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BOUNDARY INFLUENCE ON THE ENTROPY OF A PROBLEM IN CELLULAR NEURAL NETWORKS

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Abstract. Let T a two dimensional map induced from a spatial discretization of a Reaction-Difussion system. In [Afraimovich and Hsu, 2003], the following open problems were raised. Is it true that, in general, $h(T) = h_D(T) = h_N(T) = h_{\ell_{(1)},\ell_{(2)}}(T)$? Here $h(T)$ and $h_{\ell_{(1)},\ell_{(2)}}(T)$ (see Definitions 1.3 and 1.4) are, respectively, the spatial entropy of the system T and the spatial entropy of T with respect to the lines $\ell_{(1)}$ and $\ell_{(2)}$, and $h_D(T)$ and $h_N(T)$ are spatial entropy with respect to the Dirichlet and Neuman boundary conditions. If it is not true, then which parameters of the lines $\ell_{(i)}$, $i = 1, 2$, are responsible for the value of $h(T)$. What kind of bifurcations occurs if the lines $\ell_{(i)}$ move? In this paper, we shed some light on this open problem for a Lozi-type map obtained from Cellular Neural Networks.

Key words: Boundary influence, Lozi-type map, dynamics of intersection, entropy, cellular neural networks.

1. Introduction

We consider one-dimensional Cellular Neural Networks (CNNs) of the form (e.g., [Ban *et al.*, 2002, 2001; Hsu 2000]).

$$\frac{dx_i}{dt} = -x_i + z + \alpha f(x_{i-1}) + af(x_i) + \beta f(x_{i+1}),\ i \in \mathbb{Z}, \tag{1.1a}$$

where $f(x)$ is a piecewise-linear output function defined by

$$f(x) = \begin{cases} rx + 1 - r, & \text{if } x \geq 1, \\ x, & \text{if } |x| \leqslant 1, \\ lx + l - 1, & \text{if } x \leq -1. \end{cases} \tag{1.1b}$$

Here r and l are positive constants. The state of a cell C_i is denoted by x_i. The quantity z is called the threshold or bias term. The constants α, a and β are the interaction weights between neighboring cells. Such triple pair $[\alpha, a, \beta]$ of the interaction weights is called the template of the system (1.1).

CNNs were first proposed by Chua and Yang [1988a, 1988b]. Their main applications are in image processing and pattern recognition [Chua, 1998]. For additional background information, applications, and theory, see [Thiran, 1997; Chua, 1998] among others.

A basic and important class of solutions of (1.1) is the steady-state solutions. Specifically, a steady-state solution $\mathbf{x} = (x_i)_{i\in\mathbb{Z}}$ of (1.1) satisfies the following equation

$$f(x_{i+1}) = \frac{1}{\beta}(x_i - z - \alpha f(x_{i-1}) - af(x_i)). \tag{1.2}$$

Set $u_i = f(x_i)$. Then (1.2) becomes

$$u_{i+1} = \frac{1}{\beta}(-\alpha u_{i-1} - z + f^{-1}(u_i) - au_i), \tag{1.3a}$$

or, equivalently,

$$(u_i, u_{i+1}) = (u_i, \frac{1}{\beta}(-\alpha u_{i-1} - z + f^{-1}(u_i) - au_i)) =: T(u_{i-1}, u_i). \tag{1.3b}$$

Clearly, (1.3b) defines a Lozi-type map T of the form

$$(s_{i+1}, t_{i+1}) = T(s_i, t_i) = (t_i, F(t_i) - bs_i). \tag{1.4a}$$

Here

$$b = \frac{\alpha}{\beta}, \tag{1.4b}$$

and

$$F(y) = \begin{cases} a_1 y + a_0 - a_1 + \bar{a}_0 := a_1 y + \bar{a}_1, & \text{if } y \geq 1, \\ a_0 y + \bar{a}_0, & \text{if } |y| \leqslant 1, \\ a_{-1} y + a_{-1} - a_0 + \bar{a}_0 := a_{-1} y + \bar{a}_{-1}, & \text{if } y \leq -1. \end{cases} \tag{1.4c}$$

where

$$\begin{array}{ll} a_1 = \frac{1}{\beta}(\frac{1}{r} - a) > 0, & a_0 = \frac{1}{\beta}(1 - a) < 0, \\ a_{-1} = \frac{1}{\beta}(\frac{1}{l} - a) > 0, & \bar{a}_0 = \frac{-z}{\beta}. \end{array} \tag{1.4d}$$

Any bounded trajectory $(s_{j+1}, t_{j+1}) = T(s_j, t_j)$ corresponds to a bounded steady-state solution of system (1.1). The following class of steady-state solutions is of particular interest.

