行政院國家科學委員會專題研究計畫成果報告

預設均曲率方程之黏解
Viscosity solutions for the Equation of Prescribed Mean Curvature

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摘要

假設 H 為定義於 n+1 維球面上之實值連續函數。本計畫考慮 n+1 維球面上其均曲率與 H 一致之超曲面的存在性問題。探討此問題所衍生的擴散方程得到：當 H 滿足適當的成長與橫越條件，對滿足非遞減條件的始值，該擴散方程之大域黏解存在。

關鍵詞：球面、超曲面、黏解、均曲率

Abstract

Let H be a given continuous real valued function defined on the (n+1)-sphere. The problem of prescribed mean curvature is to find certain conditions on H so that there exists a hypersurface M whose mean curvature is the given function H. The purpose of this report is to show that there exists a global viscosity solutions of a related evolution equation for suitable H and initial value u_0. For using the Perron's method, we need to construct explicitly a family of special sub- and supersolutions. To do this, we make certain growth and transverse assumptions on the given function H and monotonic assumptions on the initial value u_0 .

Keywords: Spheres, hypersurface, viscosity solution, mean curvature.

1. Introduction

Let X and Y be complete Riemannian manifolds, dim(Y)= n, dim(X) = n+1. Let H be a preassigned smooth function defined on X. We consider the problem of prescribed mean curvature, that is, find conditions on H so that there exists an embedding Y in X whose mean curvature is the given function H (see [Y]). For X being the Euclidean space and Y being a spherical type, based on the theory of elliptic partial differential equation, Treibergs and Wei showed the existence and uniqueness of the problem of prescribed mean curvature if H decays faster than the mean curvature of two concentric spheres [TW]. For X being a sphere and Y being a topological sphere, using the same elliptic theory as [TW], we also showed the existence and uniqueness of the problem of prescribed mean curvature if H satisfies certain growth conditions [HSW].

In this report, we consider parabolic version of the prescribed mean curvature problem. As the mean curvature flow, one can expect there are only short time existence [H]. We consider the weak solution in the sense of [ES]. We want to show the existence of the viscosity solution for a evolution flow related to this problem. Let Y be a family of immersion from a compact n-manifold into the n+1 dimensional sphere which satisfies the following equation

For deriving the equation in the viscosity

$$Y_t = (H - M)N,$$

where N is the unit normal and

M is the mean curvature of Y.

sense, assume that Y is a level hypersurface of some function u defined on the (n+1)-sphere. Then the degenerate parabolic equation is given by

$$\text{(1.1)} \quad u_t = \sum (¢_{ij} - \frac{u_i u_j}{|\nabla u|^2}) u_{ij} - |\nabla u| \ H$$

$$\text{in } S^{+1} \times (0, \infty),$$

where the derivatives are the first and second covariant derivatives related to the (n+1)-sphere. Suppose that the initial condition is given by

$$\text{(1.2)} \quad u = u_0 \ \text{ at } \ S^{n+1} \times \{t = 0\}.$$

To show the existence of vicosity solutions for the initial value problem (1.1) and (1.2), we first construct viscosity sub- and supersolutions of (1.1) and (1.2). Let a be a constant vector in S^{n+1} . Denote by

$$\overline{H}(°) = \max H, \quad \underline{H}(°) = \min H \ \text{ and}$$

$$\overline{u_0}(°) = \max u_0, \quad \underline{u_0}(°) = \min u_0,$$

where the maximum and minimum are taken over all X with (X, a) = **ρ**. We show that if these functions related to H satisfy the transverse condition (H1) and the growth condition (H2) , and these functions related to u_0 are nondecreasing (see section 2), then there exist global viscosity sub- and supersolutions. Moreover we have a comparison between viscosity sub- and supersolutions.. It follows from the Perron's method that a global viscosity solution of (1.1) and (1.2) exists. We then can state the main result of this report as follows

Theorem. Let H be a continuous function defined on the (n + 1) - sphere satisfying (H1) and (H2). Then for any given continuous function u_0, $\overline{u}_0$ and $\underline{u}_0$ are nondecreasing, (1.1) and (1.2) has a global viscosity solution u.

