行政院國家科學委員會專題研究計畫　成果報告

概似型態的信賴區間

計畫類別：個別型計畫
計畫編號：NSC94-2118-M-009-005-
執行期間：94 年 08 月 01 日至 95 年 07 月 31 日
執行單位：國立交通大學統計學研究所

計畫主持人：陳鄰安

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中文摘要：

在這個計畫中，我們介紹含有固定母體比例的一種區間，並稱之爲眾數型態區間，我們討論這個區間的統計推論，包含信賴區間及檢定。我們證明了這個信賴區間是工業統計上應用很廣的容忍區間。而在點估計方面在品質管制圖也發展一種新的技術。

關鍵詞：信賴區間，眾數區間，休華特管制圖，容忍區間。

Confidence Interval for Mode Type Coverage Interval

Lin-An Chen

Institute of Statistics, National Chiao Tung University, 1001 Ta Hsueh Rd. Hsinchu, Taiwan

Abstract

We introduce a new population interval containing $100\gamma\%$ of a distribution called the γ mode interval and then treat this interval as an unknown parameter to be estimated. In addition, we set confidence intervals and carry out tests to make inferences about the unknown interval. The estimated γ mode interval is used to construct a new control chart while confidence intervals for the γ mode interval are γ content tolerance intervals. We establish some properties of γ mode intervals and derive γ mode intervals for some simple distributions. We show how to estimate and make inferences about γ mode intervals and, again in specific cases, compare these to control charts and tolerance intervals. Our approach provides a general way of constructing control charts and tolerance intervals for multiparameter problems and asymmetric distributions.

Keywords: confidence interval, mode interval, Shewhart control chart, tolerance interval

Confidence Interval for Mode Type Coverage Interval

1. Introduction

Intervals in the form of control charts and tolerance intervals are widely used in quality control and related prediction problems to monitor manufacturing processes, detect changes in such processes, ensure product compliance with specifications, etc. The purpose of this paper is to i) introduce a new interval (which we call a mode interval) which we use as a basis for constructing control charts and tolerance intervals and ii) to formalize a unified framework for thinking about and constructing these kinds of intervals.

Control charts were introduced by Shewhart in an internal Western Electric memo in 1924; we adopt a general formulation given in Montegomery (1991, p105). Let w be a sample statistic with mean μ_w and standard deviation σ_w that measures a quality characteristic of interest on a process. Then the Shewhart control chart for w is a graphical representation of what we call the Shewhart interval $[\mu_w - k\sigma_w, \mu_w + k\sigma_w]$, where k is a positive constant (usually 3), which is written equivalently as

$$
\begin{aligned}
UCL &= \mu_w + k\sigma_w \\
\text{Center line } &= \mu_w \\
LCL &= \mu_w - k\sigma_w,
\end{aligned}
$$

as parallel, horizontal lines on a plot of w against time. The process is said to be in (out of) control if future values of w fall in (out of) the interval $[\mu_w - k\sigma_w, \mu_w + k\sigma_w]$. When the mean and standard deviation are unknown, the interval is treated as the parameter of interest and estimated by $[\hat{\mu}_w - k\hat{\sigma}_w, \hat{\mu}_w + k\hat{\sigma}_w]$, where $\hat{\mu}_w$ and $\hat{\sigma}_w$ estimate the mean and standard deviation of w respectively.

An attraction of Shewhart control charts is that they are simple to construct: they have been produced for various distributions including continuous or discrete, symmetric or asymmetric distributions and various statistics. Under normality, the level or content of the charts for the sample mean $\bar{X}$ and a single observation are 0.9973 in the sense that these statistics are within their respective Shewhart (population) intervals with probability 0.9973. However, for other statistics w and or for other distributions, the content may be far from the nominal value. Also, as pointed out by Grant and Leavenworth (199 , p121), symmetric intervals are not appropriate for asymmetric distributions when they include values representing poor manufacturing performance. It is therefore of interest to improve on Shewhart intervals under asymmetry.

