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Project: Metric Diophantine Approximation for Formal Laurent Series

by

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Abstract

In a recent paper, Deligero and Nakada devised a new approach to investigate the number of coprime solutions of the diophantine approximation problem for formal Laurent series over a finite base field. In this project, we further developed their method and proved almost sure and distributional type invariance principles. One of the consequences of our results is a generalization of the analogue of a famous theorem of Khintchine. Despite a lot of previous research efforts, the corresponding result in the classical case has not been established yet.

1 General

This is the final report on the National Science Council project "Metric Diophantine Approximation for Formal Laurent Series" with grant number NSC-94-2115-M-009-011 and term from August 1st, 2005 to July 31st, 2006.

Before going into details, we shortly summarize the main achievements.

- The paper [1] contains the main findings of this project. It was submitted and will be published in one of the forthcoming issues of *Finite Fields and Their Applications*.
- A three week research visit at the Vienna University of Technology from July 7, 2006 to July 29, 2006 was partially financed by the project. A report on this research visit will be handled in separately.

2 Results

Subsequently, we shortly describe some of our main results and their significance. For details, we refer the reader to [1].

Let $\mathbb{F}_q((T^{-1}))$ denote the field of formal Laurent series over a finite field $\mathbb{F}_q$ with q elements equipped with the usual standard evaluation. Fix an $f \in \mathbb{F}_q((T^{-1}))$ with $|f| < 1$. We are interested in the number of solutions $P, Q \in \mathbb{F}_q[T], Q \neq 0$ of the diophantine approximation problem

$$\left| f - \frac{P}{Q} \right| < \frac{1}{q^{2n+l_n}}, \qquad \deg Q = n, \qquad (P,Q) = 1, \tag{1}$$

where (l_n) is a sequence of positive integers. Subsequently, we denote by m the unique, translation-invariant probability measure on the measure space consisting of

$$\{f \in \mathbb{F}_q((T^{-1})) | |f| < 1\}$$

and its set of Borel sets.

The following result was proved by deMathan in [3] (see [4] for a different method of proof) and is the analogue of a famous theorem of Khintchine.

Theorem 1 (deMathan 1970; Fuchs 2002). *Assume that (l_n) is non-decreasing. Then, (1) has either a finite or an infinite number of solutions almost surely; the latter holds if and only if*

$$\sum_{n=1}^{\infty} q^{-l_n} = \infty.$$

In order to refine the above result, the following sequence of random variables was introduced

$$Z_N(f) := \#\left\{P/Q : \langle P, Q\rangle \text{ is a solution of (1)},\ \deg Q \le N\right\}.$$

Then, the following extension of the above theorem was implicitly proved by Inoue and Nakada in [7].

Theorem 2. *Assume that (l_n) is non-decreasing. Then, for almost all f,*

$$Z_N(f) = \left(1 - q^{-1}\right) \sum_{n \le N} q^{-l_n} + \mathcal{O}\left(F(N)^{1/2} (\log F(N))^{3/2+\epsilon}\right),$$

where $\epsilon > 0$ is arbitrary and $F(N)$ is a suitable defined sequence.

A similar result (with a slightly weaker error term) was long known to hold for the classical case (=metric diophantine approximation over the real number field). However, despite a lot of research efforts, the question of the optimal error term remained open (only under further restrictions on (l_n), it was possibly to prove more; see [5] and references therein).

One of the consequences of our present study is now the following result that gives a definite answer to the question of the optimal error term for diophantine approximation in the field of formal Laurent series.