Definition 1.1. *A solution $\mathbf{x} = (x_i)_{i\in\mathbb{Z}}$ of equations (1.2) is called a mosaic solution if $|x_i| > 1$ for all $i \in \mathbb{Z}$. Its associated pattern $\mathbf{y} = (y_i)_{i\in\mathbb{Z}} = (f(x_i))_{i\in\mathbb{Z}}$ is called a mosaic pattern.*

To define the stability of a steady-state solution, we consider the following linearized stability. Let $\xi = (\xi_i)_{i\in\mathbb{Z}} \in \ell^2$, the linearized operator $L(\mathbf{x})$ of (1.1) at a steady-state solution $\mathbf{x} = (x_i)_{i\in\mathbb{Z}}$ is given by

$$(L(\mathbf{x})\xi)_i = -\xi_i + \alpha f'(x_{i-1})\xi_{i-1} + af'(x_i)\xi_i + \beta f'(x_{i+1})\xi_{i+1}. \tag{1.5}$$

Definition 1.2. *Let $\mathbf{x} = (x_i)_{i\in\mathbb{Z}}$ be a solution of* (1.2) *with $|x_i| \neq 1$ for all $i \in \mathbb{Z}$. The steady-state solution $\mathbf{x}$ is called* (*linearized*) *stable if all eigenvalues of $L(\mathbf{x})$ have negative real parts. The solution is called unstable if there is an eigenvalue λ of $L(\mathbf{x})$ such that λ has a positive real part.*

It is well-known, see e.g., [Juang and Lin, 2000; Hsu, 2000], that for

$$\frac{1}{|a| + |\alpha| + |\beta|} > \max\{r, \ell\} \geq 0, \tag{1.6}$$

where r, ℓ, a, α and β are defined as in (1.1), $-L(x)$ is a self-adjoint and positive operator. Therefore, if r and ℓ are sufficiently small, all mosaic solutions of (1.1) are stable.

If we have a finite number of n cells, which is usually the case in real applications, we will take the boundary conditions into account. The question then is how well such finite system as its size grows represents the original infinite system (1.1)? One quantity measures such resemblance is "spatial entropy". We next define the spatial entropy of the infinite system as well as that of the finite system. The following notion of the entropy of the system (1.1) was introduced by [Mallet-Paret and Chow, 1995]. Set $\Gamma_{n,k}(T)$ (resp., $\Gamma^M_{n,k}(T)$) to be the number of elements in the solution set $S_{n,k}$ (resp., $S^M_{n,k}$), where $S_{n,k}$ (resp., $S^M_{n,k}$)$= \{\{u_i\}_{i=k}^{n+k-1} : \{u_i\}_{i=-\infty}^{\infty}$ is a bounded steady-state solution (resp., stable mosaic solution) of (1.1)$\}$. Here $k \in \mathbb{Z}$. Since the template of system (1.1) is space invariant, the steady-state solutions of (1.1) are also space invariant. That is to say if $\{u_i\}_{i=-\infty}^{\infty}$ is a steady state solution of (1.1), so is $\{u_{i+k}\}_{i=-\infty}^{\infty}$ for any $k \in \mathbb{Z}$. Hence, $\Gamma_{n,k}(T)$ and $\Gamma^M_{n,k}(T)$ are independent of the choice of k. Thus, we may set $\Gamma_{n,k}(T) = \Gamma_n(T)$ and $\Gamma^M_{n,k}(T) = \Gamma^M_n(T)$.

