行政院國家科學委員會專題研究計畫　成果報告

不可壓縮流的有限元後驗估計與多重網格計算

計畫類別： 個別型計畫
計畫編號： NSC94-2119-M-009-003-
執行期間： 94 年 11 月 01 日至 95 年 07 月 31 日
執行單位： 國立交通大學應用數學系(所)

計畫主持人： 吳金典

報告類型： 精簡報告
報告附件： 出席國際會議研究心得報告及發表論文
處理方式： 本計畫可公開查詢

中　華　民　國 95 年 10 月 27 日

行政院國家科學委員會補助專題研究計畫 v 成果報告
□期中進度報告

（計畫名稱）

不可壓縮流的有限元後驗估計與多重網格計算

計畫類別：v 個別型計畫　　□ 整合型計畫
計畫編號：NSC 94－2119－M－009－003－
執行期間：2005 年 11 月 1 日 至　2006 年 07 月 31 日

計畫主持人：吳金典
共同主持人：
計畫參與人員：

成果報告類型(依經費核定清單規定繳交)：v 精簡報告　□完整報告

本成果報告包括以下應繳交之附件：
□赴國外出差或研習心得報告一份
□赴大陸地區出差或研習心得報告一份
v 出席國際學術會議心得報告及發表之論文各一份
□國際合作研究計畫國外研究報告書一份

執行單位：國立交通大學應用數學系

中　華　民　國　95 年 10　月　27　日

（一）　計畫中文摘要。(五百字以內)

在本計劃內，我們將發展一個解三維 Navier-Stokes方程有效能和準確的數值方法來模擬複雜區域內不可壓縮流的行為。 為了解 Navier-Stokes 方程, 首先，我们用隱式的 Euler 方法與 Crank-Nicolson 方法來對時間做離散. 因此而產生的非線性對流項則以 Picard 迭代法或者 Simo-Amero 方法將其線性化. 下一步, 再以 Streamline-Diffusion 的有限元素法 (SDFEM) 離散空間區域. 其中簡單的 P1/P 1元素分別被用來離散速度和壓力兩個變量. 此離散法由於流線方向有額外的穩定度，使得離散所得的線性系統也因此比一般由 Galerkin 有限元方法離散所得的線性系統更加的穩定.

再來，我們將集中精力探討怎樣有效地求得這個線性系統的解. 在本計劃中, 兩種方法將被考慮. 首先,我們考慮一個常用的方法, 在那裡, 一步的 block Gauss-Seidel 迭代被用來作為線性系統的 preconditioner. 在每一個 block 中則以 GMRES 或 MINRES 來求解並以多重網格法來改善系統的條件數以增加 GMRES 和 MINRES的收斂速度. 進而縮短數值計算所需的時間. 另一個方法, 我們分別于線性系統的左右兩邊乘上特殊的矩陣來改善系統的條件數. 對所

得的新系統再以 GMRES 迭代法求解. 我們發現第 2 種方法的優勢在於:(i) 每個 block 中我們只需要一個類似 Poisson 的 solver, (ii) 計算過程中只要時間的離散尺度和網格的大小固定, 在 (i) 中類似 Poisson 的矩陣並不需隨時間改變而重新計算, (iii) 代數性的多重網格解是可以使用的, 也就是說我們不需產生粗網格. 因此程式的複雜度也大大的簡化了.

最後， 為了增加數值解的準確度，自適應網格和隨網格大小自動調整的時間尺度也將一並被考慮. 我們也將用許多標準的流體測試問題 (包括 driven cavity flow, flow in a backward-facing step 和 flow around a cylinder 等), 來檢測我們的不可壓縮流數值模擬器的準確性及穩定性.

(二) 計畫英文摘要。(五百字以內)

In this proposal, we would like to develop a robust and accurate 3D Navier-Stokes (NS) solver for simulating the incompressible flow in a complex domain. For solving the NS equation, first, both the implicit backward Euler scheme and the 2nd order

Crank-Nicolson scheme will be employed on time domain. The nonlinear convection term is linearized and treated by Picard iterations or the Simo-Amero scheme. Next, the streamline-diffusion finite element method (SDFEM) will be used for discretization on the spatial domain, where the simple P1/P1 element is used to discretize the velocity and pressure variables, respectively. Due to the extra stabilization along the streamline direction, the resulting bilinear form is consistent. As a result, the discrete linear system is stable.

Then, we will focus on how to solve the linear system efficiently. Two approaches will be considered. First, we consider a heuristic approach where one step of block Gauss-Seidel iteration is used as a preconditioner of the system. The multigrid-preconditioned GMRES and MINRES are then used in the block solvers to accelerate the overall computation time. Next, our two preconditioners are applied to the linear system, one on the left and one on the right. The resulting system is again solved by GMRES. The advantages of the second

approach includes (i) only block solver which is similar to the Poisson solver is needed, (ii) no reassembling of matrices is needed for this block solver during the time evolution, as long as the time step and the mesh are fixed, and (iii) algebraic multigrid solver can be applied, i.e. no coarse grids need to be generated a priori.

Finally, in order to increase the solution accuracy, adaptive mesh refinement and adaptive time-stepping strategies will be considered. Many benchmark flow problems including driven cavity flows, flows in a backward-facing step and flows around a cylinder will be tested to demonstrate the robustness and accuracy of the developed solver.

(三) 計畫執行成果:

見下列附件。

Available online at www.sciencedirect.com
ScienceDirect
Applied Numerical Mathematics ••• (••••) •••–•••
APPLIED NUMERICAL MATHEMATICS
www.elsevier.com/locate/apnum

An efficient semi-coarsening multigrid method for variable diffusion problems in cylindrical coordinates

Ming-Chih Lai *, Chin-Tien Wu, Yu-Hou Tseng

Department of Applied Mathematics, National Chiao Tung University, 1001, Ta Hsueh Road, Hsinchu 300, Taiwan

Abstract

In this paper, we present an efficient multigrid (MG) algorithm for solving the three-dimensional variable coefficient diffusion equation in cylindrical coordinates. The multigrid V-cycle combines a semi-coarsening in azimuthal direction with the red-black Gauss–Seidel plane (radial-axial plane) relaxation. On each plane relaxation, we further semi-coarsen the axial direction with red-black line relaxation in the radial direction. We also prove the convergence of two-level MG with plane Jacobi relaxation. Numerical results show that the present multigrid method indeed is scalable with the mesh size.

Keywords: Multigrid method; V-cycle; Variable diffusion equation; Cylindrical coordinates

1. Introduction

The variable coefficient elliptic equation in three-dimensional domain arises in many physical applications. The heat transfer in heterogeneous material where the thermal conductivity depends on the position is one of the classical examples. The fluid flows with non-constant viscosity is another application. Very often, the geometry we consider is no longer Cartesian, say a circular cylinder instead. Thus, it is more convenient to write the equation in cylindrical coordinates.

In this paper, we consider the following variable coefficient diffusion problem written in cylindrical coordinates on a 3D domain $\Omega = \{(r,\theta,z) \mid 0 < r < 1,\ 0 \leqslant \theta < 2\pi,\ 0 < z < 1\}$ as

$$-\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\beta\frac{\partial u}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\frac{\beta}{r}\frac{\partial u}{\partial\theta}\right)+r\frac{\partial}{\partial z}\left(\beta\frac{\partial u}{\partial z}\right)\right]=f(r,\theta,z) \quad \text{in } \Omega, \tag{1}$$

$$u(1,\theta,z)=g(\theta,z), \qquad u(r,\theta,0)=p(r,\theta), \qquad u(r,\theta,1)=q(r,\theta). \tag{2}$$

Here the diffusion coefficient $\beta(r,\theta,z)$ is inhomogeneous and satisfies $0 < \mu \leqslant \beta(r,\theta,z) \leqslant 1$. The upper bound is not a restriction of the method discussed in this paper since we can re-scale both sides of Eq. (1) by the maximum of β.