Theorem 3 (Deligero, Fuchs, and Nakada 2006). *Assume that (l_n) is non-decreasing. Then, for almost all f,*

$$\limsup_{n\to\infty} \frac{\left|Z_N(f) - (1 - q^{-1}) \sum_{n \le N} q^{-l_n}\right|}{\sqrt{2F(N) \log\log F(N)}} = 1.$$

This result follows from the invariance principles we proved in [1]. More generally, the results in [1] yield as consequences a functional law of the iterated logarithm and a functional central limit theorem for the number of solutions of (1) for non-decreasing sequences (l_n) (the latter improves upon the main result in Deligero and Nakada [2]). For the detailed results, we direct the interested reader to [1]. We just want to make a few comments on their significance. The invariance principles in [1] improve upon our previous results in [6]. The substantial improvement is achieved via a new approach that has its origin in [2]. All previous approaches relied on metric results for the continued fraction expansion of formal Laurent series which made necessary some technical restrictions on the set of sequences (l_n). In [2], a new approach not relying on continued fraction expansion was devised. Moreover, its power was demonstrated by proving a central limit theorem for the number of solutions of the diophantine approximation problem in the setting of Khintchine's theorem. In our paper, this approach is now further developed by improving all the results in [2] as well as most of the results in [6].

Another natural and interesting question is whether or not one can even go one step further by getting rid of the the monotonicity assumption on (l_n). In fact, Inoue and Nakade proved in [2] that Theorem 1 and Theorem 2 still hold without the assumption of monotonicity. As to our Theorem 3, a proof without this assumption seems to be complicated. However, we managed to prove that the result still holds for (more or less) all convergent sequences (either with finite or infinite limit). This set of sequences clearly contains the set of sequences that is considered in Theorem 3. More generally, all results in [1] are proved for this larger set of sequences, too. Again, we do not want to go into too much details and instead refer the reader to [1].

3 Summary

In this project we further developed the method introduced in [2] and proved invariance principles for the number of coprime solutions of the diophantine approximation problem for formal Laurent series over a finite base field. Our results improve upon most of the previous results in this field. Moreover, as one of the consequences of our main findings, a long sought goal that is still an open question in the classical case was finally achieved for diophantine approximation in the field of formal Laurent series. This once more demonstrates that metric diophantine approximation over the field of formal Laurent series is easier than its counterpart over the real number field.

Many questions remain open. For instance do our results still hold when no conditions on the sequence (l_n) are imposed? How about the situation where all solutions of the diophantine approximation problem are counted (compared with the above restriction to counting coprime solutions)? How about the situation where the set of solutions is further restricted to e.g. solutions with denominator and/or enumerator belonging to a fixed arithmetic progression, etc.? All those questions are interesting and might constitute the topic of a forthcoming project.

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Invariance Principles for Diophantine Approximation of Formal Laurent Series over A Finite Base Field

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Abstract

In a recent paper, the first and third author proved a central limit theorem for the number of coprime solutions of the diophantine approximation problem for formal Laurent series in the setting of the classical theorem of Khintchine. In this note, we consider a more general setting and show that even an invariance principle holds, thereby improving upon earlier work of the second author. Our result yields two consequences: (i) the functional central limit theorem and (ii) the functional law of the iterated logarithm. The latter is a refinement of Khintchine's theorem for formal Laurent series. Despite a lot of research efforts, the corresponding results for diophantine approximation of real numbers have not been established yet.

1 Introduction

The last few years have witnessed an increasing interest in the metric theory of diophantine approximation for formal Laurent series; for recent results concerning limit laws see Deligero and Nakada [1], Fuchs [3], [5], Inoue and Nakada [6]; for recent results concerning Hausdorff dimensions of exceptional sets see Kristensen [7], Niederreiter and Vielhaber [12], Wu [15].

In this short note, we are studying invariance principles for the number of coprime solutions of the diophantine approximation problem. In the classical case, invariance principles were obtained by Fuchs in [4]; see Fuchs [5] for corresponding results for formal Laurent series. The main difference to the previous

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line of research is a new approach that does not involve continued fraction expansion. Continued fraction expansion made necessary several restrictions on earlier results which will be shown to be superfluous in this paper. This new approach was devised by Deligero and Nakada in [1] and it is the paper's aim to further demonstrate its usefulness.