Definition 1.3. (1) *The spatial entropy $h(T)$ of the system (1.1) or the map T is defined to be the limit*

$$h(T) = \overline{\lim_{n\to\infty}} \frac{\ln \Gamma_n(T)}{n}.$$

(2) *The spatial mosaic entropy $h^M(T)$ of the system (1.1) or the map T is defined to be the limit*

$$h^M(T) = \overline{\lim_{n\to\infty}} \frac{\ln \Gamma_n^M(T)}{n}.$$

Inspired by the open problems raised in [Afraimovich and Hsu 2003], we are led to define the following notation of the spatial entropy for the finite system. Define the line $\ell_{(m,k)}$ as

$$\ell_{(m,k)} = \{(x, y) \in \mathbb{R}^2 : y = mx + k\}. \tag{1.7a}$$

Here

$$\ell_{(\infty,k)} \text{ is interpreted as } \{(x, y) \in \mathbb{R}^2 : x = k\}. \tag{1.7b}$$

Denote by $\mathcal{N}(n, \ell_{(m_1,k_1)}, \ell_{(m_2,k_2)}, T)$ the number of points on the intersection of $T^n\ell_{(m_1,k_1)} \cap \ell_{(m_2,k_2)}$. Should no ambiguity arise, we will write $\ell_{(m_i,k_i)}$ as $\ell_{(i)}$. We next elaborate the meaning of the intersection $T^n\ell_{(m_1,k_1)} \cap \ell_{(m_2,k_2)}$. If $(x, y) \in T^n\ell_{(m_1,k_1)} \cap \ell_{(m_2,k_2)}$, then there exists a point $(u_0, u_1) \in \ell_{(m_1,k_1)}$ such that $(x, y) = T^n(u_0, u_1) := (u_n, u_{n+1}) \in \ell_{(m_2,k_2)}$. That is to say, in a CNN of n cells, its steady state $\{u_i\}_{i=1}^n$ satisfies the following Robbin's boundary conditions

$$u_1 = m_1u_0 + k_1$$

and

$$u_{n+1} = m_2u_n + k_2.$$

In particular, $(m_1, k_1) = (\infty, 0)$ and $(m_2, k_2) = (0, 0)$ (resp., $(m_1, k_1) = (m_2, k_2) = (1, 0)$) correspond to the Dirichlet (resp., Neumann) boundary conditions. Hence, $\mathcal{N}(n, \ell_{(m_1,k_1)}, \ell_{(m_2,k_2)}, T)$ denotes the number of steady-states of CNNs of n cells with the Robbin's boundary conditions. Likewise, we will set $\mathcal{N}^M(n, \ell_{(m_1,k_1)}, \ell_{(m_2,k_2)}, T)$ as the number of stable mosaic solutions of CNNs of n cells with the Robbin's boundary conditions.

Definition 1.4. (1) *The spatial entropy (resp., mosaic entropy) of the CNNs of n cells withe Robbin's boundary conditions is defined to be*

$$h_{n,\ell_{(1)},\ell_{(2)}}(T) = \frac{\ln \mathcal{N}(n, \ell_{(1)}, \ell_{(2)}, T)}{n} \left(\textit{resp., } h^M_{n,\ell_{(1)},\ell_{(2)}}(T) = \frac{\ln \mathcal{N}^M(n, \ell_{(1)}, \ell_{(2)}, T)}{n} \right).$$

(2) *The spatial entropy* $h_{\ell_{(1)},\ell_{(2)}}(T)$ *(resp., mosaic entropy* $h^M_{\ell_{(1)},\ell_{(2)}}(T)$*) of* T *with respect to lines* $\ell_{(1)}$ *and* $\ell_{(2)}$ *is defined as the limit of the entropy of the finite system, that is,*

$$h_{\ell_{(1)},\ell_{(2)}}(T) = \overline{\lim_{n\to\infty}} \frac{\ln \mathcal{N}(n, \ell_{(1)}, \ell_{(2)}, T)}{n} \tag{1.8a}$$

$$\textit{(resp., } h^M_{\ell_{(1)},\ell_{(2)}}(T) = \overline{\lim_{n\to\infty}} \frac{\ln \mathcal{N}^M(n, \ell_{(1)}, \ell_{(2)}, T)}{n} \textit{).} \tag{1.8b}$$