When we solve Eq. (1) numerically, the first issue called the coordinate singularity arises. This is because the equation is not valid at $r = 0$ when it is written in cylindrical coordinates. In [8], Lai et. al. have developed a FFT-based fast direct solver for the Poisson equation (a special case of $\beta(r,\theta,z) = 1$ in Eq. (1)). The solver relies on the truncated

* Corresponding author.
E-mail address: mclai@math.nctu.edu.tw (M.-C. Lai).

0168-9274/$30.00 © 2006 IMACS. Published by Elsevier B.V. All rights reserved.
doi:10.1016/j.apnum.2006.07.019

Fourier series expansion, where the partial differential equations of Fourier coefficients are solved by the standard centered difference scheme under a shifted grid in the radial direction. The method handles the coordinate singularity without special treatment and the resultant matrix equations can be solved by the fast direct solver such as those provided in public software package—FISHPACK [1]. However, when the diffusion coefficient is inhomogeneous, solving the numerical solution for Eq. (1) is a different story. Since now the elliptic equation has a variable diffusion coefficient, we are unable to write the solution as Fourier series expansion, which means the fast Fourier transform (FFT) cannot be called directly. Furthermore, the equation is not a separable type, the resultant linear system after the discretization cannot be solved by the fast direct solver as mentioned before.

Multigrid (MG) methods are known to be very efficient for solving the elliptic problems. Its main idea consists of applying simple relaxations on the fine grid to smooth the error and correcting the smooth error on the coarse grid, on which the smooth error is relatively oscillatory and can be eliminated more effectively by relaxations. It has been shown that the convergence rates of traditional MG methods are independent of the mesh size for many elliptic problems [2,3]. However, for solving problems with strong anisotropic diffusion or solving problems on anisotropic meshes, the convergence of the MG methods is seriously deteriorated [4,7,13]. The reasons contributing to the slow multigrid convergence in such problems are, first, the traditional Jacobi and Gauss–Seidel relaxations fail to smooth the errors in some directions of the domain and, second, the standard coarse grids cannot represent the errors that are highly frequent in these directions. Many cures have been proposed to improve the convergence rate. For some two-dimensional anisotropic problems on the Cartesian coordinates, Brandt, Hackbush and Mulder [4,7,11] have obtained robust convergence of multigrid in which coarse grids from semi-coarsening are employed. For three-dimensional anisotropic problems, Dendy [6], Llorente and Melson [10], and Washio and Oosterlee [12] obtained robust convergence by using semi-coarsened grids and the alternating plane relaxations in their MG methods. Schaffer [14] also developed an efficient semi-coarsening multigrid method for symmetric and nonsymmetric elliptic PDEs with highly discontinuous and anisotropic coefficients in two- and three-dimensional Cartesian domains. In this paper, we extend the discretization used in [9] to the 3D variable diffusion equation in cylindrical coordinates (1)–(2), and present an efficient semi-coarsening MG algorithm to solve the resultant linear system.

The rest of this paper is as follows. In Section 2, we introduce the finite difference discretization to Eq. (1). In Section 3, we first review some elements of the multigrid method and then provide a two-level convergence analysis for the semi-coarsening MG with Jacobi plane relaxation. The numerical results including the accuracy check and the detailed performance comparison for the present multigrid algorithm are shown in Section 4.

2. Finite difference discretization

We define the same grid points in the radial, azimuthal and axial directions as in [8] by

$$r_i = (i - 1/2)\Delta r, \qquad r_{i+1/2} = r_i + \Delta r/2, \qquad r_{i-1/2} = r_i - \Delta r/2, \tag{3}$$

$$\theta_j = j\Delta\theta, \qquad \theta_{j+1/2} = \theta_j + \Delta\theta/2, \qquad \theta_{j-1/2} = \theta_j - \Delta\theta/2, \tag{4}$$

$$z_k = k\Delta z, \qquad z_{k+1/2} = z_k + \Delta z/2, \qquad z_{k-1/2} = z_k - \Delta z/2, \tag{5}$$

where $\Delta r = 2/(2n_r + 1)$, $\Delta\theta = 2\pi/n_\theta$ and $\Delta z = 1/(n_z + 1)$. By the choice of the radial mesh width, the boundary values are defined on the grid points. Let the discrete values be denoted by $u_{ijk} \approx u(r_i, \theta_j, z_k)$, $f_{ijk} \approx f(r_i, \theta_j, z_k)$, $g_{jk} \approx g(\theta_j, z_k)$, $p_{ij} \approx p(r_i, \theta_j)$ and $q_{ij} \approx q(r_i, \theta_j)$. The values at the half-integered points are defined in a similar fashion, such as $\beta_{i+1/2,j,k} \approx \beta(r_{i+1/2}, \theta_j, z_k)$. Using the centered difference method to discretize Eq. (1), we have

$$\begin{aligned} -\Bigg[&\left(r_{i+1/2}\beta_{i+1/2,j,k}\frac{u_{i+1,j,k} - u_{i,j,k}}{\Delta r} - r_{i-1/2}\beta_{i-1/2,j,k}\frac{u_{i,j,k} - u_{i-1,j,k}}{\Delta r} \right) \Big/ \Delta r \\ &+ \left(\frac{\beta_{i,j+1/2,k}}{r_i}\frac{u_{i,j+1,k} - u_{i,j,k}}{\Delta\theta} - \frac{\beta_{i,j-1/2,k}}{r_i}\frac{u_{i,j,k} - u_{i,j-1,k}}{\Delta\theta} \right) \Big/ \Delta\theta \\ &+ r_i \left(\beta_{i,j,k+1/2}\frac{u_{i,j,k+1} - u_{i,j,k}}{\Delta z} - \beta_{i,j,k-1/2}\frac{u_{i,j,k} - u_{i,j,k-1}}{\Delta z} \right) \Big/ \Delta z \Bigg] = r_i f_{i,j,k}. \end{aligned} \tag{6}$$

Among the above representation, the numerical boundary values are given by $u_{n_r+1,j,k} = g_{j,k}$, $u_{i,j,0} = p_{i,j}$, $u_{i,j,n_z+1} = q_{i,j}$, and $u_{i,0,k} = u_{i,n_\theta,k}$, $u_{i,n_\theta+1,k} = u_{i,1,k}$ since u is 2π periodic in the azimuthal direction. At $i = 1$,

To cite this article: Ming-Chih Lai, et al., An efficient semi-coarsening multigrid method for variable diffusion problems in cylindrical coordinates, Applied Numerical Mathematics (2006), doi:10.1016/j.apnum.2006.07.019

we have immediately observed from Eq. (3) that $r_{1/2} = 0$, so the coefficient of $u_{0,j,k}$ is zero. This implies that the scheme does not need any extrapolation for the inner numerical boundary value $u_{0,j,k}$ so that there is no pole condition needed.