We give a short outline of the paper: in this section, we briefly recall metric diophantine approximation for formal Laurent series, state our new result and discuss some consequences. The proof of the main result which rests on blocking techniques and a general invariance principle obtained by Fuchs [4] will then be given in the final two sections.

Formal Laurent Series. Denote by $\mathbb{F}_q$ the finite field with q elements, where q is a power of p, p a prime. We consider the field of formal Laurent series

$$\mathbb{F}_q((T^{-1})) = \left\{ f = \sum_{n=n_0}^{\infty} a_n T^{-n} \middle| a_n \in \mathbb{F}_q, n_0 \in \mathbb{Z}, a_{n_0} \neq 0 \right\} \cup \{0\}$$

together with the valuation $|f| = q^{-n_0}$, $f \neq 0$ and $|0| = 0$. It is easy to see that $|\cdot|$ is non-Archimedean and that the polynomial ring $\mathbb{F}_q[T]$ and the field of rational functions $\mathbb{F}_q(T)$ are both contained in $\mathbb{F}_q((T^{-1}))$, where we have the chain of inclusions $\mathbb{F}_q[T] \subseteq \mathbb{F}_q(T) \subseteq \mathbb{F}_q((T^{-1}))$, a situation that closely resembles the corresponding chain $\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$.

In order to consider metric diophantine approximation, we restrict to the set

$$\mathbb{L} = \left\{ f \in \mathbb{F}_q((T^{-1})) \middle| |f| < 1 \right\}$$

as we restrict to the unit interval in the classical case. It is straightforward to prove that $\mathbb{L}$ together with the restriction of the valuation is a compact metric space. Hence, there exists a unique, translation-invariant probability measure on $(\mathbb{L}, \mathcal{L})$ ($\mathcal{L}$ denoting the set of all Borel sets) that we are going to denote by m.

Diophantine Approximation Problem and Three Sets. For f a formal Laurent series with $|f| < 1$, consider the diophantine approximation problem in unknowns $P, Q \in \mathbb{F}_q[T], Q \neq 0$,

$$\left| f - \frac{P}{Q} \right| < \frac{1}{q^{2n+l_n}}, \qquad \deg Q = n, \qquad (P,Q) = 1, \tag{1}$$

where (l_n) is a sequence of positive integers.

We are interested in studying the solution set. Results of different strengths made necessary different restrictions on the set of sequences (l_n). The sets which will be considered in this paper are as follows:

$\mathcal{A} = \{(l_n)_{n\geq 0} \mid l_n > 0 \text{ and non-decreasing}\}$;

$\mathcal{B} = \left\{(l_n)_{n\geq 0} \mid l_n > 0 \text{ and either (C1) } \lim_{n\to\infty} l_n = l < \infty, \text{ or (C2) } \lim_{n\to\infty} l_n = \infty, \lim_{i\to\infty} \sum_{i<j\leq i+l_i} q^{-l_j} \text{ exists}\right\}$;

$\mathcal{C} = \{(l_n)_{n\geq 0} \mid l_n > 0\}$.

Note that we have the following chain of proper inclusions $\mathcal{A} \subset \mathcal{B} \subset \mathcal{C}$.

0-1 Laws. In [2], deMathan proved an analogue of Khintchine's theorem: for $(l_n) \in \mathcal{A}$ the solution set of the above inequality is either finite or infinite for almost all f, the latter holding if and only if $\sum_{n=0}^{\infty} q^{-l_n} = \infty$ (see Fuchs [3] for a different approach based on continued fraction expansions).

In a recent paper, Inoue and Nakada [6] showed that the monotonicity assumption is in fact superfluous (see Section 2 for a simplified proof of their result).

