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期中報告

黃大原

作爲堵丁柱在完備圖上的不同大小的完美匹配上所建構的 Pooling 設計的推廣，我們以詹氏圖、格氏圖的不同大小的點團所構成的鄰接圖爲基礎，我們給出倆列類具有改錯能力的 Pooling 設計。

Some Error-Correcting Pooling Designs Associated with Johnson Graphs and Grassmann Graphs

(Preliminary Version 3.1)

Yujuan Bai, Tayuan Huang and Kaishun Wang*

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Abstract

Based on the inclusion matrices of t-cliques with various sizes of Johnson graphs $J(n,t)$ and Grassmann graphs $J_q(n,t)$ respectively, two families of error-correcting pooling designs are given, some of their properties including the error-correcting capability together with two parameters e_d and $e_{\leq d}$are studied. With an interpretation of matchings K_{2m} of as 2-cliques of Johnson graph $J(n,2)$, this gives a q-analogue of the pooling designs defined over matchings of K_{2m} given by Ngo and Du.

1 Introduction

Suppose there are at most d defective items among n items to be tested, and we assume some testing mechanism exists which if applied to an arbitrary subset of the

*corresponding author

population gives a *negative outcome* if the subset contains no positive and *positive outcome* otherwise. A group testing algorithm is *non-adaptive* if all tests must be specified without knowing the outcomes of other tests, which is useful in many areas such as DNA library screening.

The notion of d^e-*disjunct* matrices (defined in Sec.2) provides a mathematical model for error-correcting pooling designs. Macula [5] constructed d^e-disjunct matrices for certain values of e by the containment relation of subsets in a finite set. The q-analogue of Macula's construction is given by Ngo and Du in [7]. Moreover, the notion of pooling spaces was introduced by Huang and Weng [ ] which provides one of general frameworks for d^e-*disjunct* matrices. They showed that a d^{2e}-disjunct matrix is e-error-correcting in [3].

Recently, Ngo and Du constructed a class of disjunct matrices over the incidence matrices of matchings with various sizes of the complete graph K_{2m} in [7], and asked for its q-analogue. With an interpretation of matchings as 2-cliques of Johnson graphs $J(n, 2)$, we generalize Ngo and Du's construction to the incidence matrices of t-cliques with various sizes of Johnson graphs $J(n, t)$ and Grassmann graphs $J_q(n, t)$, respectively. We show that our pooling designs have the same capability of error-detecting and error-correcting as Ngo and Du's, however the test to item ratio of ours is much smaller. Moreover, the parameters e_d and $e_{\leq d}$ of these pooling designs are also considered.

An overview of up-to-date results on Combinatorial Group Testing algorithms was given by Du and Ngo [8]. It is interesting to note that they pointed out that this is a young and interesting field with deep connections to coding theory and design theory, and they strongly believe that the theory of association schemes , and in particular distance regular graphs, should play an important role in improving our pooling designs.

We will recall some known results regarding pooling designs in the framework of two families of distance regular graphs, the John graph and the Grassmann Graphs.

We first recall some pooling designs associated with the Johnson graphs and Grassmann graphs as well in section 2. Some basic definitions on t-cliques, $\{1, 2, K, t\}$-cliques of Johnson graphs are also given in Section 2. Two new families of pooling designs together with their capability of error-correcting are given in Section 3. Moreover, two parameters e_d and $e_{\leq d}$ over the Johnson graphs are studied in Section 4.

2 Preliminaries

The notion of d^e-*disjunct* matrices provides a mathematical model for error-correcting pooling designs.

Definition 2.1 *A binary matrix M is said to be d^e-disjunct if given any $d+1$ columns of M with one designated, there are $e+1$ rows with a* 1 *in the designated column and* 0 *in each of the other d columns.*

A d^e-disjunct matrix with $e = 0$ is said to be d-*disjunct* matrix. Let q be a positive integer, indeed a prime power in use. Given positive integers $1 \leq i \leq n$, the *Gaussian binomial coefficients with basis q* is defined by

$$\begin{bmatrix} n \\ i \end{bmatrix}_q = \begin{cases} \prod_{j=0}^{i-1} \frac{n-j}{i-j}, & \text{if } q = 1, \\ \prod_{j=0}^{i-1} \frac{q^n - q^j}{q^i - q^j}, & \text{if } q \neq 1. \end{cases}$$

In the case $q = 1$, we write $\binom{n}{i}$ instead of $\begin{bmatrix} n \\ i \end{bmatrix}_1$ for convenience.