Let us order the unknowns to a solution vector u, the resultant linear system of (6) can be written as $Au = f$. Here the matrix A is given by

$$A = I_\theta \otimes B_{rz} + C_\theta \otimes I_{rz}, \tag{7}$$

where $\otimes$ is the regular Kronecker product, B_{rz} is the matrix with a five-point stencil obtained from discretization of the operator $\frac{\partial}{\partial r}(r\beta\frac{\partial u}{\partial r}) + \frac{\partial}{\partial z}(r\beta\frac{\partial u}{\partial z})$, C_θ is the circulant matrix with a three-point stencil obtained from discretization of the operator $\frac{\partial}{\partial\theta}(\frac{\beta}{r}\frac{\partial u}{\partial\theta})$, and I_θ and I_{rz} are the identity matrix of size $n_\theta \times n_\theta$ and $n_r \cdot n_z \times n_r \cdot n_z$, respectively. Clearly, in the region close to the origin, the problem (1) is strongly anisotropic, namely, the diffusion in the azimuthal direction is much greater than that in other directions. As a result, the matrix A becomes indefinite as the grid space decreases. Therefore, a scalable and efficient solver for the linear system $Au = f$ is desired.

3. Semi-coarsening multigrid method

In this paper, we would like to solve the aforementioned linear system by the semi-coarsening MG with plane relaxations. In the following, firstly, the general multigrid V-cycle algorithm is introduced, and then the convergence of a two-level V-cycle of the MG method is proved under the assumption that the solution of Eq. (1) is smooth enough so that the discretization error of the scheme (6) is second-order accurate.

Given a sequence of meshes $\{G^\ell\}$, let V^ℓ be the vector space of the nodal variables on G^ℓ, A^ℓ the matrix obtained from the difference equation (6) on G^ℓ, and w^ℓ the initial guess. Let MG_1 represents the direct solver on the coarsest grid G^1. A typical multigrid V-cycle on ℓ-level is shown recursively in Algorithm 1. Here, f^ℓ is the right-hand side obtained from (6) on G^ℓ, $(M^\ell)^{-1}$ represents the smoothing operator and the operators $I_\ell^{\ell-1}$ (restriction) and $I_{\ell-1}^\ell$ (prolongation) represent the grid transfers between $G^{\ell-1}$ and G^ℓ. The elements of the multigrid V-cycle are described as follows.

Algorithm 1. Multigrid V-cycle $MG_\ell(w^\ell, f^\ell)$

(1) Set $u^\ell = w^\ell$;
(2) (Pre-smoothing) $u^\ell = u^\ell + (M^\ell)^{-1}(f^\ell - A^\ell u^\ell)$, ν_1 times;
(3) (Restriction) $\bar{f}^\ell = I_\ell^{\ell-1}(f^\ell - A^\ell u^\ell)$;
(4) (Coarser grid correction) $q_1 = MG_{\ell-1}(0, \bar{f}^\ell)$;
(5) (Prolongation) $\bar{q}_1 = I_{\ell-1}^\ell q_1$, and set $u^\ell = u^\ell + \bar{q}_1$;
(6) (Post-smoothing) $u^\ell = u^\ell + (M^\ell)^{-1}(f^\ell - A^\ell u^\ell)$ ν_2, times;
(7) Set $w^\ell = MG_\ell(w^\ell, f^\ell) = u^\ell$.

For the MG considered here, (i) the coarse grids $G^{\ell-1}$ are obtained from semi-coarsening the fine grid G^ℓ along the azimuthal direction, (ii) the prolongation operator $I_{\ell-1}^\ell$ is the linear interpolation along the azimuthal direction and the restriction operator $I_\ell^{\ell-1}$ is the transpose of the interpolation, (iii) the matrix $A^{\ell-1}$ is obtained from discretization scheme (6) on $(\ell-1)$-level grid $G^{\ell-1}$ and (iv) the smoothing operator M^ℓ consists of the Jacobi, or Gauss–Seidel plane relaxations. From now on, we use the notations $MG_J(\nu_1, \nu_2)$ and $MG_{\mathrm{GS}}(\nu_1, \nu_2)$ to represent the different versions of the above multigrid method in which ν_1 times of pre-smoothing and ν_2 times of post-smoothing by plane Jacobi and plane Gauss–Seidel smoothers, respectively, are applied.

3.1. The convergence of two-level MG

In the following, we present the convergence analysis for the two-level V-cycle MG method with Jacobi plane relaxations. To simplify our convergence proof, we assume the number of grid points used in r, θ and z directions are $n_r = n_\theta/2 = n_z = n$. Let $\|\cdot\|_2$ denote the standard vector 2-norm or matrix 2-norm depending on which quantities are considered. Let $\|w\|_Q = \sqrt{(Qw, w)}$ denote the Q-weighted Euclidean norm for $w \in V^\ell$, where Q is a symmetric

To cite this article: Ming-Chih Lai, et al., An efficient semi-coarsening multigrid method for variable diffusion problems in cylindrical coordinates, Applied Numerical Mathematics (2006), doi:10.1016/j.apnum.2006.07.019

positive definite matrix and $(\cdot,\cdot)$ is the inner product on V^ℓ. For any linear transformation T from V^ℓ to V^ℓ, the matrix Q-norm, denoted by $\|T\|_Q$, is the matrix 2-norm associated with the Q-weighted Euclidean norm.

First, let us show that the Jacobi plane relaxation is convergent. Recall that the Jacobi plane relaxation is obtained from the matrix splitting

$$A^\ell = M^\ell - N^\ell, \tag{8}$$

where

$$M^\ell = I_\theta \otimes B_{rz} + D_\theta \otimes I_{rz} \quad \text{and} \quad N^\ell = (D_\theta - C_\theta) \otimes I_{rz}, \tag{9}$$

and the matrix D_θ is the diagonal part of C_θ. The matrices M^ℓ and N^ℓ can also be written in the following block form

$$M^\ell = \begin{bmatrix} M_1 & 0 & & & \\ 0 & M_2 & 0 & & \\ & \ddots & \ddots & \ddots & \\ & & 0 & M_{2n-1} & 0 \\ & & & 0 & M_{2n} \end{bmatrix}, \qquad N^\ell = \begin{bmatrix} 0 & N_1 & 0 & 0 & N_{2n} \\ N_1 & 0 & N_2 & 0 & 0 \\ 0 & N_2 & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & 0 & N_{2n-1} \\ N_{2n} & 0 & 0 & N_{2n-1} & 0 \end{bmatrix}, \tag{10}$$

where M_j are the block tridiagonal matrices with the stencils

$$\begin{cases} \Big[-\frac{r_{i-1/2}\beta_{i-1/2,j,k}}{\Delta r^2}, \frac{r_{i-1/2}\beta_{i-1/2,j,k}+r_{i+1/2}\beta_{i+1/2,j,k}}{\Delta r^2} + \frac{r_i\beta_{i,j,k-1/2}+r_i\beta_{i,j,k+1/2}}{\Delta z^2} + \frac{\beta_{i,j-1/2,k}+\beta_{i,j+1/2,k}}{r_i\Delta\theta^2}, \\ -\frac{r_{i+1/2}\beta_{i+1/2,j,k}}{\Delta r^2}\Big]_{i,k} & \text{in the main diagonal blocks,} \\ \Big[-\frac{r_i\beta_{i,j,k-1/2}}{\Delta z^2}\Big]_{i,k} & \text{in the upper off-diagonal blocks,} \\ \Big[-\frac{r_i\beta_{i,j,k+1/2}}{\Delta z^2}\Big]_{i,k} & \text{in the lower off-diagonal blocks,} \end{cases} \tag{11}$$

and the matrices N_j, $j = 1 \ldots 2n$, are the block diagonal matrices with stencils $[\frac{\beta_{i,j+1/2,k}}{r_i\Delta\theta^2}]_{i,k}$, respectively. Let $S_J^\ell = (M^\ell)^{-1}N^\ell$ be the error reduction operator of the Jacobi plane relaxation. It is clear that, due to the given Dirichlet boundary conditions, M^ℓ and A^ℓ are irreducible M-matrices [15]. Therefore, $(M^\ell)^{-1}$ and $(A^\ell)^{-1}$ are positive. Since N^ℓ is non-negative, the splitting (8) is a regular splitting. As a result, the Jacobi plane relaxation S_J is convergent since $\|S_J^\ell\|_2 < 1$ ([15] Theorem 3.29).