Theorem 1 (Inoue and Nakada [6]). *Let $(l_n) \in \mathcal{C}$. (1) has either finitely many or infinitely many solutions for almost all f; the latter holds if and only if*

$$\sum_{n=0}^{\infty} q^{-l_n} = \infty.$$

Central Limit Theorems. Define a sequence of random variables as

$$Z_N(f) := \#\left\{P/Q : \langle P, Q\rangle \text{ is a solution of (1)}, \ \deg Q \le N\right\}.$$

Assuming that $(l_n) \in \mathcal{A}$, $\sum_{n=0}^{\infty} q^{-l_n} = \infty$ and under some further technical conditions on (l_n), Fuchs [3] proved the central limit theorem for (Z_N). His approach was based on continued fraction expansions which made the additional conditions seemingly hard to drop.

A new approach, not relying on continued fraction expansions, was devised by Deligero and Nakada in [1]. With this approach they succeeded in dropping the additional conditions in Fuchs's result, thereby generalizing the central limit theorem to Khintchine's setting, i.e., to all sequences $(l_n) \in \mathcal{A}$ with $\sum_{n=0}^{\infty} q^{-l_n} = \infty$. Note that a similar result for the real number field has not been proved yet; see LeVeque [9], [10] and Philipp [13] for similar but weaker results in the real case.

The Invariance Principle. In [5], Fuchs obtained the invariance principle for sequences $(l_n) \in \mathcal{A}$ that satisfy $\sum_{n=0}^{\infty} q^{-l_n} = \infty$ and some technical extra conditions. Here, we are going to explore further the approach of Deligero and Nakada in order to extend Fuchs's result to all sequences $(l_n) \in \mathcal{B}$ with $\sum_{n=0}^{\infty} q^{-l_n} = \infty$.

In order to state the result we fix some notation. Set

$$F(N) := \begin{cases} q^{-2l-2}\left(q^{l+1}(q-1) - (2l+1)(q-1)^2\right)N, & \text{if (C1);} \\ q^{-1}(q-1)\sum_{n\le N} q^{-l_n}, & \text{if (C2),} \end{cases}$$

and

$$N_t := \begin{cases} \max\{n : F(n) \le t\}, & \text{if } t \ge F(0); \\ 0, & \text{otherwise,} \end{cases}$$

for $t \ge 0$. Define on $(\mathbb{L}, \mathcal{L}, m) \times ([0,1], \bar{\mathcal{B}}, \lambda)$ the following stochastic process

$$Z(t) := Z(t; f, x) := Z_{N_t}(f) - \left(1 - \frac{1}{q}\right)\sum_{n=0}^{N} q^{-l_n},$$

where $\bar{\mathcal{B}}$ denotes the set of Borel sets on $[0,1]$ and λ is the Lebesgue measure. Note that the definition does not depend on the second variable. However, adjoining a uniformly distributed random variable is necessary to guaranteeing that the probability space is rich enough (see Remark 6 in Fuchs [4]).

Theorem 2. *There exists a sequence $(Y_n)_{n\ge 0}$ of independent, standard normal random variables on $(\mathbb{L}, \mathcal{L}, m) \times ([0,1], \bar{\mathcal{B}}, \lambda)$ such that, as $N \to \infty$,*

$$\left| Z(N) - \sum_{n\le N} Y_n \right| = o\left((N \log\log N)^{1/2}\right), \quad \textit{a.s.}$$

and

$$(m \times \lambda)\left[\frac{1}{\sqrt{N}} \max_{n\le N} \left| Z(n) - \sum_{k\le n} Y_k \right| \ge \epsilon \right] \to 0$$

for all $\epsilon > 0$.

Consequences. The above result implies the functional central limit theorem which generalizes the result of Deligero and Nakada [1].

Corollary 1. *As* $N \to \infty$*,*

$$\left\{ \frac{Z(F(N)t)}{\sqrt{F(N)}}, 0 \le t \le 1 \right\} \longrightarrow \{W(t), 0 \le t \le 1\},$$

where $W(t)$ *denotes the standard Brownian motion.*

Moreover, we have the functional law of the iterated logarithm.