For any positive integer n we use $[n]$ to denote the set $\{1, 2, \ldots, n\}$. For any positive integer k, $\binom{[n]}{k}$ denotes the collection of all k-subsets of $[n]$, and $\begin{bmatrix} GF(q)^n \\ k \end{bmatrix}_q$

denotes the collection of all k-subspaces of $GF(q)^n$.

Definition 2.2

1. *The* Johnson graph $J(n,t)$ *is the graph defined on* $\binom{[n]}{t}$ *such that* A *and* B *are adjacent if* $|A \cap B| = t-1$.

2. *The* Grassmann graph $J_q(n,t)$ *is the graph defined on* $\begin{bmatrix} GF(q)^n \\ t \end{bmatrix}_q$ *such that* A *and* B *are adjacent if* $\dim(A \cap B) = t-1$.

Definition 2.3 *A clique* $\mathcal{C}$ *of* $J(n,2)$ *is a subfamily of* $\binom{[n]}{2}$ *such that* $|A \cap B| = 1$ *for any two distinct* $A, B \in \mathcal{C}$.

Note that $J(n,2)$ is a strongly regular graph, i.e.a distance regular graph of diameter 2. Both Johnson graphs and Grassmann graph are distance-regular, refer to [1] for details.

Hence an l-matching in [7] is a 2-clique of $J(n,2)$ with size l. With this interpretation, its q-analogue extensions are available.

Definition 2.4

1. *A* t-clique *of* $J(n,t)$ *with size* l *is a subfamily* $\{A_1, A_2, \ldots, A_l\}$ *of* $\binom{[n]}{t}$ *such that* $|A_1 \cup A_2 \cdots \cup A_l| = tl$, *i.e.,* $A_i \cap A_j = \emptyset$ *for any two distinct* i *and* j.

2. *A* t-clique *of* $J_q(n,t)$ *with size* l *is a subfamily* $\{A_1, A_2, \ldots, A_l\}$ *of* $\begin{bmatrix} GF(q)^n \\ t \end{bmatrix}_q$ *such that* $\dim(A_1 + A_2 + \cdots + A_l) = tl$.

Definition 2.5 *A family of* k*-subsets in* $[n]$ *with* $|K \bigcap K'| \leq k-t$ *for all* K *and in* K' *in* $\mathcal{K}$ *is called a* $\{1,2,K,t\}$*-clique of* $J(n,k)$.

The notations for disjunct matrices: Let $d < k < n$,
$J(n,d,k)$: the incidence matrix of the system $(\binom{[n]}{d}, \binom{[n]}{k}; \subseteq)$

(J is for Johnson Schemes)

$G_q(n, d, k)$: the incidence matrix of the system $(\binom{GF(q)^n}{d}, \binom{GF(q)^n}{k}; \subseteq)$

(G is for Grassmann Schemes)

$M(2n, d, k)$: the incidence matrix of the system

(M is for matchings)

$M_q(2n, d, k)$: the incidence matrix of the system of q-analog of

(M_q is for q-analogues of matchings) The error-correcting capability of d^e - disjunct matrices is summarized in the following.

Theorem 2.1 *Suppose a d^e - disjunct matrix M of order $N \times t$ is used for a pooling design, and P is the positive set to be identified,*

1. *if it is known that $|P| = d$, then M can correct e-errors;*
2. *if it is known that $|P| \leq d$, then M can correct $\left\lfloor \frac{e}{2} \right\rfloor$-errors; moreover, M can correct e-errors in addition to another d-confirmation tests.*

Moreover, the q-analogue of $G(m, t, k, r)$ can be obtained naturally by Definition 2.2.