Throughout this section, we define the matrix R^ℓ as

$$R^\ell = I_\theta \otimes R_{rz}^\ell = I_\theta \otimes \big(I_z \otimes R_r^\ell\big), \tag{12}$$

where $R_r^\ell = \operatorname{diag}(r_i)$. Obviously, one also has $\|S_J^\ell\|_{R^\ell} < 1$.

Lemma 3.1. *Suppose M_j, $j = 1, \ldots, 2n$ and $N_l, l = 1, \ldots, 2n$ are the block matrices defined in* (10)*. The inequality*

$$\big\|M_j^{-1}N_l\big\|_{R_{rz}^\ell} < \frac{\kappa}{2\mu} \tag{13}$$

holds, where $\kappa < 1$ is a constant depending on $\beta(r,\theta,z)$ and μ is the lower bound of $\beta(r,\theta,z)$.

Proof. The matrix M_j can be rewritten as

$$M_j = \frac{1}{\Delta\theta^2}\big(R_{rz}^\ell\big)^{-1}\overline{B}_j(I_{rz} + C),$$

where $\overline{B}_j$ is a diagonal matrix whose diagonal entries $[\bar{\beta}_j]_{i,k}$ are defined by $\bar{\beta}_j = \beta_{i,j-1/2,k} + \beta_{i,j+1/2,k}$, and the matrix $C = \Delta\theta^2\overline{B}_j^{-1}R_{rz}^\ell M_j - I_{rz}$. Clearly, one has

$$\big\|M_j^{-1}N_l\big\|_{R_{rz}^\ell} \leqslant \Big\|\big[I_{rz} + \big(R_{rz}^\ell\big)^{1/2}C\big(R_{rz}^\ell\big)^{-1/2}\big]^{-1}\Delta\theta^2\overline{B}_j^{-1}N_l\Big\|_2 \tag{14}$$

$$= \left\|\big[I_{rz} + \big(R_{rz}^\ell\big)^{1/2}C\big(R_{rz}^\ell\big)^{-1/2}\big]^{-1}\left[\frac{\beta_{i,l+1/2,k}}{\bar{\beta}_j}\right]_{i,k}\right\|_2. \tag{15}$$

To cite this article: Ming-Chih Lai, et al., An efficient semi-coarsening multigrid method for variable diffusion problems in cylindrical coordinates, Applied Numerical Mathematics (2006), doi:10.1016/j.apnum.2006.07.019

From stencils (11) and the given Dirichlet boundary conditions in (2), one can conclude that C is an irreducibly diagonal dominant matrix. Therefore, the matrix C is positive definitive. Moreover, $(R^{\ell}_{rz})^{1/2}C(R^{\ell}_{rz})^{-1/2}$ is also positive definite. Let $\Lambda_{\min}(\overline{C})$ denote the smallest eigenvalue of the matrix $\overline{C} = (R^{\ell}_{rz})^{1/2}C(R^{\ell}_{rz})^{-1/2}$. Obviously, the assumption that $\mu < \beta \leqslant 1$ and (14) imply

$$\left\| M_j^{-1}N_l \right\|_{R^{\ell}_{rz}} \leqslant \frac{1}{1+\Lambda_{\min}(\overline{C})} \left\| \left[\frac{\beta_{i,l+1/2,k}}{\bar{\beta}_j} \right]_{i,k} \right\|_2 \tag{16}$$

$$< \frac{\kappa}{2\mu}. \tag{17}$$

Hence, the inequality (13) holds. □

Corollary 3.2. *For the constant diffusion case, suppose the mesh width satisfies the relation as* $\Delta r = \eta_1 \Delta z = \eta_2 \Delta\theta \ll 1$, *where* $\eta_1 = \frac{2n_z+2}{2n_r+1}$ *and* $\eta_2 = \frac{n_\theta}{(2n_r+1)\pi}$ *are the grid aspect ratios of* Δz *and* $\Delta\theta$ *with respect to* Δr, *and* $\eta_1 \approx \eta_2 \approx 1$. *Then, we have*

$$\left\| M_j^{-1}N_l \right\|_{R^{\ell}_{rz}} \leqslant \frac{\kappa}{2}, \tag{18}$$

and $\kappa \approx (1 - c\Delta r^2)$ *for some constant* $c > 0$ *independent of mesh size. As a result of* (18)*, the plane Jacobi relaxation converges with*

$$\left\| S^{\ell}_J \right\|_{R^{\ell}} \leqslant 1 - c\Delta r^2. \tag{19}$$

Proof. First, the stencil of the matrix $\overline{C}$ in the proof of Lemma 3.1 can be rewritten as

$$\overline{C} = \left(\frac{\Delta\theta}{\Delta r}\right)^2 \widetilde{C}, \tag{20}$$

where $\widetilde{C}$ is a block tridiagonal matrix with $[-r_i\sqrt{r_i r_{i-1/2}}, (2+2\eta_1^2)r_i^2, -\eta_1^2 r_i\sqrt{r_i r_{i+1/2}}]$ in the diagonal blocks, and $[-r_i^2]$ and $[-\eta_1^2 r_i^2]$ in the lower and upper off-diagonal blocks, respectively. By the Gersgorin theorem, we obtain the following estimates as

$$\begin{aligned} \left|\Lambda_{\min}(\widetilde{C}) - \left(2+2\eta_1^2\right)r_i^2\right| &\leqslant r_i\left(\left(1+\eta_1^2\right)r_i + \sqrt{r_i r_{i-1/2}} + \eta_1^2\sqrt{r_i r_{i+1/2}}\right) \\ &\leqslant \left(2+2\eta_1^2\right)r_i^2 - c_0 r_i\Delta r, \quad \text{for some constant } c_0 > 0 \\ &\leqslant \left(2+2\eta_1^2\right)r_i^2 - 0.5c_0\Delta r^2. \end{aligned} \tag{21}$$

Clearly, from (20) and (21), there exists a constant $c_1 \approx 0.5c_0(\frac{\Delta\theta}{\Delta r})^2 = 0.5c_0(\frac{1}{\eta_2})^2 > 0$ such that $\Lambda_{\min}(\overline{C}) \geqslant c_1\Delta r^2$. Next, let $e_1 = [1, 0, \ldots, 0]$. Since

$$\begin{aligned} \|\overline{C}e_1\|_2 &= \left(\frac{\Delta\theta}{\Delta r}\right)^2 \left(\left((2+2\eta_1)r_1^2\right)^2 + \eta_1^2 r_2^2 r_2 r_{3/2} + r_1^4\right)^{1/2} \\ &= \frac{\sqrt{16\eta_1^2 + 60\eta_1 + 5}}{4\eta_2^2}\Delta r^2 = c_2\Delta r^2 \end{aligned}$$

from the positive definiteness of $\overline{C}$, one can conclude that $\Lambda_{\min}(\overline{C}) \leqslant c_2\Delta r^2$, for some constant $c_2 > 0$. By using the upper and lower bounds of $\Lambda_{\min}(\overline{C})$, the inequality (16) can now be shown easily as

$$\begin{aligned} \left\| M_j^{-1}N_l \right\|_{R^{\ell}_{rz}} &\leqslant \frac{1}{2(1+\Lambda_{\min}(\overline{C}))} \leqslant \frac{1}{2}\left(1 - \Lambda_{\min}(\overline{C}) + \Lambda_{\min}(\overline{C})^2\right) \\ &\leqslant \frac{1}{2}\left(1 - c_1\Delta r^2 + c_2^2\Delta r^4\right) \leqslant \frac{1}{2}\left(1 - c\Delta r^2\right), \end{aligned}$$

for some constant $c > 0$. The last inequality arises easily from the assumption that the radial mesh Δr is much smaller than one. The inequality (19) can now be derived directly from the Gersgorin theorem. □

