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辛幾何的誘導方法(3/3)結案報告

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關鍵詞：誘導方法、誘導表現、幾何量化

摘要：

　　令 G 為一半單李羣。本計畫在 G 的半單共伴軌道上採用辛誘導方法，並研究產生的 G 辛流型。我們也在它上面進行辛約化，以便了解辛誘導和辛約化之間的關係。在表現理論中，我們利用 G 的拋物子羣來誘導 G 的表現。針對裂解的 G，我們發現辛誘導的過程類似誘導表現。我們的構想是利用幾何量化來證明量化過程和誘導過程是可互換順序的。

Induction Method in Symplectic Geometry

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Abstract

Let G be a semisimple Lie group. We perform symplectic induction on semisimple coadjoint orbits of G and study the resulting symplectic G-manifolds. We also perform symplectic reduction to them, in order to understand the relations between symplectic induction and symplectic reduction. Recall that in representation theory, one uses the representation of a parabolic subgroup of G to induce a G-representation. For split G, we show that our process of symplectic induction is analogous to the induced representation. The idea is to use geometric quantization to show that quantization commutes with induction.

Keywords: symplectic induction, induced representation, geometric quantization.

Let G be a real semisimple Lie group. In this project, we study certain symplectic G-manifolds X which are induced from the semisimple coadjoint orbits of G. We also study the effect of symplectic reduction on X. For split Lie groups, we also use geometric quantization to show that this process is analogous to induction in representation theory, in the sense that quantization commutes with induction.

The Lie algebras are denoted by the lower case German letters, so for instance the Lie algebra of G is $\mathfrak{g}$. Let C be a Cartan subgroup of G. Taking the coadjoint representation of G on $\mathfrak{g}^*$, let L be the stabilizer of some element of $\mathfrak{c}^*$, so that G/L has the structure of a semisimple coadjoint orbit. Let $L^{\mathrm{ss}} = (L, L)$ be its commutator subgroup. Let H be the centralizer of L in C. Let

$$(0.1) \qquad\qquad X = G/L^{\mathrm{ss}} \times \mathfrak{h}.$$

By letting G act on the first component of X, we obtain the left G-action on X. Since H centralizes L^{ss}, it also acts on X from the right.

We show that a mapping

$$\beta : \mathfrak{h} \longrightarrow \mathfrak{h}^*$$

may be extended to a $G \times H$-invariant 1-form on X. We are interested in the corresponding 2-form $\omega = d\beta$ on X. The next theorem studies the condition for ω to be symplectic. We gather the notations here for convenience:

$$(0.2) \qquad \begin{array}{c} X = G/L^{\mathrm{ss}} \times \mathfrak{h} \ , \ \beta : \mathfrak{h} \longrightarrow \mathfrak{h}^* \ , \\ \omega = d\beta \ \text{ is a } G \times H\text{-invariant 2-form on } X. \end{array}$$

Theorem 1 *The 2-form $\omega = d\beta$ is symplectic if and only if β is a local diffeomorphism and $\mathrm{Im}(\beta) \subset \mathfrak{h}^*_{\mathrm{reg}}$.*

Here $\mathrm{Im}(\beta)$ denotes the image of β, and the regular elements $\mathfrak{h}^*_{\mathrm{reg}}$ are those in $\mathfrak{h}^*$ which are orthogonal only to the roots in $\mathfrak{c}^*$ which annihilate $\mathfrak{h}$. So $\mathfrak{h}^*_{\mathrm{reg}}$ is a union of open cones.

In general, if R is a closed connected subgroup of G, then the G-invariant q-forms on G/R can be identified with

$$(0.3) \qquad \wedge^q(\mathfrak{g}, \mathfrak{r})^* = \{\gamma \in \wedge^q \mathfrak{g}^* \ ; \ \mathrm{ad}^*_x \gamma = \iota(x)\gamma = 0 \text{ for all } x \in \mathfrak{r}\}.$$

Here $\mathrm{ad}^* : \mathfrak{g} \longrightarrow \mathrm{End}(\wedge^q \mathfrak{g}^*)$ is the coadjoint representation, and $\iota(x) : \wedge^q \mathfrak{g}^* \longrightarrow \wedge^{q-1} \mathfrak{g}^*$ is the interior product.

Suppose that $\nu \in \mathfrak{h}^*$ is in the image of the moment map of ω. In particular let $\Phi_r : X \longrightarrow \mathfrak{h}^*$ be the moment map of the right H-action, and let

$$X_\nu = (\Phi_r^{-1}(\nu))/H.$$

Let $\imath : \Phi_r^{-1}(\nu) \hookrightarrow X$ be the inclusion, and let $\pi : \Phi_r^{-1}(\nu) \longrightarrow X_\nu$ be the natural quotient. Then there is a unique symplectic form ω_ν on X_ν such that $\pi^*\omega_\nu = \imath^*\omega$. The process

$$(0.4) \qquad\qquad (X, \omega) \rightsquigarrow (X_\nu, \omega_\nu)$$

is known as symplectic reduction with respect to $\nu \in \mathrm{Im}(\Phi_r)$. The space (X_ν, ω_ν) is called the symplectic quotient. Each step of (0.4) commutes with the left G-action, so ω_ν is G-invariant. Using the notation in (0.3), the following theorem describes the symplectic quotient.

Theorem 2 *The symplectic quotient X_ν consists of copies of G/L, each of which has the symplectic form $d\nu \in \wedge^2(\mathfrak{g}, \mathfrak{l})^*$.*

A semisimple coadjoint orbit is the coadjoint orbit of an element of $\mathfrak{c}^*$, so it is equivalent to some G/L. The above theorem shows that symplectic reduction on X leads to G-invariant symplectic forms on the semisimple orbits. The next theorem shows that the converse is true.