Notation 1.1. *In the case of Dirichlet (resp., Neumann) boundary conditions, we write* $h_{\ell_{(1)},\ell_{(2)}}(T) = h_D(T)$ *(resp.,* $h_N(T)$*) and* $h^M_{\ell_{(1)},\ell_{(2)}}(T) = h^M_D(T)$ *(resp.,* $h^M_N(T)$*).*

In case that the growth rate of $\mathcal{N}(n, \ell_{(1)}, \ell_{(2)}, T)$ is super exponential, $h_{\ell_{(1)},\ell_{(2)}}(T)$ is defined to be ∞. Let T be a local holomorphic mapping, preserving the origin, and two lines $\ell_{(1)}$ and $\ell_{(2)}$ passing the origin. Suppose all the images $T^n\ell_{(1)}$ are smooth [4] or that everything is algebraic (see [2], [3]). Then $h_{\ell_{(1)},\ell_{(2)}}(T)$ exists and is finite. In our case, $\mathcal{N}(n, \ell_{(1)}, \ell_{(2)}, T) \leq 3^n$, see Section 2.

Let T a two dimensional map induced from a spatial discretization of a Reaction-Difussion system. In [Afraimovich and Hsu, 2003], the following open problems were raised.

- **(P1):** Is it true that, in general, $h(T) = h_D(T) = h_N(T) = h_{\ell_{(1)},\ell_{(2)}}(T)$ or $h^M(T) = h^M_D(T) = h^M_N(T) = h^M_{\ell_{(1)},\ell_{(2)}}(T)$?
- **(P2):** If it is not true, then which parameters m_i and k_i, $i = 1, 2$, are responsible for the values of $h(T)$ or $h^M(T)$. What kind of bifurcations occurs if the lines $\ell_{(m,b)}$ move?

The purpose of this paper is to shed some light on those two problems for T being given as in (1.4). Specifically, under some mild conditions, we show that for any $\ell_{(1)}$ and $n \in \mathbb{N}$, except possibly a few pieces of $T^n\ell_{(1)}$, $T^n\ell_{(1)}$ is contained in an N-shaped tube for which its boundary points are ω-limit points of $\ell_{(1)}$ for T. Moreover, we show under a stronger condition, see (3.4), that the entropy $h_{\ell_{(1)},\ell_{(2)}}(T)$ of T with respect to $\ell_{(1)}$ and $\ell_{(2)}$ is independent of the choice of $\ell_{(1)}$. It is also shown that $h_D(T) = h_N(T) = \ln 3$, and that $h_{\ell_{(1)},\ell_{(2)}}(T)(= h_{\ell_{(2)}}(T))$ takes on two distinct values $\ln 3$ and 0. The necessary and sufficient conditions on $\ell_{(2)}$ for which $h_{\ell_{(2)}}(T) = \ln 3$ are also obtained. Those results are recorded in Section 3. Similar results for the spatial mosaic entropy are given in Section 4. In Section 2, we study the dynamics of a certain two-dimensional map induced from $T^n\ell_{(1)}$. We conclude this introductory section by mentioning some related work. Shih [2000] studied the influence of periodic, Neumann and Dirichlet boundary conditions on a problem arising in two dimensional CNNs. Since their output function f, as given in (1.1b), is flat at infinity, i.e., $r = l = 0$, the formulation of the problem is much different from those in [Afraimovich and Hsu 2003]. Consequently, the techniques used in both situations are also quite different. We also remark that the problem of the asymptotic behavior of the number of points on the intersection $f^k L_1 \cap L_2$, where L_1, L_2 are submanifolds of a smooth manifold, and f is a smooth map, is said to be a problem of dynamics of the intersection. These problems arise in various branches of analysis. There are some general results (see, e.g., p.261 of [Arnold 1993]) obtained for such problems. However, no approaches are available to solve specific problems.