To cite this article: Ming-Chih Lai, et al., An efficient semi-coarsening multigrid method for variable diffusion problems in cylindrical coordinates, Applied Numerical Mathematics (2006), doi:10.1016/j.apnum.2006.07.019

Now, we are ready to prove the convergence of the 2-level MG. First, the error reduction operator of the 2-level MG for solving the linear system obtained from (6) can be written as

$$E_{\mathrm{MG}} = \big((A^{\ell})^{-1} - I_{\ell-1}^{\ell}\big(A^{\ell-1}\big)^{-1} I_{\ell}^{\ell-1}\big) R^{\ell} A^{\ell} \big(S_J^{\ell}\big)^{\nu},$$

where ν is the number of relaxations. To prove the convergence of MG is independent of mesh size, we need to show that there is some number $\nu > 0$ such that

$$\|E_{\mathrm{MG}}\|_{R^{\ell}} < 1. \tag{22}$$

In the following, we let e^{ℓ} be an arbitrary error vector in V^{ℓ} satisfying $\|e^{\ell}\|_{R^{\ell}} = 1$, and $(f^{\ell})^{s} = A^{\ell}(S_J^{\ell})^{\nu} e^{\ell}$ be the corresponding residual of the relaxed error. Clearly, one has

$$\begin{aligned}
&\big(R^{\ell}\big[\big((A^{\ell})^{-1} - I_{\ell-1}^{\ell}(A^{\ell-1})^{-1} I_{\ell}^{\ell-1}\big) R^{\ell} A^{\ell} (S_J^{\ell})^{\nu}\big] e^{\ell}, e^{\ell}\big) \\
&\quad = \big(R^{\ell}\big[\big((R^{\ell})^{-1} A^{\ell}\big)^{-1} - I_{\ell-1}^{\ell}\big(\big(R^{\ell-1}\big)^{-1} A^{\ell-1}\big)^{-1} I_{\ell}^{\ell-1}\big]\big(f^{\ell}\big)^{s}, e^{\ell}\big) \\
&\quad \leqslant \big\|\big[\big((R^{\ell})^{-1} A^{\ell}\big)^{-1} - I_{\ell-1}^{\ell}\big(\big(R^{\ell-1}\big)^{-1} A^{\ell-1}\big)^{-1} I_{\ell}^{\ell-1}\big]\big(f^{\ell}\big)^{s}\big\|_{R^{\ell}}.
\end{aligned}$$

Since $(R^{\ell})^{-1}A^{\ell}$ and $(R^{\ell-1})^{-1}A^{\ell-1}$ represent the discrete difference operators in (6) on grids G^{ℓ} and $G^{\ell-1}$, respectively, the truncation errors are $O(\Delta r^2)$ (the meshes $\Delta\theta$ and Δz are the same order as Δr) if the solution is smooth. Thus, there is a constant $c_3 > 0$ such that

$$\begin{aligned}
\|E_{\mathrm{MG}}\|_{R^{\ell}} &\leqslant c_3 \Delta r^2 \sup_{e^{\ell}} \big(R^{\ell} A^{\ell} \big(S_J^{\ell}\big)^{\nu} e^{\ell}, A^{\ell} \big(S_J^{\ell}\big)^{\nu} e^{\ell}\big)^{1/2} \\
&\leqslant c_3 \big\|\Delta r^2 A^{\ell} \big(S_J^{\ell}\big)^{\nu}\big\|_{R^{\ell}} \leqslant c_3 \big\|\Delta r^2 A^{\ell} S_J^{\ell}\big\|_{R^{\ell}} \big\|\big(S_J^{\ell}\big)^{\nu-1}\big\|_{R^{\ell}}.
\end{aligned} \tag{23}$$

Next, since $A^{\ell} S_J^{\ell} = N^{\ell}(I - S_J^{\ell})$ and

$$\big\|\Delta r^2 N^{\ell}\big\|_{R^{\ell}} \leqslant 2 \left\| \left[\frac{\beta_{i,l+1/2,k}}{\pi^2} \right]_{i,k} \right\|_2 \leqslant \frac{2}{\pi^2},$$

clearly, the inequality (23) implies

$$\|E_{\mathrm{MG}}\|_{R^{\ell}} \leqslant \frac{2c_3}{\pi^2} \big\|I - S_J^{\ell}\big\|_{R^{\ell}} \big\|\big(S_J^{\ell}\big)^{\nu-1}\big\|_{R^{\ell}} \leqslant \frac{4c_3}{\pi^2} \big\|S_J^{\ell}\big\|_{R^{\ell}}^{\nu-1}. \tag{24}$$

Therefore, from the fact $\|S_J^{\ell}\|_{R^{\ell}} < 1$, the inequality (22) holds when the number of Jacobi plane relaxations is large enough. We conclude our result of the multigrid convergence in the following theorem.

Theorem 3.3. *For the variable diffusion problem* (1) *discretized by the finite difference scheme* (6) *on the cylindrical uniform grids G^{ℓ}, let S_J^{ℓ} and E_{MG} be the error reduction operator of plane Jacobi relaxation and the present* 2*-level MG, respectively. There exists a positive constant c_4 independent of the grid size such that*

$$\|E_{\mathrm{MG}}\|_{R^{\ell}} < c_4 \big\|S_J^{\ell}\big\|_{R^{\ell}}^{\nu-1}, \tag{25}$$

where ν is the number of smoothing steps in the MG. If ν is large enough, the 2*-level MG is convergent independent of the mesh size.*

Remark 3.4. From the convergence rate of the plane Jacobi relaxation estimated by the inequality (19) of Corollary 3.2, Theorem 3.3 suggests that for the constant diffusion case, the number of Jacobi plane relaxations should increase four times when the mesh is refined to achieve grid-independent convergence of MG. On the other hand, since the plane Jacobi matrix $S_J^{\ell} > 0$ with zero diagonal blocks and $(R^{\ell})^{1/2} S_J^{\ell} (R^{\ell})^{-1/2}$ is block-symmetric, irreducible and convergent, by the Perron–Frobenius theorem, and Theorem 4.15 of the Varga's book [15], the relation of the asymptotic convergence rates between the plane Jacobi S_J^{ℓ} and plane Gauss–Seidel S_{GS}^{ℓ} can be derived as follows.

$$\big\|S_{\mathrm{GS}}^{\ell}\big\|_{R^{\ell}} \leqslant \frac{\|S_J^{\ell}\|_{R^{\ell}}}{2 - \|S_J^{\ell}\|_{R^{\ell}}} \leqslant \frac{1 - c\Delta r^2}{1 + c\Delta r^2} \approx 1 - 2c\Delta r^2.$$

To cite this article: Ming-Chih Lai, et al., An efficient semi-coarsening multigrid method for variable diffusion problems in cylindrical coordinates, Applied Numerical Mathematics (2006), doi:10.1016/j.apnum.2006.07.019