The simplest general γ content intervals for w are the γ median intervals for w given by $C_{w,med}(\gamma) = [F_w^{-1}\{(1-\gamma)/2\}, F_w^{-1}\{(1+\gamma)/2\}]$, where F_w is the distribution function of w. We call these median intervals because they always contain the median of F_w and, if the median is unique, reduce to it as $\gamma \to 0$. They place equal probability $(1-\gamma)/2$ in the tails of the distribution and have level γ for any F_w. Moreover, they are often used as a starting point for tolerance interval calculations and to construct a process capability index which is the ratio between the width $F_w^{-1}(0.996\ \ 5) - F_w^{-1}(0.00135)$ of the $\gamma = 0.9973$ median interval and the process specification limit. However, when F_w is asymmetric, the median intervals are not necessarily well located - they may exclude the mode and other regions of high probability for w. We therefore expand the class of intervals by introducing the larger class of γ intervals $\{[F_w^{-1}(\delta), F_w^{-1}(\gamma+\delta)] : 0 < \delta < 1-\gamma\}$ in which δ is now a location constant to be chosen by the user. This class reduces to the quantile intervals when $\delta = (1-\gamma)/2$ but is more 8exible. We are particularly interested in the shortest interval in this class which is

$$C_w(\gamma) = [F_w^{-1}(\delta^*), F_w^{-1}(\gamma + \delta^*)], \tag{1.1}$$

where $\delta^* = \delta^*(\gamma) = \arg_\delta \min_{0<\delta<1-\gamma}\{F_w^{-1}(\gamma+\delta) - F_w^{-1}(\delta)\}$. Whereas the median intervals reduce to the set containing the median as γ decreases to zero, we will show that this interval reduces to the set containing the mode of F_w as γ decreases to zero; we call (1.1) a γ mode interval. It is natural then to define a modal control chart as

$$\begin{aligned} UCL_{mode} &= F_w^{-1}(0.9973 + \delta^*) \\ \text{Center line } &= \text{mode}(F_w) \\ LCL_{mode} &= F_w^{-1}(\delta^*), \end{aligned} \tag{1.2}$$

where $\delta^* = \arg_\delta \min_{0<\delta<1-\gamma}\{F_w^{-1}(0.9973 + \delta) - F_w^{-1}(\delta)\}$.

In practice, exactly as with Shewhart intervals and control charts, we usually need to estimate F_w: if k samples are available from a process which is under control and $\hat{F}_{wi}^{-1}$, $i = 1, ..., k$ are estimates of the population quantile F_w^{-1} from these samples, then the population quantile can be estimated by $\bar{F}_w^{-1} = \frac{1}{k}\sum_{i=1}^{k} \hat{F}_{wi}^{-1}$. We can then go further and discuss setting confidence intervals and testing hypotheses about population intervals.

As is intuitively reasonable, if the population interval contains a given proportion of the distribution, then a confidence interval for the population interval is a tolerance interval (see for example Wilks (1941), Wald (1943), Paulson (1943), Guttman (1970) and, for a recent review, Patel (196)). That is, confidence intervals for median and mode intervals are both tolerance intervals; the latter should perform better under asymmetry. There is a vast literature on tolerance intervals but, as noted by Bucchianico, Einmahl and Mushkudiani (2001), both the mathematically and the engineering oriented statistics textbooks hardly deal with this topic explictly, and, if they do, the treatment is often confined to tolerance intervals for the normal distribution. This is partly because tolerance intervals can be difficult to construct for particular distributions (although nonparametric tolerance intervals based on order statistics can be obtained for particular values of the content) and, perhaps, partly because as Carroll and Ruppert (1991) suggest, the idea of conditional coverage probability is considered to be too difficult for beginning students. The advantages of constructing tolerance intervals as confidence intervals for particular population intervals include wide applicability (including multiparameter and asymmetric distributions), simplicity of interpretation (without conditional probability) and a nice link between control charts and tolerance intervals.

There are alternatives to the terminology we have used for intervals containing a known proportion of a distribution. In the population, we use the general terminology of γ content intervals and refer to specific examples of such intervals as Shewhart, median and mode intervals. Our terms for the last two intervals links them to measures of location and reinforces the idea that the intervals generalise point measures of location. We also refer to confidence intervals for the population intervals and identify these as tolerance intervals. In metrology (the study of measurement), the γ content interval is called a coverage interval and tolerance intervals are called statistical coverage intervals (see for example Willink (2004)). These are also sometimes called statistical tolerance regions.

The idea of treating population intervals as unknown parameters which we can estimate and make inferences about is already used implicitly in the literature (for example by Carroll and Ruppert (1991)) but seems to us to be of sufficient importance and value to make it explicit. We therefore present the general approach to estimation and inference for intervals in abstract form in Section 2. We then establish the properties of (population) γ

mode intervals in Section 3 and derive (population) γ mode intervals for several standard distributions in Section 4. We discuss the estimation of γ mode intervals and corresponding modal control charts in Section 5 before discussing confidence intervals for γ mode intervals in Section and compare these intervals to other tolerance intervals. We present a brief discussion of hypothesis tests for γ mode intervals in Section 7. Throughout this paper, we focus on parametric methods for setting two-sided intervals. We consider simple distributions to show how the approach performs in familiar cases but it is important to keep in mind that the approach is most advantageous for complicated (i.e multiparameter) problems and asymmetric distributions.