2. Dynamics of Certain Maps Induced From $T^n\ell_{(m,k)}$

We begin with the calculation of $T^n\ell_{(m,k)}$. Now, for $m \neq 0$,

$$T(x^{'}, mx^{'} + k) = (mx^{'} + k, F(mx^{'} + k) - bx^{'}).$$

Set $x = mx^{'} + k$, $y = F(mx^{'} + k) - bx^{'}$, we see immediately that

$$y = F(x) - \frac{b(x-k)}{m} = \begin{cases} (a_1 - \frac{b}{m})x + (\bar{a}_1 + \frac{bk}{m}), & \text{if } x \geq 1, \\ (a_0 - \frac{b}{m})x + (\bar{a}_0 + \frac{bk}{m}), & \text{if } |x| \leqslant 1, \\ (a_{-1} - \frac{b}{m})x + (\bar{a}_{-1} + \frac{bk}{m}), & \text{if } x \leq -1. \end{cases} \tag{2.1}$$

From (2.1), we see immediately that T consists of three dynamics. Each dynamics acts on the following regions:

$$R_1 = \{(x,y) : x \geq 1\},\ R_0 = \{(x,y) : |x| \leq 1\} \text{ and } R_{-1} = \{(x,y) : x \leq -1\}. \tag{2.2}$$

The dynamics on regions R_i, $i = 1, 0, -1$, are to be termed the ith-dynamics, respectively. Given a straight line/ line segment/ half-line ℓ, the image of ℓ consists of at most two half-lines/ line segments ℓ_1 and ℓ_{-1} and one line segment ℓ_0. That is,

$$T\ell = \begin{cases} \ell_1, & \text{if } (x,y) \in \ell \text{ and } (x,y) \in R_1, \\ \ell_0, & \text{if } (x,y) \in \ell \text{ and } (x,y) \in R_0, \\ \ell_{-1}, & \text{if } (x,y) \in \ell \text{ and } (x,y) \in R_{-1}. \end{cases} \tag{2.3a}$$

Note that some of ℓ_1, ℓ_0, or ℓ_{-1} could be empty. We then define the notations $\ell_{i_1,i_2,\cdots,i_{n-1}}$, $i_j \in \{-1,0,1\}$, $j = 1,2,\cdots,n-1$, inductively as the followings.

$$T\ell_{i_1,i_2,\cdots,i_{n-1}} = \begin{cases} \ell_{i_1,i_2,\cdots,i_{n-1},1}, & \text{if } (x,y) \in \ell_{i_1,i_2,\cdots,i_{n-1}} \text{ and } (x,y) \in R_1, \\ \ell_{i_1,i_2,\cdots,i_{n-1},0}, & \text{if } (x,y) \in \ell_{i_1,i_2,\cdots,i_{n-1}} \text{ and } (x,y) \in R_0, \\ \ell_{i_1,i_2,\cdots,i_{n-1},-1}, & \text{if } (x,y) \in \ell_{i_1,i_2,\cdots,i_{n-1}} \text{ and } (x,y) \in R_{-1}. \end{cases} \tag{2.3b}$$

Therefore, $T^n\ell_{(m,k)} = T^n\ell$ has at most 3^n pieces of half-lines and segments. Using the above notation, we see that $T^n\ell = \{\ell_{i_1,i_2,\cdots,i_n} : u_j \in \{-1,0,1\}, j = 1,2,\cdots,n\}$. To understand $T^n\ell$, it is natural to first consider the cases that $i_1 = i_2 = \cdots = i_n$. That is the cases that ℓ has been applied by same dynamics repeatedly. To this end, we define the following two dimensional maps of the form,

$$G_i(x,y) = (a_i - \frac{b}{x}, \bar{a}_i + \frac{b}{x}y) =: (g_{i,1}(x), g_{i,2}(x,y)). \tag{2.4}$$

We call $g_{i,1}(x)$, $i = 1,0,-1$, the slope maps of T. Since $g_{i,1}(x)$, $i = 1,0,-1$, denote, respectively, the slopes of ℓ_i. Here $\ell = \ell_{(x,y)}$. Moreover, $g_{i,2}(x,y)$ are to be termed the intercept maps. Because if we let $\ell_{(x,y)} = \ell$, then $g_{i,2}(x,y)$ denote, respectively, $i = -1,0,1$, the y-intercepts of ℓ_i. We next consider the dynamics of the slope and intercept maps $g_{i,1}$ and $g_{i,2}$.