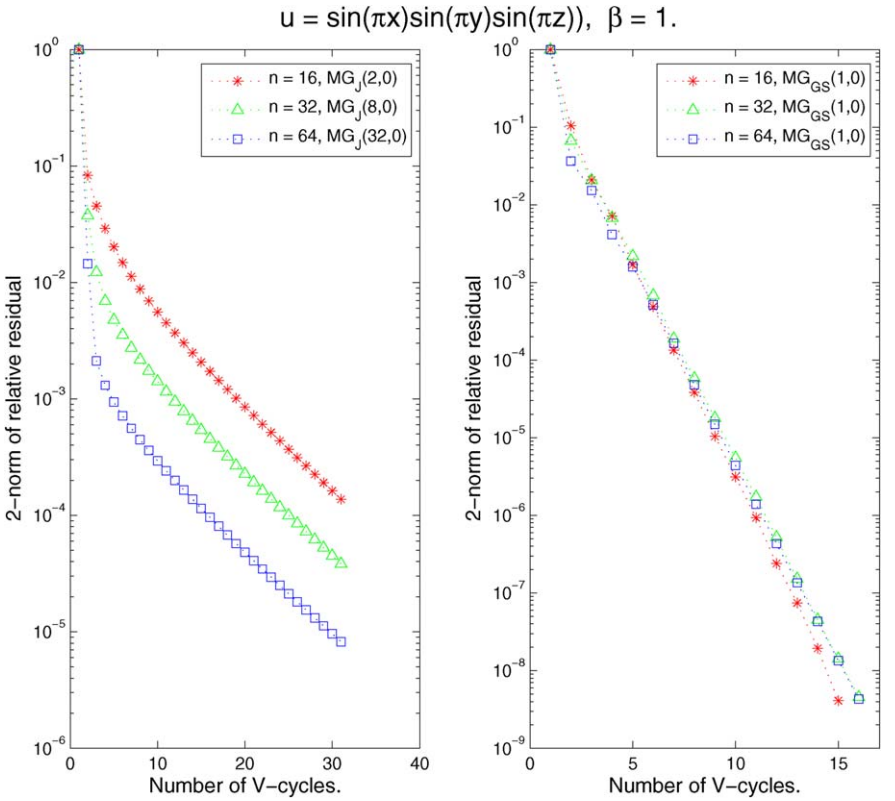


Fig. 1. The 2-norm of relative residuals with grid sizes $n = 16(*), 32(\triangle), 64(\square)$ for MG_{J} (left) and MG_{GS} (right).

Moreover, the estimate of the error reduction rate of MG with S^{ℓ}_{GS} smoother in (24) becomes $\|E_{\mathrm{MG}}\|_{R^{\ell}} \leqslant \frac{\sqrt{10}c}{\pi^2}\|S^{\ell}_{\mathrm{GS}}\|^{\nu-1}_{R^{\ell}}$. Hence, the grid-independent convergence of MG should be easier to achieve when plane Gauss–Seidel relaxations are employed in the pre-smoothing and post-smoothing steps of the MG.

Fig. 1 shows the convergence behaviors of MG for the constant diffusion case $\beta = 1$. One can see that when the plane Jacobi smoother is employed in MG, in order to have grid-independent convergence rate, the number of smoothing steps must increase four times as the grid number n doubles. This is exactly what we expect from the previous remark. In practice, only one step of the plane Gauss–Seidel relaxation is required for convergence and the convergence rate is independent of the problem size as shown in Fig. 1.

4. Numerical results

In this section, we report several numerical tests for the multigrid method. On a given $n_r \times n_z \times n_\theta$ mesh size with $n_r = n_z = n_\theta/2$, the present multigrid algorithm is a complete $\log_2 n_\theta$-level V-cycle which employs semi-coarsening in the azimuthal (θ) direction, and one step of red-black plane (r–z plane) Gauss–Seidel relaxation for both the pre-smoothing and the post-smoothing. It is important to note that the present MG method semi-coarsens the grid in the θ direction instead of the r direction. This is because if the radial grid is coarsened, then the coarser grid would not coincide with the fine grid based on our radial grid arrangement (3). The other reason is that Eq. (1) is on r, θ, z rectangular domain, where the strongly anisotropic diffusion occurs along the θ direction near the z-axis. It is well known that the semi-coarsening direction is determined by which direction is strongly anisotropic [5].

In order to save the computation time, each plane relaxation step is further approximated by one step of $\log_2 n_z$-level multigrid V-cycle iteration where coarse grids are obtained by semi-coarsening in the axial (z) direction on each plane and employing a red-black line Gauss–Seidel relaxation in the radial (r) direction. The restriction (fine to coarse)

To cite this article: Ming-Chih Lai, et al., An efficient semi-coarsening multigrid method for variable diffusion problems in cylindrical coordinates, Applied Numerical Mathematics (2006), doi:10.1016/j.apnum.2006.07.019

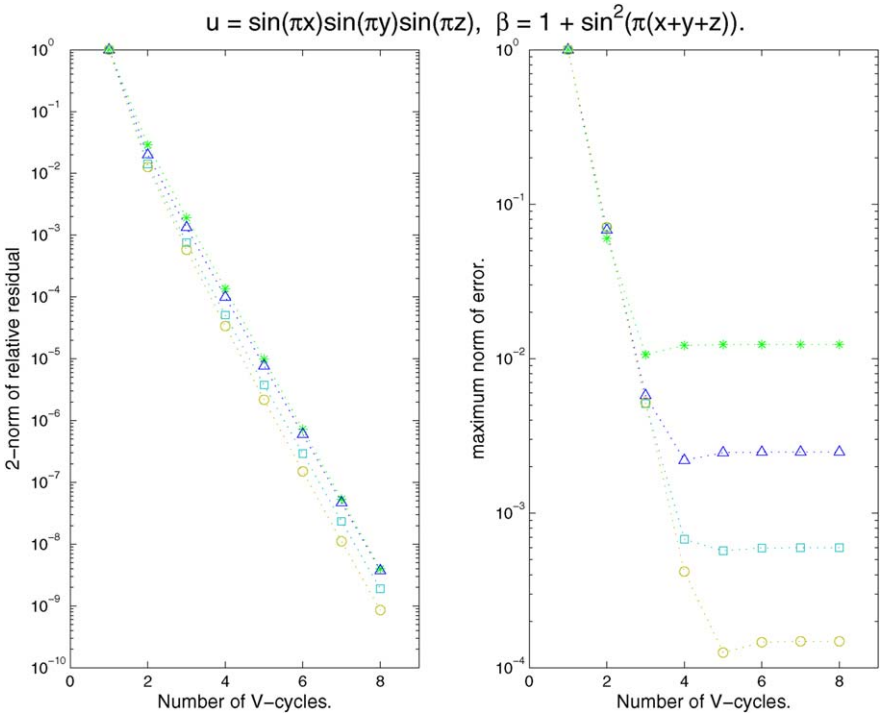


Fig. 2. The 2-norm of relative residuals of $MG_{\mathrm{RB}}(1,1)$ and the maximum norm of errors for Example 1 (smooth diffusion coefficient) with grid sizes $n = 16(*), 32(\triangle), 64(\square), 128(\circ)$.

and prolongation (coarse to fine) operators are the conventional full weighting and linear interpolation, respectively. Hereafter, we denote this version of the multigrid method as $MG_{\mathrm{RB}}(1,1)$.

In this paper, we have tested the convergence behavior and the numerical accuracy of the present MG algorithm by the following examples. In those examples, we choose the analytical solution $u(x, y, z) = \sin(\pi x)\sin(\pi y)\sin(\pi z)$ and vary the diffusion coefficients based on different smoothing characteristics as

(1) $\beta = 1 + \sin^2(\pi(x + y + z))$ (smooth case);
(2) $\beta = 1 + \sin^2(10\theta)$ (moderately oscillatory in θ direction);
(3) $\beta = 1 + \sin^2(10\theta)\sin^2(20\pi z)$ (moderately oscillatory in θ and z directions).