Definition 2.6 *Given positive integers $m \geq k > r \geq 1$,*

1. *$G(m, t, k, r)$ be the binary-matrix M with row-indexed (resp. column-indexed) by t-cliques of size r (resp. k) of $J(tm, t)$ such that $M(A, B) = 1$ if $A \subseteq B$ and 0 otherwise.*
2. *$G_q(m, t, k, r)$ be the binary-matrix M be with row-indexed (resp. column-indexed) by t-cliques of size r (resp. k) of $J_q(tm, t)$ such that $M(A, B) = 1$ if $A \subseteq B$ and 0 otherwise.*

Lemma 2.2 *Let W be a k-subspace of ${\mathbb{F}_q}^n$. Then the number of d-subspaces of ${\mathbb{F}_q}^n$ intersecting trivially with W is $\begin{bmatrix} n-k \\ d \end{bmatrix}_q q^{dk}$.*

Proof. Let

$$\mathcal{D} = \{A | A \in \begin{bmatrix} V \\ d \end{bmatrix}_q, A \cap W = 0\}.$$

Counting the set $\{(v_1, v_2, \ldots, v_d) |\ v_i \notin \langle W, v_1, v_2, \ldots, v_{i-1} \rangle\}$ in two ways, we have

$$(q^m - q^k)(q^m - q^{k+1}) \cdots (q^m - q^{k+d-1}) = |\mathcal{D}| \cdot (q^d - 1)(q^d - q) \cdots (q^d - q^{d-1}).$$

Hence $|\mathcal{D}| = \begin{bmatrix} n-k \\ d \end{bmatrix}_q q^{dk}$ as required. □

Lemma 2.3 *1. The number $u[m,l]_1 = u(m,l)$ of t-cliques of $J(tm,t)$ with size l is $u[m,l]_1 = u(m,l) = \binom{tm}{tl}(tl)!/(t!)^l l!$.*

2. The number $u_q(m,l)$ of t-cliques of $J_q(tm,t)$ with size l is

$$u_q(m,l) = \begin{bmatrix} tm \\ tl \end{bmatrix}_q \prod_{i=1}^{t} \begin{bmatrix} it \\ i \end{bmatrix}_q \cdot \frac{q^{t^2 l(l-1)/2}}{l!}.$$

, where $1 \le l \le m$ and

Proof. By Definition 2.2, $\{A_1, A_2, \ldots, A_l\}$ is a t-clique of $J_q(tm,t)$ with size l if and only if $A_1 + A_2 + \cdots + A_l$ is a tl-subspace of $GF(q)^{tm}$.

Let $L(m,l)$ be the number of ordered tuples $(A_1, A_2, \ldots, A_l)$ of t-subspaces of $GF(q)^{tm}$ such that $\dim(A_1 + A_2 + \cdots A_l) = tl$. Notice that the number of tl-subspaces of $GF(q)^{tm}$ is $\begin{bmatrix} tm \\ tl \end{bmatrix}_q$. Counting $L(m,l)$ directly, there are $\begin{bmatrix} tl \\ t \end{bmatrix}_q$ ways to choose A_1, then $\begin{bmatrix} tl-t \\ t \end{bmatrix}_q q^{t^2}$ ways to choose A_2 by Lemma 2.1 and so on. Thus

$$L(m,l) = \begin{bmatrix} tm \\ tl \end{bmatrix}_q \begin{bmatrix} tl \\ t \end{bmatrix}_q q^{t \cdot 0} \begin{bmatrix} tl-t \\ t \end{bmatrix}_q q^{t \cdot t} \cdots \begin{bmatrix} 2t \\ t \end{bmatrix}_q q^{t \cdot (l-2)t} \begin{bmatrix} t \\ t \end{bmatrix}_q q^{t \cdot (l-1)t}. \quad (1)$$