Proposition 2.1. *Let $b > 0$, $a_i > 2\sqrt{b}$, $i = 1,-1$ and $-a_0 > 2\sqrt{b}$. Then* (i) *for $i = 1,-1$, $m^{\pm}_{i,\infty} := \frac{a_i \pm \sqrt{a_i^2 - 4b}}{2}$ are two fixed points of the slope maps $g_{i,1}$. For $i = 0$, $m^{-}_{0,\infty} := \frac{a_0 + \sqrt{a_0^2 - 4b}}{2}$ and $m^{+}_{0,\infty} := \frac{a_0 - \sqrt{a_0^2 - 4b}}{2}$ are two fixed points of the slope maps $g_{0,1}$* (ii) *Moreover, the attracting interval of $m^{+}_{i,\infty}$, $i = 1,0,-1$, is $R - \{m^{-}_{i,\infty}\}$. That is to say if $x \in R - \{m^{-}_{i,\infty}\}$, then, for $i = 1,0,-1$, $\lim_{n\to\infty} g^n_{i,1}(x) = m^{+}_{i,\infty}$.* (iii) *Suppose $a_i = 2\sqrt{b}$. Then $m^{+}_{i,\infty} = m^{-}_{i,\infty}$ is the globally attracting fixed point of $g_{i,1}$, $i = 1,0,-1$.* (vi) *If $x \notin (m^{-}_{i,\infty}, m^{+}_{i,\infty})$, $i = 1,-1$, (resp., $x \notin (m^{+}_{0,\infty}, m^{-}_{0,\infty})$), then $g_{i,1}(x)$, $i = 1,0,-1$, converges to $m^{+}_{i,\infty}$ uniformly. That is, given $\varepsilon > 0$, there exists an N_ε, independent of x, such that $|g^n_{i,1}(x) - m^{+}_{i,\infty}| < \varepsilon$ where $n \geq N_\varepsilon$.*

Proof. We illustrate only $i = 1$. Clearly, two fixed points of $g_{1,1}$ are $m^{\pm}_{1,\infty}$. The attracting interval of $g_{1,1}$ can be easily concluded by using graphical analysis on Figure 2.1. To prove (vi), let $x = a_1$. For $\varepsilon > 0$, then there exists an N such that $|g^n_{1,1}(a_1) - m^{+}_{1,\infty}| < \varepsilon$ whenever $n \geq N$. Let $x \in (m^{+}_{1,\infty}, a_1)$, clearly, for $\varepsilon > 0$,

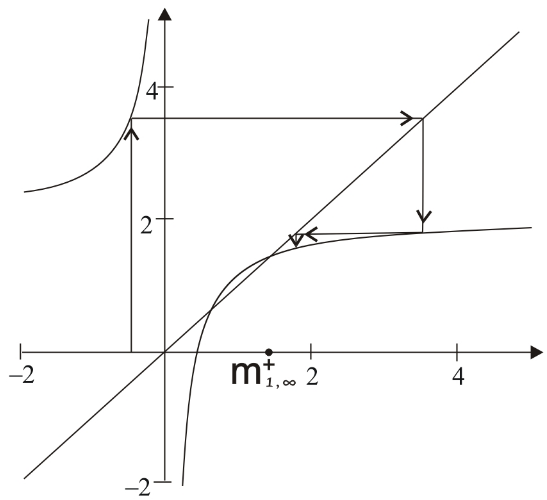

FIGURE 2.1

$|g^n_{1,1}(x) - m^+_{1,\infty}| < |g^n_{1,1}(a_1) - m^+_{1,\infty}| < \varepsilon$ whenever $n \geq N$. Now for $x \in (-\infty, m^-_{1,\infty})$, we see that $g^3_{1,1} \in (m^+_{1,\infty}, a_1)$. Thus, the assertion of Proposition 2.1-(vi) for $i = 1$ holds by choosing $N_\varepsilon = N + 3$. The other part of the proof is similar and is thus omitted. □

Remark 2.1. *Given $\ell_{(m,k)} = \ell$, we see from Figure 2.1, that the slopes of $\ell_{i_1,i_2,\cdots,i_n}$ remain positive (resp., negative) for all $n \geq 3$. Here $i_1, i_2, \cdots, i_n \in \{-1, 1\}$ (resp., $\in \{0\}$).*

Proposition 2.2. *Suppose*

$$b > 0,\ a_i > 1 + b,\ i = 1, -1 \ \textit{and} \ -a_0 > 1 + b. \tag{2.5}$$

For fixed $x = m^+_{i,\infty}$, $i = 1, 0, -1$, then $k_{i,\infty} := \frac{m^+_{i,\infty}\bar{a}_i}{m^+_{i,\infty} - b}$ is a globally attracting fixed point of the intercept maps $g_{i,2}(m^+_{i,\infty}, y)$.