The right-hand side functions are obtained by substituting the solutions into Eq. (1).

The stopping tolerance of the iterations is set to be 10^{-8} for all of our test cases. All numerical runs are carried out on a PC with 1 GB RAM and 1.3 GHz CPU, and the program is written in MATLAB. All the following figures show the 2-norm of the relative residuals and the maximum norm of the errors versus the number of V-cycles. From Fig. 2, we can see that the residual norms decrease as the same rate for all mesh sizes and it takes about 8 V-cycles to reach the stopping criterion. However, it takes only 5 V-cycles to reach the discretization error for all meshes. Furthermore, the maximum norm of errors decrease by a factor of four as the resolution doubles which indicates that our finite difference discretization for Eq. (1) is indeed second-order accurate. Fig. 3 shows the convergence and accuracy performance for the case of moderately oscillatory diffusion in θ direction. This example takes about 3–4 more V-cycles than the smooth case to reach the residual stopping criterion and the discretization error. It leads to the same convergence rate for all meshes as well. Note that, since the diffusion coefficient is moderately oscillatory, the mesh $n = 16$ is under-resolving for this case. Fig. 4 shows the performance of our multigrid method for the case of moderately oscillatory diffusion in θ and z directions. Although the meshes $n = 16, 32$ show under-resolution accuracy, the residual convergence rates are still the same for all meshes. All three examples show our present multigrid V-cycle is scalable with the mesh size perfectly.

To cite this article: Ming-Chih Lai, et al., An efficient semi-coarsening multigrid method for variable diffusion problems in cylindrical coordinates, Applied Numerical Mathematics (2006), doi:10.1016/j.apnum.2006.07.019

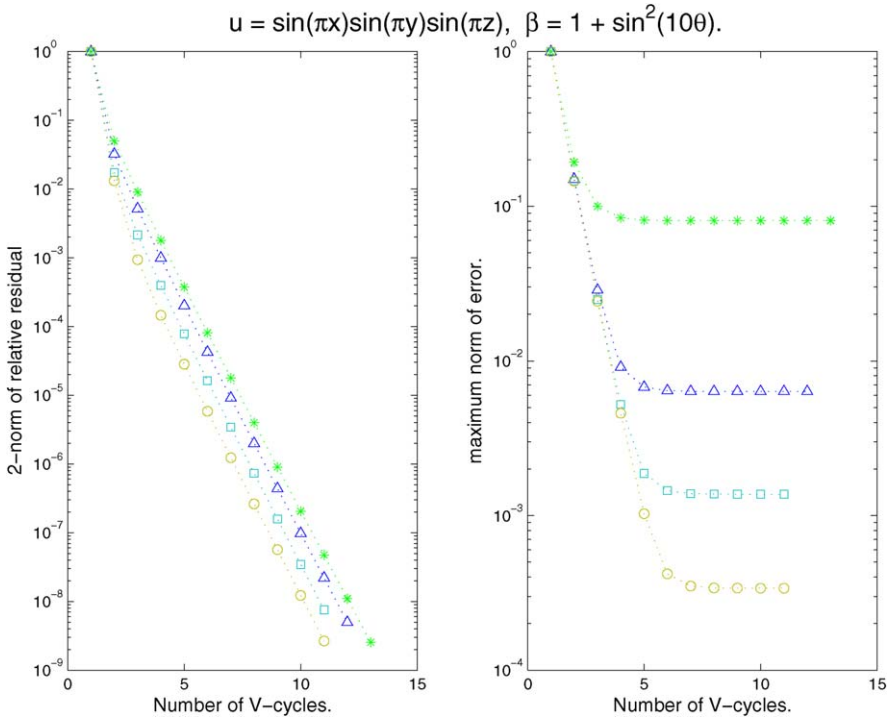


Fig. 3. The 2-norm of relative residuals of $MG_{\mathrm{RB}}(1,1)$ and the maximum norm of errors for Example 2 (diffusion coefficient moderately oscillatory in θ direction) with grid sizes $n = 16(*), 32(\triangle), 64(\square), 128(\circ)$.

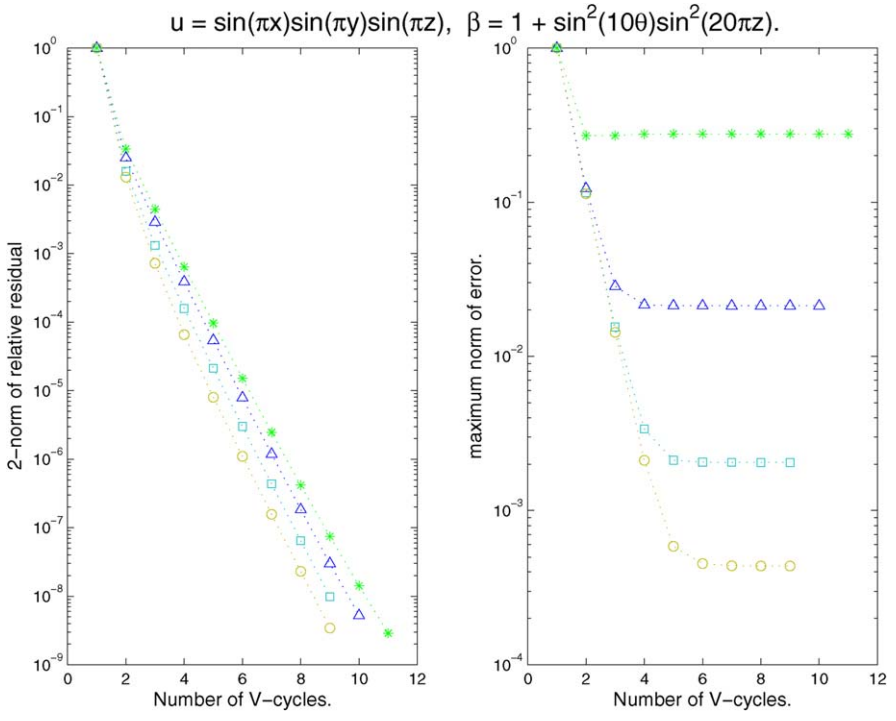


Fig. 4. The 2-norm of relative residuals of $MG_{\mathrm{RB}}(1,1)$ and the maximum norm of errors for Example 3 (diffusion coefficient moderately oscillatory in θ and z directions) with grid sizes $n = 16(*), 32(\triangle), 64(\square), 128(\circ)$.

To cite this article: Ming-Chih Lai, et al., An efficient semi-coarsening multigrid method for variable diffusion problems in cylindrical coordinates, Applied Numerical Mathematics (2006), doi:10.1016/j.apnum.2006.07.019

Acknowledgements

The first author was supported by National Science Council of Taiwan under research grant NSC-94-2115-M-009-004. The second author was supported by National Science Council of Taiwan under research grant NSC-94-2119-M-009-003.