Corollary 2. *The sequence of functions*

$$\left\{ \frac{Z(F(N)t)}{(2F(N)\log\log F(N))^{1/2}},\ 0 \le t \le 1 \right\}_{N \ge 0}$$

is a.s. relatively compact in the topology of uniform convergence and has Strassen's set as its set of limit points.

Since our set of sequences (l_n) contains the sequences of Khintchine's theorem, we note the following consequence of the latter result which is a refinement of Khintchine's theorem for formal Laurent series.

Corollary 3 (Law of the iterated logarithm for Khintchine's setting). *Assume that* $(l_n) \in \mathcal{A}$ *and* $\sum_{n=0}^{\infty} q^{-l_n} = \infty$*. Then, for almost all* f*,*

$$\limsup_{n \to \infty} \frac{\left| Z_N(f) - (1 - q^{-1}) \sum_{n \le N} q^{-l_n} \right|}{\sqrt{2F(N) \log\log F(N)}} = 1.$$

Note that a similar result for the real number field has so far not been established; see Philipp [13] and Fuchs [4] for similar but weaker results in the real case. Moreover note that the above result also gives the optimal bound in the law of large numbers:

Let $(l_n) \in \mathcal{A}$*. Then, for almost all* f*,*

$$Z_N(f) = \left(1 - q^{-1}\right) \sum_{n \le N} q^{-l_n} + \mathcal{O}\left((F(N)\log\log F(N))^{1/2}\right).$$

The previous best bound was of order $F(N)^{1/2}(\log F(N))^{3/2+\epsilon}, \epsilon > 0$ which more generally even holds for all $(l_n) \in \mathcal{C}$; see a remark by Inoue and Nakada [6].

2 Blocking

Define a sequence of sets as

$$F_n := \{f \in \mathbb{L} : \exists \langle P, Q \rangle \text{ such that (1) holds}\}.$$

The measure of these sets was computed by Inoue and Nakada [6],

$$m(F_n) = q^{-l_n}\left(1 - \frac{1}{q}\right). \tag{2}$$

Moreover, as was proved by Inoue and Nakada [6] as well, two distinct sets F_i and F_j are either independent or have empty intersection, the first case occurring if and only if $i + l_i < j$.

Note that the latter implies

$$m(F_i \cap F_j) \leq m(F_i)m(F_j) \qquad (i \neq j). \tag{3}$$

Sequences of sets satisfying this condition are called negative quadrant dependent (see Lehmann [8]). This gives a simplified proof of Theorem 1.

Proof of Theorem 1. Since $\sum_{n\leq N} m(F_n) = (1-q^{-1})\sum_{n\leq N} q^{-l_n}$ the result follows from the Borel-Cantelli lemma for negative quadrant dependent sequences of sets (see Matula [11] or Rényi [14]). ∎

In the sequel, we use the notation

$$X_n := \mathbf{1}_{F_n} - m(F_n),$$

where $\mathbf{1}_A$ denotes the indicator function of the set A. Furthermore, we set $\lim_{n\to\infty} l_n = l$ regardless whether we have (C1) or (C2). Subsequently, we shall interpret all expressions in terms of l for (C2) as the corresponding value obtained by taking the limit, e.g. $q^{-\infty} = 0$. Finally, the constant c is defined in the following lemma.

