}$$

$$\nabla \cdot \mathbf{V}^\epsilon = \mathbf{Q}^\epsilon. \tag{2.3}$$

$\Theta^\epsilon \in [0,1]$ stands for invading fluid concentration, $\mathbf{V}^\epsilon$ for the net fluid velocity, K^ϵ for absolute permeability, $\mu(\Theta^\epsilon)$ for the fluid viscosity, P^ϵ for the pressure, $\rho(\Theta^\epsilon, x)$ for a function depending on concentration, density, gravity, and position, $\mathbf{D}(\Theta^\epsilon, \mathbf{V}^\epsilon)$ for dispersion tensor, and $\mathbf{Q}^\epsilon$ for the external source. Porosity in (2.1) is neglected for simplicity of presentation. In this work,

$$\mathbf{D}(\Theta, \mathbf{V}) \equiv \Lambda(\Theta)I + d_\ell|\mathbf{V}|\widetilde{T}(\mathbf{V}) + d_t|\mathbf{V}|(I - \widetilde{T}(\mathbf{V})),$$

where Λ is a strictly positive continuous function, I is the identity matrix, $\widetilde{T}$ is a tensor with (i,j) component $\widetilde{T}_{i,j}(\mathbf{V}) \equiv \frac{\mathbf{V}_i\mathbf{V}_j}{|\mathbf{V}|^2}$, and d_ℓ, d_t are positive constants [6, 14, 21, 23, 25]. In Ω_m^ϵ, $t > 0$, the governing equations are

$$\partial_t\theta^\epsilon - \epsilon\nabla\cdot\big(\mathbf{D}(\theta^\epsilon, \mathbf{v}^\epsilon)\epsilon\nabla\theta^\epsilon - \mathbf{v}^\epsilon\theta^\epsilon\big) - \mathbf{q}^\epsilon\theta^\epsilon = 0, \tag{2.4}$$

$$\mathbf{v}^\epsilon = \frac{-k^\epsilon}{\mu(\theta^\epsilon)}(\epsilon\nabla p^\epsilon - \rho(\theta^\epsilon, x)), \tag{2.5}$$

$$\epsilon\nabla\cdot\mathbf{v}^\epsilon = \mathbf{q}^\epsilon. \tag{2.6}$$

θ^ϵ, $\mathbf{v}^\epsilon$, k^ϵ, $\mu(\theta^\epsilon)$, p^ϵ, $\rho(\theta^\epsilon, x)$, $\mathbf{D}(\theta^\epsilon, \mathbf{v}^\epsilon)$, $\mathbf{q}^\epsilon$ in the matrix subdomain Ω_m^ϵ are used to represent the same quantities as those denoted by upper case symbols in the fracture subdomain. Fluxes, pressure, and concentration are continuous on interface Γ^ϵ, $t > 0$,

$$\begin{cases} \mathbf{D}(\Theta^\epsilon, \mathbf{V}^\epsilon)\nabla\Theta^\epsilon\cdot\vec{\nu}^\epsilon = \epsilon^2\mathbf{D}(\theta^\epsilon, \mathbf{v}^\epsilon)\nabla\theta^\epsilon\cdot\vec{\nu}^\epsilon, \\ \mathbf{V}^\epsilon\cdot\vec{\nu}^\epsilon = \epsilon\mathbf{v}^\epsilon\cdot\vec{\nu}^\epsilon, \\ \Theta^\epsilon = \theta^\epsilon, \\ P^\epsilon = p^\epsilon, \end{cases} \tag{2.7}$$

where $\vec{\nu}^\epsilon$ is the unit vector outward normal to Γ^ϵ. Boundary conditions are

$$\begin{cases} \mathbf{D}(\Theta^\epsilon, \mathbf{V}^\epsilon)\nabla\Theta^\epsilon\cdot\vec{n}|_{\partial\Omega} = 0, \\ \mathbf{V}^\epsilon\cdot\vec{n}|_{\partial\Omega} = 0, \end{cases} \tag{2.8}$$

where $\vec{n}$ is the unit vector outward normal to $\partial\Omega$. Initial conditions are

$$\begin{cases} \Theta^\epsilon(x,0) = \Theta_0^\epsilon(x) & \text{in } \Omega_f^\epsilon, \\ \theta^\epsilon(x,0) = \theta_0^\epsilon(x) & \text{in } \Omega_m^\epsilon. \end{cases} \tag{2.9}$$