On the other hand, $(A_1, A_2, \ldots A_l)$ may be obtained by first picking a t-clique of $J_q(tm,t)$ with size l in $u[m,l]_q$ ways, then for each t-clique, there are $l!$ ways to get the ordered tuples $(A_1, A_2, \ldots, A_l)$. This yields

$$L(m,l) = u[m,l]_q l!. \quad (2)$$

Combining (1) and (2) gives $u[m,l]_q$ as desired. □

Theorem 2.4 ([6, Theorem 2]) *Let $\mathcal{K}$be a family of k-subsets of $[n]$ and $\alpha_d = \min(t^d, k-d)$, i.e., a 1,2,K,t-clique of J(n,k). If the minimum Hamming distance $d_H(\mathcal{K})$ between any pair of k-sets in $\mathcal{K}$ is at least $2t$, then $J(n,d,\mathcal{K})$ is $d^{\alpha_d - 1}$-disjunct.*

Theorem 2.5 *$J(n,k,d)$ is s^e - disjunct for $1 \leq s \leq d$, where e is the function of s defined by $e = \binom{k-s}{d-s} - 1$.*

Proof. For those columns of $J(n,k,d)$ indexed by $K_0, K_1, \ldots, K_s \in \binom{[n]}{k}$, let $x_i \in K_0 - K_i$, $1 \leq i \leq s$, and let S be a s-subset of K_0 containing $\{x_i | 1 \leq i \leq s\}$. Then each row indexed by $D \in \binom{[n]}{d}$ with $S \subseteq D \subseteq K_0$ is of the form $1 \cdots 1$ over K_0, and $0 \cdots 0$ over $K_1, \cdots, K_d$. Indeed, there are $\binom{k-s}{d-s}$ many such choice of D, as required.

3 New families of d^e-disjunct matrices

Given positive integers $m \geq k > d \geq 1$, . Let $M(2m,d,k)$ be the binary-matrix M with row-indexed (resp. column-indexed) by d-matchings (resp. k-matchings) of K_{2m} such that $M(A,B) = 1$ if $A \subseteq B$ and 0 otherwise. In [7], Ngo and Du proved the following results:

Theorem 3.1 ([7, Theorem 11]) *Let $g(m,l) = (\binom{2m}{2l})(2l)!/2^l l!, v = g(m,d)$ and $n = g(m,k)$. For $m \geq k > d \geq 1$, $M(m,k,d)$ is a $v \times n$ d-disjunct matrix with row weight $g(m-d, k-d)$ and column weight $\binom{k}{d}$.*

Theorem 3.2 ([7, Corollary 12]) *Given integers $m > d \geq 1$, the following hold:*

(1) *$M(m,m,d)$ is d-error-detecting and $\lfloor d/2 \rfloor$-error-correcting. Moreover,*

(2) *If the number of positives is known to be exactly d, then $M(m,m,d)$ is $(2d+1)$-error-detecting and d-error-correcting.*

With an interpretation of matchings as 2-cliques of the Johnson graph $J(n,2)$, we will give some generalizations of Ngo and Du's construction.

Find examples of $\Gamma \subseteq \binom{[n]}{k}$ with $d_H(\Gamma) \geq 2r$? study their properties?

Theorem 3.3 *Let $m \geq k > r \geq d \geq 1$, then the matrix $G(m,t,k,r)$ is a d^e-disjunct matrix of order $v \times n$ where $(v,n) = (u(m,r), u(m,k))$ and $e = \binom{k-d}{r-d} - 1$ with a constant row weight $u(m-r, k-r)$ and a constant column weight $\binom{k}{r}$.*

Proof. By Lemma 2.2, $G(m,t,k,r)$ is a $v \times n$ matrix with row weight $u(m-r,k-r)$ and column weight $\binom{k}{r}$.

Let $C_{j_0}, C_{j_1}, \ldots, C_{j_d}$ be any $d+1$ distinct columns of $G(m,t,k,r)$. For each $i \in [d]$, there is a t-subset V_i of $[tm]$ such that $V_i \in C_{j_0} \backslash C_{j_i}$. Let $E = \{V_i | i \in [d]\}$. Then $|E| \leq d$ and $E \subset C_{j_0}$ but $E \not\subseteq C_{j_i}$ for each $i \in [d]$. If $|E| = i$, the number of r-subsets of C_{j_0} containing E is $\binom{k-i}{r-i}$. Since $\binom{k-i}{r-i} \geq \binom{k-d}{r-d}$, the number of t-cliques of size r contained in C_{j_0} but not contained in C_{j_i} for each $i \in [d]$ is at least $\binom{k-d}{r-d}$. □

The following corollary shows the above e is optimal if $m > k$.