Lemma 1. *With the assumptions from the introduction,*

$$c := \lim_{i\to\infty} \sum_{i<j\leq i+l_i} q^{-l_j} = lq^{-l}.$$

Proof. If we assume (C1), then the assertion follows from the fact that $l_n = l, n \geq N$ for a sufficiently large N. For (C2), since the limit is assumed to exist, it suffices to prove that

$$\liminf_{i\to\infty} \sum_{i<j\leq i+l_i} q^{-l_j} = 0.$$

Assume that this is wrong. Then there is an $\epsilon > 0$ such that for all $i \geq i(\epsilon)$,

$$\sum_{i<j\leq i+l_i} q^{-l_j} \geq \epsilon.$$

If $l_i \leq l_{i+1} \leq \cdots \leq l_{i+l_i}$ then

$$\sum_{i<j\leq i+l_i} q^{-l_j} \leq l_i q^{-l_i}.$$

Since $l_n \to \infty$, the above chain of inequalities cannot hold if $i(\epsilon)$ is chosen large enough. Hence, starting with any fixed $i_0 \geq i(\epsilon)$, we can find an $i_1 > i_0$ such that $l_{i_0} > l_{i_1}$ etc. This gives a contradiction. ∎

Blocking I: 2-dependent process. Define the sequence τ_n recursively as $\tau_0 = 0$ and

$$\tau_{n+1} := \max_{\tau_n\leq j\leq \tau_n+l_{\tau_n}} \{j : j + l_j \geq i + l_i \text{ for all } \tau_n \leq i \leq \tau_n + l_{\tau_n}\}.$$

Furthermore, denote by

$$Y_n := \sum_{j=\tau_n}^{\tau_{n+1}-1} X_j, \qquad (n \geq 0).$$

We gather some properties of the sequence (Y_n).

Lemma 2. *(i)* $(Y_n)_{n\geq 0}$ *is a 2-dependent process.*

(ii)

$$\mathbb{V}\left(\sum_{n\leq N} Y_n\right) \sim F\left(\tau_{N+1}-1\right). \tag{4}$$

Proof. Due to the properties of the sets F_n, the first part follows from

$$\max_{\tau_n\leq j<\tau_{n+1}} (j+l_j) < \tau_{n+3}.$$

In order to prove the latter, observe that the left hand side is bounded by $\tau_{n+1}+l_{\tau_{n+1}}$. Moreover, we have

$$\tau_{n+2}+l_{\tau_{n+2}} < \tau_{n+3}+l_{\tau_{n+3}}. \tag{5}$$

Assuming that $\tau_{n+3}\leq \tau_{n+1}+l_{\tau_{n+1}}$ would now imply that

$$\tau_{n+2}+l_{\tau_{n+2}} \geq \tau_{n+3}+l_{\tau_{n+3}}$$

which however contradicts (5). Hence, we have proved the first part of the lemma.

For the second part, we first observe that

$$\mathbb{V}\left(\sum_{n\leq N} Y_n\right) = \sum_{n<\tau_{N+1}} m(F_n) - \sum_{n<\tau_{N+1}} m(F_n)^2 + 2\sum_{i<j<\tau_{N+1}} \left(m(F_i\cap F_j) - m(F_i)m(F_j)\right).$$

From the assumptions on (l_n) and (2),

$$\sum_{n<\tau_{N+1}} m(F_n)^2 \sim q^{-l}\left(1-\frac{1}{q}\right)^2 \sum_{n<\tau_{N+1}} q^{-l_n}.$$

Moreover, from the property of the sequence F_n mentioned in the paragraph preceding (3),

$$\begin{aligned}\sum_{i<j<\tau_{N+1}} \left(m(F_i\cap F_j) - m(F_i)m(F_j)\right) &= -\sum_{i<\tau_{N+1}} m(F_i) \sum_{i<j\leq \min\{i+l_i,\tau_{N+1}-1\}} m(F_j)\\ &\sim -c\left(1-\frac{1}{q}\right)^2 \sum_{i<\tau_{N+1}} q^{-l_i},\end{aligned}$$

the last step following from the assumptions on (l_n), Lemma 1, and (2).