3. Notation and Assumption

$\Omega(2\epsilon) \equiv \{x \in \Omega : dist(x, \partial\Omega) > 2\epsilon\}$, $\Omega_m^\epsilon \equiv \{z : z \in \epsilon(Y_m + j) \subset \Omega(2\epsilon) \text{ for } j \in Z^3\}$, $\Omega_f^\epsilon \equiv \Omega \setminus \Omega_m^\epsilon$, and $\Omega^\epsilon \equiv \{z : z \in \epsilon(Y + j), \epsilon(Y_m + j) \subset \Omega(2\epsilon) \text{ for some } j \in Z^3\}$. $\mathcal{Q}_m^\epsilon \equiv \Omega^\epsilon \times Y_m$, $\mathcal{Q} \equiv \Omega \times Y$, $\mathcal{Q}_i \equiv \Omega \times Y_i, i = f, m$. $\mathcal{B}^t \equiv \mathcal{B} \times (0,t)$ if $\mathcal{B} = \partial\Omega, \Gamma^\epsilon, \Omega, \Omega_i^\epsilon, \mathcal{Q}_m^\epsilon, \mathcal{Q}, \mathcal{Q}_i, i = m, f$.

We shall use the notations in Chapter I [17] for Hölder spaces and Sobolev spaces. Let $\mathcal{B}$ be a space domain in $\Re^3$. $C(\mathcal{B})$ (resp. $C(\mathcal{B}^T)$) is the set of continuous functions in $\mathcal{B}$ (resp. $\mathcal{B}^T$). $C^\infty(\mathcal{B})$ (resp. $C^\infty(\mathcal{B}^T)$) contains infinitely differentiable functions in $\mathcal{B}$ (resp. $\mathcal{B}^T$). $C_0^\infty(\mathcal{B})$ (resp. $C_0^\infty(\mathcal{B}^T)$) is the subset of $C^\infty(\mathcal{B})$ (resp. $C^\infty(\mathcal{B}^T)$) with compact support in $\mathcal{B}$ (resp. $\mathcal{B}^T$). For $j \in N$ and $r \in [1,\infty]$, $L_r(\mathcal{B})$ (resp. $L_r(\mathcal{B}^T)$, $W_r^j(\mathcal{B})$) denotes a Sobolev space with norm $\|\psi\|_{r,\mathcal{B}}$ (resp. $\|\psi\|_{r,\mathcal{B}^T}$, $\|\psi\|_{r,\mathcal{B}}^{(j)}$). $W_{r,0}^j(\mathcal{B})$ is the closure of $C_0^\infty(\mathcal{B})$ in space $W_r^j(\mathcal{B})$. $W_r^{-j}(\mathcal{B})$ is the dual space of $W_{r,0}^j(\mathcal{B})$. $W_r^{j,0}(\Omega^T)$ (resp. $W_2^{1,1}(\Omega^T)$) is the Hilbert space with norm $\|\psi\|_{W_r^{j,0}(\Omega^T)} = \sum_{i=0}^j \|D_x^i\psi\|_{r,\Omega^T}$ (resp. $\|\psi\|_{W_2^{1,1}(\Omega^T)} = \sum_{i=0}^1 \|D_x^i\psi\|_{2,\Omega^T} + \|D_t\psi\|_{2,\Omega^T}$). For a nonintegral positive number α, $H^\alpha(\mathcal{B})$ (resp. $H^{2\alpha,\alpha}(\mathcal{B}^T)$) is a Hölder space with norm $\|\psi\|_{\mathcal{B}}^{(\alpha)}$ (resp. $\|\psi\|_{\mathcal{B}^T}^{(2\alpha)}$). $H_0^\alpha(\mathcal{B})$ (resp. $H_0^{\alpha,\alpha/2}(\mathcal{B}^T)$) is the closure of $C_0^\infty(\mathcal{B})$ (resp. $C_0^\infty(\mathcal{B}^T)$) in space $H^\alpha(\mathcal{B})$ (resp. $H^{\alpha,\alpha/2}(\mathcal{B}^T)$).