Theorem 3 *Every G-invariant symplectic form on G/L has the expression $d\nu \in \wedge^2(\mathfrak{g}, \mathfrak{l})^*$, where $\nu \in \mathfrak{h}_{\mathrm{reg}}^*$. Therefore, they can be obtained by symplectic reduction from (X, ω), where ν is in the image of the moment map of ω.*

In what follows, we shall use the method of geometric quantization to construct a unitary representation out of the symplectic form. The intended representation will be given by the G-representations which are induced by its parabolic subgroups, namely the principal series representations. Therefore, we suppose that G is split, namely it has a Cartan subgroup of the form MA, where M is a finite abelian group, and the Lie group A is equivalent to the Euclidean space. We shall take H to be MA, so that its centralizer is $L = H = MA$. In this case X becomes $G \times \mathfrak{a}$. Therefore, X admits a $G \times G$-action. Here $\beta : \mathfrak{a} \longrightarrow \mathfrak{a}^*$ is $G \times G$-invariant. So (0.2) becomes

$$(0.5) \qquad\qquad \begin{aligned} X = G \times \mathfrak{a} \ , \ \beta : \mathfrak{a} \longrightarrow \mathfrak{a}^* \ , \\ \omega = d\beta \ \text{ is a } G \times G\text{-invariant 2-form on } X. \end{aligned}$$

By the method of geometric quantization, ω leads to a holomorphic Hermitian line bundle $\mathbf{L}$ on X. We use certain sections of $\mathbf{L}$ to construct a unitary $G \times MA$-representation

$$\mathcal{H} = \mathcal{H}(X, \omega).$$

Let $\hat{M}$ and $\hat{A}$ respectively denote the spaces of unitary irreducible representations of M and A. Since G is split, $M = \hat{M}$ is finite abelian. Choose a positive system in $\mathfrak{a}^*$, and let $N \subset G$ be the subgroup corresponding to the positive root spaces. We

shall show that the irreducible G-representations which occur in $\mathcal{H}$ are given by the representations $I(\sigma \otimes \nu) = \mathrm{Ind}_{MAN}^G(\sigma \otimes \nu \otimes 1)$ induced by the parabolic subgroup MAN. Here $\sigma \in \hat{M}$ and $\nu \in \hat{A}$. Since $\hat{M}$ is finite, we write $\mathcal{H} = \sum_\sigma \mathcal{H}_\sigma$ with respect to the M-action. The parameter ν is continuous under the Plancherel measure $d\nu$ on $\hat{A}$. We shall use the notion of direct integral and write $\mathcal{H}_\sigma = \int_{\hat{A}}^{\oplus} \mathcal{H}_{\sigma,\nu}\, d\nu$, where $\mathcal{H}_{\sigma,\nu} \cong I(\sigma \otimes \nu)$. The measure $d\nu$ has no point-mass. So we first say that an open ball $U \subset \hat{A}$ occurs in $\mathcal{H}_\sigma$ if there exists some $\int_{\hat{A}} s_\nu\, d\nu \in \mathcal{H}_\sigma$ such that $s_\nu \neq 0$ for all $\nu \in U$. Then $\mathcal{H}_{\sigma,\nu}$ is said to occur in $\mathcal{H}_\sigma$ if some open ball U containing ν occurs in $\mathcal{H}_\sigma$.

If $F : \mathfrak{a} \longrightarrow \mathbf{R}$, we write its gradient function $\frac{\partial F}{\partial x}$ as

$$(0.6) \qquad\qquad\qquad F' : \mathfrak{a} \longrightarrow \mathfrak{a}^*.$$

We shall compare F' with β of (0.5). If the Hessian matrix $\left(\frac{\partial^2 F}{\partial x_i \partial x_j}\right)$ is positive definite everywhere, we say that F is strictly convex.

The moment map $\Phi : X \longrightarrow \mathfrak{g}^*$ of $\omega = d\beta$ is G-equivariant, and so can be regarded as a map $\mathfrak{a} \longrightarrow \mathfrak{g}^*$. We shall see that this map is simply β itself, and so Φ becomes $\mathfrak{a} \longrightarrow \mathfrak{a}^*$. Further, $\mathfrak{a}^* \cong \hat{A}$, where $\lambda \in \mathfrak{a}^*$ can be identified with $\nu \in \hat{A}$ by $e^{\lambda(\xi)} = \nu(e^\xi)$ for all $\xi \in \mathfrak{a}$. So the image $\mathrm{Im}(\Phi)$ lies inside $\hat{A}$. The following theorem shows that $\mathrm{Im}(\Phi)$ determines the occurrence of the principal series in $\mathcal{H}$. Let ρ denote half the sum of positive roots.

Theorem 4 *Suppose that $\beta = F'$ and F is strictly convex. The unitary $G \times MA$-representation $\mathcal{H} = \mathcal{H}(X,\omega)$ decomposes as a direct integral of G-representations $\mathcal{H} = \sum_\sigma \int_\nu \mathcal{H}_{\sigma,\nu}\, d\nu$. Here $\mathcal{H}_{\sigma,\nu} = I(\sigma \otimes \nu)$ when $\nu + \rho$ is in the image of the moment map Φ, and $\mathcal{H}_{\sigma,\nu} = 0$ otherwise.*

We say that $I(\sigma \otimes \nu)$ is regular if $\nu \in \mathfrak{a}_{\mathrm{reg}}^*$. We shall apply Theorems 1 and 4 to construct a regular principal model $\mathcal{H}$, in the sense that every regular principal series representation occurs exactly once in $\mathcal{H}$.