Putting everything together yields the claimed result. ∎

Blocking II: Linear variance. For any positive integer n define the integer j_n by

$$F\left(\tau_{j_n+1}-1\right) \leq n < F\left(\tau_{j_n+2}-1\right)$$

and set $j_0=-1$. Note that

$$\begin{aligned}&F\left(\tau_{n+2}-1\right) - F\left(\tau_{n+1}-1\right)\\ &\qquad\leq \left(\left(1-\frac{1}{q}\right) - \left(2c+q^{-l}\right)\left(1-\frac{1}{q}\right)^2\right) \sum_{\tau_{n+1}\leq j\leq \tau_{n+1}+l_{\tau_{n+1}}} q^{-l_j}\\ &\qquad< 1,\end{aligned}$$

where the last line holds if n is chosen large enough. Hence, the above definition makes sense.

Now, we define

$$\xi_n := \sum_{j=j_n+1}^{j_{n+1}} Y_j, \qquad (n \geq 0).$$

Some properties of (ξ_n) are summarized in the next lemma.

Lemma 3. *We have*

(i) $(\xi_n)_{n\geq 0}$ *is a 2-dependent process.*

(ii)

$$\mathbb{E}|\xi_n|^3 \ll 1.$$

(iii)

$$\mathbb{V}\left(\sum_{n\leq N} \xi_n\right) \sim N.$$

Proof. Property (i) is clear. For the proof of (ii), we first apply the multinomial theorem,

$$\begin{aligned}\mathbb{E}|\xi_n|^3 &\leq \mathbb{E}\left(\sum_{j=\tau_{j_n+1}}^{\tau_{j_{n+1}+1}-1} |X_j|\right)^3 \\ &= \sum_{e_{\tau_{j_n+1}}+\cdots+e_{\tau_{j_{n+1}+1}-1}=3} \binom{3}{e_{\tau_{j_n+1}},\ldots,e_{\tau_{j_{n+1}+1}-1}} \mathbb{E}|X_{\tau_{j_n+1}}|^{e_{\tau_{j_n+1}}} \cdots |X_{\tau_{j_{n+1}+1}-1}|^{e_{\tau_{j_{n+1}+1}-1}}. \qquad (6)\end{aligned}$$

In order to estimate the right hand side, we use property (3), a property that more generally holds for any finite number of pairwise distinct F_i's as was proved by Deligero and Nakada [1].

Now, observe

$$\sum_{j=\tau_{j_n+1}}^{\tau_{j_{n+1}+1}-1} \mathbb{E}|X_j|^3 \ll \sum_{j=\tau_{j_n+1}}^{\tau_{j_{n+1}+1}-1} m(F_j) \ll 1,$$

where the last estimate follows by the definition of j_n.

Next, we treat the following sum

$$\sum_{\tau_{j_n+1}\leq i<j\leq\tau_{j_{n+1}+1}-1} \mathbb{E}|X_i|^2|X_j| \ll \sum_{\tau_{j_n+1}\leq i<j\leq\tau_{j_{n+1}+1}-1} m(F_i)m(F_j) \ll \left(\sum_{j=\tau_{j_n+1}}^{\tau_{j_{n+1}+1}-1} m(F_j)\right)^2 \ll 1.$$

Similarly, we have

$$\sum_{\tau_{j_n+1}\leq i<j\leq\tau_{j_{n+1}+1}-1} \mathbb{E}|X_i||X_j|^2 \ll 1.$$

Hence, we are left with

$$\sum_{\tau_{j_n+1}\leq i<j<l\leq\tau_{j_{n+1}+1}-1} \mathbb{E}|X_i||X_j||X_l| \ll \left(\sum_{j=\tau_{j_n+1}}^{\tau_{j_{n+1}+1}-1} m(F_j)\right)^3 \ll 1.$$

Plugging the last three estimates into (6) gives property (ii).