There are more sections for this report. The detailed content is ommited.

2. Estimation and inference for population intervals

3. Population mode intervals

4. Examples of population mode intervals

5. Estimation of mode intervals

6. Confidence intervals for γ mode intervals

7. Testing hypotheses about mode intervals

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計畫成果自評：

　　本計畫之主題可以做多方向的探討。本人的學生在容忍區間的有效性及探討可靠度問題的容忍區間有相當的突破。這個計畫的結果目前寫成兩篇文章。主要成果部分目前在一統計雜誌審稿中。這個主題所延伸討論 Coverage interval 與 Statistical Coverage interval 的文章正被”Metrologia”雜誌發表。文章附於下面。

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SHORT COMMUNICATION

Extending the discussion on coverage intervals and statistical coverage intervals

Lin-An Chen and Hui-Nien Hung

Institute of Statistics, National Chiao Tung University, Hsinchu, Taiwan

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Abstract
Willink (2004 *Metrologia* **41** L5–6) is concerned that, in the society of metrology, there is potential for confusion between coverage interval and statistical coverage interval and he makes a precise interpretaion of these two terms. We further clarify that the confidence of a coverage interval is actually a statistical coverage interval.

Quite often a scientist is less interested in estimating parameters and more concerned about gaining a notion about where individual observations or measurements might fall. There are two attempts to determine bounds from this notion. The coverage interval (also called the reference interval in clinical chemistry) refers to population-based reference values obtained from a well-defined (normal or healthy people) group of reference individuals. This is an interval with two confidence limits that covers the individual values in the population in some probabilistic sense. One method of establishing a bound on single values in the population is to determine a confidence interval for a fixed proportion of the measurements. A common problem in clinical chemistry is to determine the coverage interval for a particular test. This reference interval represents a region of the distribution of normal or healthy people. Once the reference interval is determined, any patient with a suspected disease may have the test and the result of the test can be compared with the reference interval. A result outside the reference interval may then be taken as confirmation of the disease.

The statistical coverage intervals are statistical intervals that contain (or cover) at least a proportion p of a population with a stated confidence $1-\alpha$. In recent years, the International Organization for Standardization (ISO 3534-2 1993, ISO 16269-6 2005) and several expert scientists have advocated calculation of the coverage interval and the statistical coverage interval. Due to the potential for confusion between the terms 'coverage interval' and 'statistical coverage interval', Willink (2004) provided a clear interpretation of the roles that these two terms play in the literature on metrology. The interpretation has been taken further by Perruchet (2004) in describing the differences between the terms 'confidence interval', 'coverage interval' and 'statistical coverage interval'. In this paper, we want to clarify the relation between 'coverage interval' and 'statistical coverage interval'.

Suppose that a quantity (called a random variable in statistics) X has a distribution with probability density function $f(x)$, and generally this function involves an unknown parameter θ. A $100p\%$ coverage interval for this quantity is any interval (a, b) such that $p = P(X \in (a, b))$. On the other hand, Wilks (1941) introduces a p-content statistical coverage interval with confidence $1-\alpha$ as any random interval (T_1, T_2) that satisfies

$$P\{P_X[(T_1, T_2)] \geqslant p\} \geqslant 1-\alpha. \qquad (1)$$

There is a vast literature (see, for example, Wald (1943), Paulson (1943), Guttman (1970) and, for a recent review, Patel (1986)) that introduces techniques in constructing a p-content statistical coverage interval with confidence $1-\alpha$. However, this variety of techniques generally involves approximation or simulation for the construction of a statistical coverage interval. Thus, the connection between a statistical coverage interval and a coverage interval has been unclear.

Goodman and Madansky (1962) implicitly applied the concept that a $100(1-\alpha)\%$ confidence interval (T_1, T_2) of a $100p\%$ coverage interval (a, b) in the sense that

$$P\{T_1 \leqslant a < b \leqslant T_2\} = 1-\alpha$$

is a p-content statistical coverage interval with confidence $1-\alpha$. Here we formally prove that any statistical coverage interval is a confidence interval of a coverage interval with some confidence. Suppose that (a, b) is a $100p\%$ coverage interval and we have a sample $X_1, \ldots, X_n$ from a distribution with distribution F_X. Let (T_1, T_2) be a $100(1-\alpha)\%$ confidence interval of (a, b). The following statements will help us clarify some relations between the coverage interval and the statistical coverage interval:

$$\begin{aligned} P\{P_X[(T_1, T_2)] \geqslant p\} &\qquad (2)\\ = P\{F_X(T_2) - F_X(T_1) \geqslant p\}&\\ = P\{F_X(T_2) - F_X(T_1) \geqslant F_X(b) - F_X(a)\}&\\ \geqslant P\{F_X(T_1) \leqslant F_X(a) < F_X(b) \leqslant F_X(T_2)\}&\\ \geqslant P\{T_1 \leqslant a < b \leqslant T_2\}& \end{aligned}$$

as F_X is non-decreasing. □

Points of interest include the following:

(a) If we choose a random interval (T_1, T_2) that is a $100(1-\alpha)\%$ confidence interval of (a, b), this indicates that $P\{P_X[(T_1, T_2)] \geqslant p\} \geqslant 1-\alpha$ such that (T_1, T_2) is a p-content statistical coverage interval at confidence $1-\alpha$.

(b) Suppose that (T_1, T_2) is only a random interval. Then it is still a p-content statistical coverage interval at confidence $1-\alpha = P\{T_1 \leqslant a < b \leqslant T_2\}$. The fact that every random interval is also a statistical coverage interval is noteworthy.

This connection contributes the construction of a statistical coverage interval through the use of a coverage interval. Choosing any a p-content coverage interval, then any random interval which is a $100(1-\alpha)\%$ confidence interval of it is a p-content statistical coverage interval at confidence coefficient $1-\alpha$.

Perhaps the most significant observation is that in (*a*). Suppose we have a quantity X that has a normal distribution $N(\mu, \sigma^2)$ where σ is a known constant. With a sample $X_1, ..., X_n$, let $\hat{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. Consider the p-content coverage interval

$$(\mu - z_{\frac{1+p}{2}}\sigma < \mu + z_{\frac{1+p}{2}}\sigma). \qquad (3)$$

The following shows that

$$\left(\hat{X} - z_{\frac{1+p}{2}}\sigma - z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}},\ \hat{X} + z_{\frac{1+p}{2}}\sigma + z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right) \qquad (4)$$

is a $100(1-\alpha)\%$ confidence interval of the coverage interval in (3):

$$\begin{aligned} &P\Big(\hat{X} - z_{\frac{1+p}{2}}\sigma - z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}} \leqslant \mu - z_{\frac{1+p}{2}}\sigma\\ &\quad < \mu + z_{\frac{1+p}{2}}\sigma \leqslant \hat{X} + z_{\frac{1+p}{2}}\sigma + z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\Big)\\ &= P\Big(-z_{\frac{1+p}{2}}\sigma - z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}} \leqslant \hat{X} - \mu - z_{\frac{1+p}{2}}\sigma\\ &\quad < \hat{X} - \mu + z_{\frac{1+p}{2}}\sigma \leqslant z_{\frac{1+p}{2}}\sigma + z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\Big)\\ &= P\left(-z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}} \leqslant \hat{X} - \mu \leqslant z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right)\\ &= P\left(-z_{1-\frac{\alpha}{2}} \leqslant \frac{\hat{X}-\mu}{\sigma/\sqrt{n}} \leqslant z_{1-\frac{\alpha}{2}}\right)\\ &= 1-\alpha. \end{aligned}$$

The random interval of (4) is a γ-content statistical coverage interval at confidence $1-\alpha$ because it is a $100(1-\alpha)\%$ confidence interval of a p coverage interval.

We further assume that both parameters μ and σ are unknown. Wald and Wolfowitz (1946) first introduced the normal tolerance interval of the form

$$(\bar{X} - kS, \bar{X} + kS). \qquad (5)$$

As noted in Guttman (1970), it is exceedingly complicated to derive k to meet the requirement (1) for preassigned p and $1-\alpha$. This leads to the approximation techniques by Wald and Wolfowitz (1946), Weissberg and Beatty (1960) and Odeh and Owen (1980). Let $T = t(r, c)$ represent a random variable with a non-central t distribution with r degrees of freedom and a non-centrality parameter c. We also let $t_\alpha(r, c)$ satisfy $\alpha = P(T \geqslant t_\alpha(r, c))$. We may show that

$$\left(\bar{X} - t_{1-\frac{\alpha}{2}}\left(n-1, \sqrt{n}z_{\frac{1+p}{2}}\right)\frac{S}{\sqrt{n}},\right.$$
$$\left.\bar{X} + t_{1-\frac{\alpha}{2}}\left(n-1, \sqrt{n}z_{\frac{1+p}{2}}\right)\frac{S}{\sqrt{n}}\right)$$

is a $100(1-\alpha)\%$ confidence interval for the coverage interval of (3) and then it is also a p-content tolerance interval at confidence $1-\alpha$. Moreover, from (2), an interval of (5) for any $k > 0$ is also a p-content tolerance interval at some confidence.

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