$\psi(y) \in L_{r,per}(Y)$ coincides with a function in $L_r(Y)$ extended by Y-periodicity to $\Re^3$. For $\mathcal{B} = Y_f, Y_m$, we define $L_{r,per}(\mathcal{B}) \equiv \{\psi \in L_{r,per}(Y) : \psi(y) = 0 \text{ if } y \in Y \setminus \mathcal{B}\}$. $C_{per}^\infty(Y)$ is the subset of $C^\infty(\Re^3)$ of Y-periodic functions. $W_{r,per}^j(Y)$ is the closure of $C_{per}^\infty(Y)$ under W_r^j norm. $W_{r,per}^j(Y_f) = W_{r,per}^j(Y) \cap L_{r,per}(Y_f)$. If $\hat{B}$ is a Banach space with norm $\|\cdot\|_{\hat{B}}$, $L_r(\mathcal{B};\hat{B})$ for $r \in [1,\infty]$ is the space of measurable functions $\psi : \mathcal{B} \to \hat{B}$ such that $\|\psi(\cdot)\|_{\hat{B}}$ belongs to $L_r(\mathcal{B};\Re)$. $C^\infty(\mathcal{B};\hat{B}) \subset L_1(\mathcal{B};\hat{B})$ contains infinitely differentiable functions in $\mathcal{B}$. $C_0^\infty(\mathcal{B};\hat{B}) \subset C^\infty(\mathcal{B};\hat{B})$ contains functions with compact support in $\mathcal{B}$. $W_r^1(\mathcal{B};\hat{B})$ is the closure of $C^\infty(\mathcal{B};\hat{B})$ under W_r^1 norm.

Time difference is defined to be $\partial^{-\delta}\psi(t) \equiv \frac{\psi(t)-\psi(t-\delta)}{\delta}$. $\mathcal{X}_{\mathcal{B}}(x)$ is a characteristic function of $\mathcal{B}$. For $\alpha \in (0,1)$ and $j \in \{0,1,2\}$,

$$\begin{cases} \|\!|g|\!\|_{\mathcal{B}}^{j+\alpha,*} \equiv \|g(\epsilon x)\|_{\frac{1}{\epsilon}\mathcal{B}}^{(j+\alpha)}, \\ \|\!|g|\!\|_{\mathcal{B}^T}^{j+\alpha,*} \equiv \|g(\epsilon x,t)\|_{\frac{1}{\epsilon}\mathcal{B}\times(0,T)}^{(j+\alpha)}. \end{cases} \tag{3.1}$$

Next let us make the following assumptions:

A1. $\Omega, Y_m \subset \Re^3$ are bounded, smooth, and connected domains,

A2. $\|\!|K^\epsilon|\!\|_\Omega^{\alpha,*}$ is bounded independently of ϵ and K^ϵ is convergent in $L^2(\Omega)$,

A3. k is a smooth Y-periodic function, $k^\epsilon = k(\frac{x}{\epsilon})$, $k \in [\gamma_1,\gamma_2]$,

A4. $\|\mathbf{Q}^\epsilon\|_{2,\Omega_f^{\epsilon,T}}$, $\|\mathbf{q}^\epsilon\|_{2,\Omega_m^{\epsilon,T}}$ are Cauchy sequences, $\|\!|\mathbf{Q}^\epsilon|\!\|_{\Omega_f^{\epsilon,T}}^{\alpha,*} + \|\!|\mathbf{q}^\epsilon|\!\|_{\Omega_m^{\epsilon,T}}^{\alpha,*}$ is bounded independently of ϵ, and $\int_{\Omega_f^{\epsilon,T}} \mathbf{Q}^\epsilon + \int_{\Omega_m^{\epsilon,T}} \mathbf{q}^\epsilon = 0$,