Corollary 3.4 *Let $m > k > r \geq d \geq 1$, the matrix $G(m,t,k,r)$ is d^e-disjunct, but not a d^{e+1}-disjunct matrix with $e = \binom{k-d}{r-d} - 1$.*

Proof. In order to prove that $G(m,t,k,r)$ is not a d^{e+1}-disjunct matrix, we need only to show that the maximum size of E is obtained. Since $m > k$, there exists a t-clique $T = \{A_1, A_2, \ldots, A_{k+1}\}$ with size $k+1$. Let $C_{j_i} = T \backslash \{A_i\}$ for each $i \in [d+1]$. Then $|E| = |\{A_i \mid i \in [d]\}| = d$. □

The results in Theorem 3.3 and Corollary 3.4 hold for its q-analogues too as shown below, their proofs are similar, and will be omitted.

Theorem 3.5 *Let $m \geq k > r \geq d \geq 1$, then the matrix $G_q(m,t,k,r)$ is a d^e-disjunct matrix of order $v \times n$ where $(v,n) = (u[m,r]_q, u[m,k]_q)$ and $e = \binom{k-d}{r-d} - 1$, with a constant row weight $u[m-d,k-d]_q$ and a constant column weight $\binom{k}{r}$.*

Corollary 3.6 *Let $m > k > r \geq d \geq 1$, then the matrix $G_q(m,t,k,r)$ is d^e-disjunct, but not a d^{e+1}-disjunct matrix with $e = \binom{k-d}{r-d} - 1$.*

An d-matching of K_{2m} is simply a family of size d of 2 -subsets of [n] which are pairwise disjoint. A 2-clique of $J_q(2m,2)$ of size l is the q-analogue of an l-matching of K_{2m}.

Similar to Corollary 12 in [7], $G(m,t,m,d)$ is d-error-detecting and $\lfloor d/2 \rfloor$ error-correcting.

For fixed integers $m \geq k > r$, the test to item ratio (v/n) of $G(m,t,k,r)$ (resp. $G_q(m,t,k,r)$) is a strictly decreasing function in t.

Some more examples of d^e-disjunct matrices.

Theorem 3.7 *Let $1 \leq s \leq d \leq k \leq n$. Let $1 \leq q$ and $e = \binom{k-s}{d-s} - 1$. $J(n,d,k)$ is s^e-disjunct. proofs.*

Note that $\binom{k-s}{d-s} = \binom{k-s}{k-d}$, it is a decreasing sequence.

Theorem 3.8

1. *$G_q(n,d,k)$ is s^e-disjunct.*

2. *$I_q()n,d,k$ is s^e-disjunct for $1 \leq s \leq p$, where*

$$p = \left\lceil \left(\begin{bmatrix} k \\ d \end{bmatrix}_q - \begin{bmatrix} k-1 \\ d \end{bmatrix}_q \right) \left(\begin{bmatrix} k-1 \\ d \end{bmatrix}_q - \begin{bmatrix} k-2 \\ d \end{bmatrix}_q \right)^{-1} \right\rceil, \text{ and}$$

$$e = \begin{bmatrix} k \\ d \end{bmatrix}_q - \begin{bmatrix} k-1 \\ d \end{bmatrix} - (s-1) \left(\begin{bmatrix} k-1 \\ d \end{bmatrix}_q - \begin{bmatrix} k-2 \\ d \end{bmatrix}_q \right) - 1$$

Theorem 3.9 *([],[]) For $1 \le d \le k \le n$ and $1 \le r \le k$, let $\mathcal{K}$ be a family of k-subsets of $[n]$ with the minimum Hamming distance $d_H(K)$ between any pair of k-sets in $\mathcal{K}$ is at least $2r$, then*