For property (iii), observe that by (4),

$$\mathbb{V}\left(\sum_{n\le N}\xi_n\right)=\mathbb{V}\left(\sum_{n\le j_{N+1}}Y_n\right)=F\left(\tau_{j_{N+1}+1}-1\right).$$

Moreover, by the definition of j_n and the remark succeeding the definition, we have

$$N<F\left(\tau_{j_{N+1}+2}-1\right)+\left(F\left(\tau_{j_{N+1}+1}-1\right)-F\left(\tau_{j_{N+1}+2}-1\right)\right)=F\left(\tau_{j_{N+1}+1}-1\right)\le N+1.$$

This yields the desired result. ∎

3 Proof of the invariance principle

The proof of Theorem 2 will rest on the following extension of a theorem of Philipp and Stout (see Fuchs [4]). We state the result in a simplified form that will be sufficient for our purpose.

Proposition 1. *Let ξ_n denote a 2-dependent process of centered random variables on the probability space $(\Omega,\mathcal{A},P)$ and suppose that*

$$\mathbb{E}|\xi_n|^3\ll 1$$

and

$$\mathbb{V}\left(\sum_{n\le N}\xi_n\right)\sim N.$$

Define a stochastic process $\xi(t)$ on $(\Omega,\mathcal{A},P)\times([0,1],\bar{\mathcal{B}},\lambda)$ by

$$\xi(t)=\sum_{n\le t}\xi_n.$$

Then, as $t\longrightarrow\infty$,

$$\xi(t)-W(t)=o\left((t\log\log t)^{1/2}\right),\quad \textit{a.s.}$$

and

$$(P\times\lambda)\left[\frac{1}{\sqrt{t}}\sup_{s\le t}|\xi(s)-W(s)|\ge\epsilon\right]\longrightarrow 0$$

for all $\epsilon>0$.

Due to the Lemma 3, the sequence ξ_n of the previous section satisfies all the assumptions of the above proposition. Therefore, we obtain, as $t\longrightarrow\infty$,

$$\xi(t)-W(t)=o\left((t\log\log t)^{1/2}\right),\quad \text{a.s.}$$

and

$$(m\times\lambda)\left[\frac{1}{\sqrt{t}}\sup_{s\le t}|\xi(s)-W(s)|\ge\epsilon\right]\longrightarrow 0$$

for all $\epsilon>0$, where $\xi(t)=\sum_{n\le t}\xi_n$.

The invariance principle for $Z(t)$. We prove the following lemma.

Lemma 4. *As $t \to \infty$,*

$$Z(t) - \xi(t) \ll t^{1/2-\epsilon}, \quad \textit{a.s.}$$

for all $0 < \epsilon < 1/6$.

Proof. We have

$$m\left[\sum_{j=\tau_{j_n+1}}^{\tau_{j_{n+1}+1}-1} |X_j| \geq n^{1/2-\epsilon}\right] \leq n^{-3/2+3\epsilon}\mathbb{E}\left(\sum_{j=\tau_{j_n+1}}^{\tau_{j_{n+1}+1}-1} |X_j|\right)^3 \ll n^{-3/2+3\epsilon}.$$

Consequently, by the Borel-Cantelli lemma,

$$\sum_{j=\tau_{j_n+1}}^{\tau_{j_{n+1}+1}-1} |X_j| \ll n^{1/2-\epsilon}, \qquad \text{a.s.}. \tag{7}$$

Now, observe

$$|Z(t) - \xi(t)| = \left|\sum_{n \leq N_t} X_n - \sum_{n \leq \tau_{j_{[t]+1}+1}-1} X_n\right| \leq \sum_{j=\tau_{j_{[t]}+1}}^{\tau_{j_{[t]+1}+1}-1} |X_j|$$

and combining with (7) concludes the proof of the desired result. ▮

The above lemma yields, as $t \to \infty$,

$$Z(t) - W(t) = o\left((t \log\log t)^{1/2}\right), \quad \text{a.s.}$$

and

$$(m \times \lambda)\left[\frac{1}{\sqrt{t}} \sup_{s \leq t} |Z(s) - W(s)| \geq \epsilon\right] \longrightarrow 0$$

for all $\epsilon > 0$. Reformulation gives Theorem 2. ▮

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