A5. $\Theta_0^\epsilon, \theta_0^\epsilon \in [0,1]$, $\Theta_0^\epsilon \mathcal{X}_{\Omega_f^\epsilon} + \theta_0^\epsilon \mathcal{X}_{\Omega_m^\epsilon} \in C^1(\Omega)$, $\|\!|\Theta_0^\epsilon|\!\|_{\Omega_f^\epsilon}^{2+\alpha,*} + \|\!|\theta_0^\epsilon|\!\|_{\Omega_m^\epsilon}^{2+\alpha,*}$ is bounded independently of ϵ, $\|\theta_0^\epsilon\|_{2,\Omega_m^\epsilon}$ is a convergent sequence, and compatibility condition of order 0 holds (see p. 319 [17]),

A6. $\max_{z\in[0,1]} |\frac{K^\epsilon}{\mu(z)} - \widehat{K}| \le \gamma_3 \widehat{K}$, $\max_{z\in[s_l,s_r]} |\frac{k^\epsilon}{\mu(z)} - \widehat{k}| \le \gamma_3 \widehat{k}$,

A7. $\mu, \Lambda \ge \gamma_1$, $|\frac{\partial\rho}{\partial\theta}| + |\mu'| + |\Lambda'| \le \gamma_4$, $\mathbf{D} - \Lambda I$ is a nonnegative symmetric matrix, and $\|\rho(\theta,\cdot)\|_\Omega^{(\alpha)} + \|\rho(\theta,\cdot)\|_{2,\Omega}^{(1)}$ is bounded for any fixed $\theta \in [0,1]$, where $\alpha \in (0,1), \widehat{K}, \widehat{k}, \ \gamma_i, i = 1,2,3,4$ are positive constants and γ_3, γ_4 are small numbers depending on given data.

Remark 3.1 *By A2,6, we know K^ϵ are bounded below and above by two positive constants independent of ϵ. γ_3, γ_4 in A6,7 are small numbers, and the explicit restriction for γ_3 can be found in p.224 Theorem 4.2 [7] and the restriction for γ_4 is in the proof below. In [21], it was shown that there exists a unique smooth solution for miscible displacement in single porosity case provided that μ' is not too large and data are small. Here we also require that μ' is not too large (see A7) in order to show the smoothness of solutions in fractured media. By A7, it is easy to see $\mathbf{D} \in W^{1,\infty}_{loc}(\Re^4)$.*

4. The Main Result

We claim as ϵ goes to 0, the equations (2.1–2.9) for the microscopic model converge to a dual-porosity model. As mentioned in §1, in this macroscopic model domain acts as a porous medium consisting of two superimposed continua, a continuous fracture system Ω and a discontinuous system of matrix blocks $\mathcal{Q}_m$. The macroscopic model includes two systems of equations; one is for flow in fracture system and the other is for flow in matrix block system. The two systems are coupled through a nonlinear source.

Let $\Omega \subset \Re^3$ be a fractured medium. Equations for the fracture system are, for $x \in \Omega, y \in Y_f, t > 0$,

$$\partial_t \Theta - \nabla \cdot \int_{Y_f} \frac{\mathbf{D}(\Theta, \mathbf{V})(\nabla\Theta + \partial_y \Theta_1) - \Theta \mathbf{V}}{|Y_f|} dy - \Theta \int_{Y_f} \frac{\mathbf{Q}}{|Y_f|} dy = \mathcal{J}, \tag{4.1}$$

$$\mathbf{V} = \frac{-K}{\mu(\Theta)}(\nabla P - \rho(\Theta, x)), \tag{4.2}$$

$$\nabla \cdot \int_{Y_f} \mathbf{V} dy = \int_{Y_f} \mathbf{Q} \, dy + \int_{Y_m} \mathbf{q} \, dy, \tag{4.3}$$