1. *$J(n, d, \mathcal{K})$ is $d^{\alpha_d - 1}$ - disjunct where $\alpha_d = \min(r^4, k-d)$. (Theorem 2).*

2. *$J(n, d, k, K, r)$ is s^e-disjunct if $1 \le s \le p$, where*
$$p = \left\lceil \left(\begin{bmatrix} k \\ d \end{bmatrix} - \begin{bmatrix} k-r \\ d \end{bmatrix} \right) \left(\begin{bmatrix} k-r \\ d \end{bmatrix} - \begin{bmatrix} k-2r \\ d \end{bmatrix} \right)^{-1} \right\rceil, \text{ and}$$
$$e = \begin{bmatrix} k \\ d \end{bmatrix} - \begin{bmatrix} k-r \\ d \end{bmatrix} - (s-1) \left(\begin{bmatrix} k-r \\ d \end{bmatrix} - \begin{bmatrix} k-2r \\ d \end{bmatrix} \right) - 1.$$

The following lemma is used in the proof of the following theorem.

Lemma 3.10 *Let $\mathcal{K}$ be a family of k-subsets in $[n]$ with $|K \bigcap K'| \le k - t$ for all K and K' in $\mathcal{K}$. Let $d \ge 1$ with $t \ge 1 + t/(k-d)$ and set $\alpha_d = \min(t^d, k-d)$. Then given $d+1$ k-sets $\{K_i\}_{i=0}^{d} \subset \mathcal{K}$, there are α_d d-sets $\{D_j\}_{j=1}^{\alpha_d}$ in $[n]$ such that each D_j is contained in K_0 and no D_j is connected in K_i for $1 \le i \le d$.*

4 Parameters e_d and $e_{\le d}$ for error-correcting

For a binary matrix M of order $t \times n$, let $B(D)$ denote the Boolean sum of those columns indexed by elements of $D \subseteq [n]$, and let $d_H(B(D), B(D'))$ denote the Hamming distance between $B(D)$ and $B(D')$ whenever D and D' are two distinct subsets of $[n]$. Suppose $B_d(M)$ is the binary matrix consists of columns $B(S)$ for all $S \subseteq [n]$ with $|S| \le d$. Let $d_H(B_d(M))$ be the minimum Hamming distance over all pairs of columns of $B_d(M)$. The minimum Hamming distance $d_H(B_d(M))$ is interesting for error tolerance; for example, Macula proved the following result:

Let

$$e_d = \min_{|D|=|D'|=d} d_H(B(D), B(D')),$$

and

$$e_{\leq d} = \min_{|D|=|D'|\leq d \ D,D' \text{ are antichains}} d_H(B(D), B(D')).$$

The larger the parameter $e_{\leq d}$ is, the better its capacity of error correcting is. Their values for the matrices $G(m,t,k,r)$ and $G_q(m,t,k,r)$ will be considered in this section. We first treat the case for $G(m,t,k,r)$ by giving a specific example.

Example 4.1 Let $m > k$, and let $T = \{A_1, A_2, \ldots, A_{k+1}\}$ be a t-clique of $J(tm,t)$ with size $k+1$. For each $i \in [d+1]$, suppose $B_i = T \backslash \{A_i\}$. Then each B_i is a t-clique of $J(tm,t)$ with size k. Let

$$D = \{B_1, B_2, \ldots, B_{d-1}, B_d\} \text{ and } D' = \{B_1, B_2, \ldots, B_{d-1}, B_{d+1}\}.$$

Then

$$\begin{aligned} d_H(B(D), B(D')) &= |\{R | R \in \binom{B_d}{r}, R \not\subseteq B_1, B_2, \ldots, B_{d-1}, B_{d+1}\}| \\ &+ |\{R | R \in \binom{B_{d+1}}{r}, R \not\subseteq B_1, B_2, \ldots, B_{d-1}, B_d\}| \\ &= |\{R | \{A_1, A_2, \ldots, A_{d-1}, A_{d+1}\} \subseteq R \subseteq B_d\}| \\ &+ |\{R | \{A_1, A_2, \ldots, A_{d-1}, A_d\} \subseteq R \subseteq B_{d+1}\}| \\ &= 2\binom{k-d}{r-d}. \end{aligned}$$

Theorem 4.1 *Let $m > k > r \geq d \geq 1$. Then $e_d = e_{\leq d} = 2\binom{k-d}{r-d}$ for $M = G(m,t,k,r)$.*

Proof. Given any two antichains $D = \{A_1, A_2, \ldots, A_d\}$ and $D' = \{A'_1, A'_2, \ldots, A'_d\}$.