2. Existence Theorem

Suppose that H and u_0 are functions of **ρ**,where **ρ** =(X, a), X is the position vector of S^{n+1} and a is a constant vector in S^{n+1}. Let

$$H_0(²) = -n \frac{²}{\sqrt{1 - ²^2}},$$

be the mean curvature of the hypersphere **τ** = **ρ**, for -1< **ρ** <1. Assume that there exist finitely many

$$°_0 = -1 < °_1 < °_2 < \Lambda < °_{2l+1} < 1 = °_{2l}$$

such that

$$\text{(H1)} \quad H < H_0 \ \text{ for } ° \in (°_{2i}, °_{2l+1}),$$
$$H > H_0 \ \text{ for } ° \in (°_{21-1}, °_{2l}).$$

To construct a global solution, we need H satisfying the following (H2) conditions

$$\int_*^{°_i} \frac{d²}{\sqrt{1 - ²^2}(H_0 - H)} = \infty \ \text{ for } i = 1, 2, \Lambda, 2l - 1.$$

Suppose that u_0 is nondecreasing. Then it follows that u = u_0 (F^{-1}(F(**ρ**)+t) is a global viscosity solution for the initial value problem (1.1) and (1.2), where

$$\text{(1.3)} \quad F(°) = \int_*^{°} \frac{d²}{\sqrt{1 - ²^2}(H_0 - H)},$$

for **ρ** in [$**ρ**_i$, $**ρ**_{i+1}$] . In fact, there are four ways to construct such a viscosity solution.

Now we go back to the general case; H and u_0 are functions of X. Assume that

$$\overline{H} < H_0 \ \text{ for } \circ \in (\overline{\circ}_{2i}, \overline{\circ}_{2i+1}),$$
$$\overline{H} > H_0 \ \text{ for } \circ \in (\overline{\circ}_{2i+1}, \overline{\circ}_{2i+2}),$$
$$\underline{H} < H_0 \ \text{ for } \circ \in (\underline{\circ}_{j}, \underline{\circ}_{2j+1}),$$
$$\underline{H} > H_0 \ \text{ for } \circ \in (\underline{\circ}_{2j+1}, \underline{\circ}_{2j+2}),$$

for i = 0,1,2,…,k-1 and j = 0,1,2,…,m-1, $\overline{u}_0$ and $\underline{u}_0$ are nondecreasing functions of $\circ$.

Under these conditions, we construct sub- and supersolutions of the original equation (1.1) and (1.2) as follows: Let

$$\overline{F}(\circ) = \int_*^{\circ} \frac{d^2}{\sqrt{1-{}^{2^2}}\,(H_0 - \underline{H})},$$
$$\underline{F}(\circ) = \int_*^{\circ} \frac{d^2}{\sqrt{1-{}^{2^2}}\,(H_0 - \overline{H})},$$

$\overline{u} = u_0\ (\overline{F}^{-1}(\overline{F}(\circ) + \mathrm{t}))$ and

$\underline{u} = u_0\ (\underline{F}^{-1}(\underline{F}(\circ) + \mathrm{t}))$.

Then $\underline{u}$ is a subsolution and $\overline{u}$ is a supersolution respectively. Furthermore, $\underline{u} \le \overline{u}$ for all t. This follows directly from the construction by comparison.

We appeal to the Perron's method. Let S be the nonempty set

$$S = \{v;\ v \text{ is a subsolution of (1.1) and (1.2) and } v \le \overline{u}\}.$$

Then a standard argument as [CGG] shows that

$$u(X) = \sup\{v(X);\ v \in S\}$$

is a viscosity solution of (1.1) and (1.2).

3. Final Comments

In this report we show that existence of global viscosity solution for the degenerate parabolic equation related to the problem of prescribed mean curvature. It is clear that this is a geometric equation [AS]. In our elliptic result, Y must be spherical type. In the parabolic case, we make no assumption on the topology of Y. However this result seems only make sense in the spherical case. Is there have other sub- and supersolutions which can observe another topological type ? On the other hand, there still have many elementary works. Does this solution is unique in the sense of [ES] ? What is the relationship between the classical solution and weak solution ? The most important and deep question at least for me is : How to observe the behavior of the singular set from the weak solution ? These problems are interesting to answer.

4. References

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