where $\Theta, P, \mathcal{J}$ are functions in Ω^T, functions $\Theta_1, \mathbf{V}, K, \mathbf{Q}$ are in $\mathcal{Q}_f^T$, and $\mathbf{q}$ are in $\mathcal{Q}_m^T$. Θ is fluid concentration and Θ_1 is a Y-periodic function satisfying

$$\begin{cases} \partial_y \cdot (\mathbf{D}(\Theta, \mathbf{V}) \partial_y \Theta_1) = -\partial_y \cdot (\mathbf{D}(\Theta, \mathbf{V}) \nabla\Theta - \Theta \mathbf{V}) & \text{in } Y_f, \\ \mathbf{D}(\Theta, \mathbf{V}) \partial_y \Theta_1 \cdot \vec{\nu}_y = -(\mathbf{D}(\Theta, \mathbf{V}) \nabla\Theta - \Theta \mathbf{V}) \cdot \vec{\nu}_y & \text{on } \partial Y_m, \end{cases} \tag{4.4}$$

where $\vec{\nu}_y$ is the unit vector outward normal to ∂Y_m. $\mathbf{V}$ is fluid velocity, K is absolute permeability field, $\mu(\Theta)$ is viscosity, P denotes pressure, $\rho(\Theta, x)$ is a function depending on concentration, density, gravity, and position, $\mathbf{D}$ is the dispersion tensor, $\mathbf{Q}$ is the external source in fractured system, $\mathbf{q}$ is the external source in matrix block system, and $\mathcal{J}$ is the matrix-fracture source due to matrix blocks. $|Y_f|$ denotes the volume of Y_f.

Above each point $x \in \Omega$ is suspended topologically a matrix block $Y_m \subset \Re^3$. Equations for flow in each matrix block are, for $x \in \Omega, \ y \in Y_m, \ t > 0$,

$$\partial_t \theta - \partial_y \cdot (\mathbf{D}(\theta, \mathbf{v}) \partial_y \theta - \mathbf{v}\theta) - \mathbf{q}\theta = 0, \tag{4.5}$$

$$\mathbf{v} = \frac{-k}{\mu(\theta)}(\partial_y p - \rho(\theta, x)), \tag{4.6}$$

$$\partial_y \cdot \mathbf{v} = \mathbf{q}. \tag{4.7}$$

Lower case symbols denote the same quantities on Y_m corresponding to those denoted by upper case symbols in the fracture system equations.

The matrix-fracture source is given by, for $x \in \Omega$, $t > 0$,

$$\mathcal{J}(x,t) = \frac{-1}{|Y_f|}\int_{Y_m}(\partial_t\theta - \theta\mathbf{q})(x,y,t)dy. \tag{4.8}$$

The boundary conditions for the fracture system are, for $t > 0$,

$$\begin{cases} \left(\int_{Y_f}\mathbf{D}(\Theta,\mathbf{V})\left(\nabla\Theta + \partial_y\Theta_1\right)dy\right)\cdot\vec{n} = 0 & \text{on } \partial\Omega, \\ \left(\int_{Y_f}\mathbf{V}dy\right)\cdot\vec{n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.9}$$

where $\vec{n}$ is the unit vector outward normal to $\partial\Omega$. On interface, concentration and pressure are continuous, that is, for $x \in \Omega$, $y \in \partial Y_m$, $t > 0$,

$$\begin{cases} \theta(x,y,t) = \Theta(x,t), \\ p(x,y,t) = P(x,t). \end{cases} \tag{4.10}$$

Initial conditions are

$$\begin{cases} \Theta(x,0) = \Theta_0(x) & \text{for } x \in \Omega, \\ \theta(x,y,0) = \theta_0(x,y) & \text{for } x \in \Omega, y \in Y_m. \end{cases} \tag{4.11}$$

Theorem 4.1 *Under $A1-7$, a subsequence of solutions of the microscopic models (2.1–2.9) converges in two-scale sense to a solution of (4.1–4.11).*

Acknowledgement

The author is supported by the grant number NSC 94-2115-M-009-008 from the research program of National Science Council.

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