We have

$$\begin{aligned}
e_d &= \min_{|D|=|D'|=d} d_H(B(D), B(D')) \\
&\geq \min|\{R \subseteq A_i \text{ for some } i \in [d] \text{ and } R \not\subseteq A'_j \; for \; j \in [d]\}| \\
&+ \min|\{R \subseteq A'_i \text{ for some } i \in [d] \text{ and } R \not\subseteq A_j \; for \; j \in [d]\}| \\
&\geq 2\binom{k-d}{r-d}
\end{aligned}$$

by Theorem 3.3.

On the other hand, Example 1 shows $e_d \leq 2\binom{k-d}{r-d}$. Hence $e_d = 2\binom{k-d}{r-d}$ as required.

To show $e_{\leq d} = 2\binom{k-d}{r-d}$, we consider two antichains $D = \{A_1, A_2, \ldots, A_u\}$ and $D' = \{A'_1, A'_2, \ldots, A'_v\}$ where $u, v \leq d$. Without loss of generality, we may assume that $D_u \notin D'$ and $D'_v \notin D$. By Theorem 3.3 there exist at least $\binom{k-v}{r-v}$ t-cliques with size r contained in A_u but not in A'_j for each $j \in [v]$. By the symmetry, we have $e_{\leq d} \geq \binom{k-v}{r-v} + \binom{k-u}{r-u}$. Note that $\binom{k-s}{r-s} \geq \binom{k-d}{r-d}$ if $s \leq d$. Hence $e_{\leq d} \geq 2\binom{k-d}{r-d}$. On the other hand, by definition, $e_{\leq d} \leq e_d = 2\binom{k-d}{r-d}$. This yields $e_{\leq d} = 2\binom{k-d}{r-d}$. □

Similar result holds for $G_q(m, t, k, r)$. The proof is similar to that of Theorem 4.2 and will be omitted.

Theorem 4.2 *Let $m > k > r \geq d \geq 1$. Then $e_d = e_{\leq d} = 2\binom{k-d}{r-d}$ for $M = G_q(m, t, k, r)$.*

References

[1] A. E. Brouwer, A. M. Cohen and A. Neumaier, *Distance-Regular Graphs*, Springer Verlag, Berlin, Heidelberg, 1989.

[2] A.G. D'yachkov, F.K. Hwang, A.J. Macula, P.A. Vilenkin, C. Weng, A construction of pooling designs with some happy surprises, J. Computational Biology, 12 (2005), 1129-1136.

[3] T. Huang and C. Weng, A note on decoding of superimposed codes, J. Comb. Optim. 7 (2003), no. 4, 381-384.

[4] T. Huang and C. Weng, Pooling spaces and non-adaptive pooling designs, Discrete Math. 282 (2004), 163-169.

[5] A.J. Macula, A simple construction of d-disjunct matrices with certain constant weights, Discrete Math. 162 (1996), 311-312.

[6] A.J. Macula, Error-corerecting nonadaptive group testing with d^e-disjunct matrices, Discrete Appl. Math. 80 (1997), 217-222.

[7] H. Ngo and D. Du, New constructions of non-adaptive and error-tolerance pooling designs, Discrete Math. 243 (2002), 161-170.

[8] H. Ngo and D. Zu, A survey on combinatorial group testing algorithms with applications to DNA library screening, DIMACS Ser. Discrete Math. Theoretical Comp. Sci. 55 (2000), 171-182.

[9] A.D'yachkov and P. Vilenkin, A. Macula, Nonadaptive Group Testing with Error-Correcting d^e-Disjunct Inclusion Matrices, Bolyai Society Mathematical Studies.