References

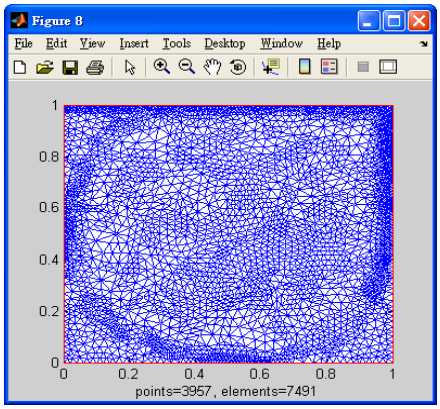
[1] J. Adams, P. Swarztrauber, R. Sweet, Fishpack—a package of Fortran subprograms for the solution of separable elliptic partial differential equations, 1980. Available in: http://www.netlib.org/fishpack.
[2] J.H. Bramble, J.E. Pasciak, J. Xu, The analysis of multigrid algorithms for nonsymmetric and indefinite elliptic problems, Math. Comp. 51 (1988) 398–414.
[3] J.H. Bramble, D.A. Kwak, J.E. Pasciak, Uniform convergence of multigrid V-cycle iterations for indefinite and nonsymmetric problems, SIAM J. Numer. Anal. 31 (1994) 1746–1763.
[4] A. Brandt, Multi-level adaptive solutions to boundary value problems, Math. Comp. 31 (1977) 333–390.
[5] W.L. Briggs, V.E. Henson, S.F. McCormick, A Multigrid Tutorial, second ed., SIAM, Philadelphia, PA, 1999.
[6] J.E. Dendy Jr., Two multigrid methods for three dimensional problems with discontinuous and anisotropic coefficients, SIAM J. Sci. Stat. Comput. 8 (1987) 673–685.
[7] W. Hackbush, Multigrid Methods and Applications, Springer, Berlin, 1985.
[8] M.-C. Lai, W.-W. Lin, W. Wang, A fast spectral/difference method without pole conditions for Poisson-type equations in cylindrical and spherical geometries, IMA J. Numer. Anal. 22 (2002) 537–548.
[9] M.-C. Lai, Y.-H. Tseng, A fast iterative solver for the variable coefficient diffusion equation on a disk, J. Comput. Phys. 208 (2005) 196–205.
[10] I.M. Llorente, N.D. Melson, Robust multigrid smoothers for three-dimensional elliptic equations with strong anisotropies, NASA/CR-1998-208700 ICASE Report No. 98–37.
[11] W.A. Mulder, A high-resolution Euler solver based on multigrid, semi-coarsening, and defect-correction, J. Comput. Phys. 100 (1992) 209–228.
[12] T. Washio, C. Oosterlee, Flexible multiple semicoarsening for three dimensional singularly pertubered problems, SIAM J. Sci. Comput. 19 (1998) 1646–1666.
[13] A. Reusken, On a robust multigrid solver, Computing 56 (1996) 303–322.
[14] S. Schaffer, A semicoarsening multigrid method for elliptic partial differential equations with highly discontinuous and anisotropic coefficients, SIAM J. Sci. Comput. 20 (1998) 228–242.
[15] R.S. Varga, Matrix Iterative Analysis, Springer Series in Computational Mathematics, Springer, Berlin, 2000.

To cite this article: Ming-Chih Lai, et al., An efficient semi-coarsening multigrid method for variable diffusion problems in cylindrical coordinates, Applied Numerical Mathematics (2006), doi:10.1016/j.apnum.2006.07.019

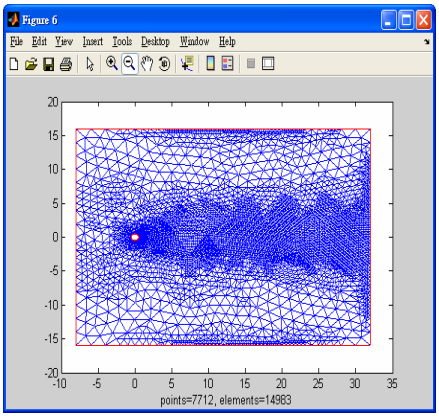
(四)計畫成果自評：

本計劃已完成二維不可壓縮流的數值計算.其中時間區域已達二階精度而空間的 Streamline-Diffusion 離散法也使我們能計算高雷諾數的流體. 我們也成功的將多重網格法及自適性網格運用於二維不可壓縮流的計算. 其中多重網格於三維變係數擴散方程的應用已發表於國際期刊. 其他部分成果請見下三頁. 雖然本計劃最終目標雖尚未完成，但考慮三維的流體計算是非常困難與重要，我們寄望未來能繼續在國科會的支持下完成未竟之業.

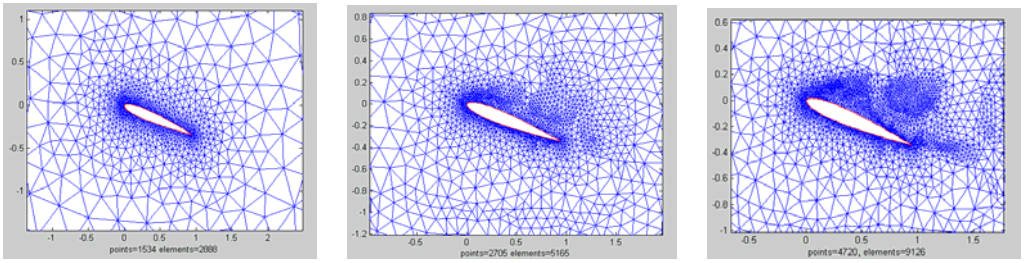
(a) Adaptive meshes for various benchmark problems:



Driven Cavity Flow ($Re=10^4$)



Flow around Cylinder ($Re=200$)



Flow around airfoil NACA0012 at various times

($Re=10^6$, attack angle $=20^o$)

(b) Accuracy are achieved for the benchmark problem (flow around cylinder)

In the following, C_D=drag coefficient= $\frac{2F_d}{\rho U_\infty^2 L}$ C_L= lift coefficient= $\frac{2F_l}{\rho U_\infty^2 L}$

ST= Strouhal number= $\frac{freq * L}{\nu}$

Re	20		50		100		200	
C_D	2.11	2.22[1] 2.19[2]	1.46	1.41[1] 1.38[3]	1.38±0.01	1.24[1] 1.35±0.012[4]	1.38±0.05	1.16[1] 1.31±0.049[4]
C_L	-	-	-	-	±0.35	±0.339[4]	±0.7	±0.69[4]
ST	-	-	0.125	0.12~0.13[1] 0.139[3]	0.167	0.167[1] 0.164[4]	0.2	0.19[4]

(1) for Re=100 and Re=200 are obtained experimentally by Clift and Reshko respectively.
(2) is computed on a 640x320 grid by Donna Calhoun, Courant Institute of mathematics science, in 2002.
(3) is computed on a 267x147 grid by Saki and Biringen, Dept. of Areospace Engineering, Univ. of Colorado, in 1996.
(4) is computed on a 256x256 grid by Liu, Zheng and Sung in 1998.

The numbers in blue color are obtained from our simulations. Clearly, we have achieved the same accuracy as other researchers. However, the mesh points generated in our calculation are just about 7000 to 8000 points which is much less than the mesh points used in all the other calculation.

(c) Efficiency is observed for the benchmark problem (flow around cylinder)

AMG is efficient

Tables on the right show the average number of Iterations for various flows.

200 time steps are performed on each grid. The initial time step is 0.2 and step size is halved when mesh is refined.

Re=20

nodes	1448	2662	4664
GMRES-GMG	13.1	29.4	25.6
GMRES-AMG	13.1	14.9	10.3

Re=50

nodes	1448	2662	4664
GMRES-GMG	14.3	33.4	28.6
GMRES-AMG	14.3	18.0	12.0

Re=100

nodes	1448	2662	4664
GMRES-GMG	16.3	43.7	35.7
GMRES-AMG	16.3	22.4	17.1

Re=200

nodes	1448	2662	4664
GMRES-GMG	19.3	50.5	40.1
GMRES-AMG	19.3	28.1	21.6

For this benchmark problem, we also found that AMG (algebraic multigrid) preconditioned GMRES (general minimal residual method) is very robust.