Proof. It suffices to show that $0 < \frac{b}{m^+_{i,\infty}} < 1$, $i = 1, -1$, and $-1 < \frac{b}{m^+_{0,\infty}} < 0$.

We illustrate only $i = 1$. Now,

$$0 < \frac{b}{m^+_{1,\infty}} = \frac{2b}{a_1 + \sqrt{a_1^2 - 4b}} = \frac{a_1 - \sqrt{a_1^2 - 4b}}{2} < 1. \tag{2.6}$$

The last inequality is justified by the fact that $a_1 > 1 + b \geq 2\sqrt{b} > 0$. □

Theorem 2.1. *Suppose (2.5) holds.* (i) *The two dimensional map G_i, as defined in (2.4), $i = 1, 0, -1$, have two fixed points $(m^{\pm}_{i,\infty}, \frac{m^{\pm}_{i,\infty}\bar{a}_i}{m^{\pm}_{i,\infty}-b}) =: A^{\pm}_i$.* (ii) *Moreover, the attracting regions of $A^{\pm}_i$, $i = 1, 0, -1$, are $\mathbb{R}^2 - \{(x, y) : x = m^{-}_{i,\infty}\}$. That is to say, for any $(m, k) \in \mathbb{R}^2 - \{(x, y) : x = m^{-}_{i,\infty}\}$, $i = 1, 0, -1$, $\lim\limits_{n\to\infty} G^n_i(m, k) = A^{+}_i$.*

Proof. We only illustrate $i = 1$. The cases for $i = 0, -1$ are similar. Define $g^n_{1,1}(m) = m_{1,n}$ and $G^n_1(m, k) = (m_{1,n}, k_{1,n})$. If $m \neq m^{-}_{1,\infty}$, then given $\varepsilon > 0$, there exists an $N_\varepsilon \in \mathbb{N}$ such that for every $n \geq N_\varepsilon$, we have

$$m^{+}_{1,\infty} - \varepsilon < m_{1,n} < m^{+}_{1,\infty} + \varepsilon. \tag{2.7}$$

It follows from (2.7) that for any $k \in \mathbb{R}$, and n sufficiently large,

$$\begin{aligned}\min\{\bar{a}_1 + \frac{bk}{m^{+}_{1,\infty} - \varepsilon},\ \bar{a}_1 + \frac{bk}{m^{+}_{1,\infty} + \varepsilon}\} &< \bar{a}_1 + \frac{bk}{m_{1,n}} \\ &< \max\{\bar{a}_1 + \frac{bk}{m^{+}_{1,\infty} - \varepsilon},\ \bar{a}_1 + \frac{bk}{m^{+}_{1,\infty} + \varepsilon}\}.\end{aligned} \tag{2.8}$$

It follows from (2.6) and Proposition 2.2 that for all sufficiently small $\varepsilon > 0$, $\lim\limits_{n\to\infty} g^n_{1,2}(m^{+}_{1,\infty} \pm \varepsilon, k)$ exist and that

$$\lim_{n\to\infty} g^n_{1,2}(m^{+}_{1,\infty} \pm \varepsilon, k) = \frac{\bar{a}_1(m^{+}_{1,\infty} \pm \varepsilon)}{m^{+}_{1,\infty} \pm \varepsilon - b} =: k_{1\pm\varepsilon}.$$

Using (2.8), we see inductively that

$$\begin{aligned}\min\{g^n_{1,2}(m^{+}_{1,\infty} + \varepsilon, k),\ g^n_{1,2}(m^{+}_{1,\infty} - \varepsilon, k)\} &< g^n_{1,2}(m_{1,n}, k) \\ &< \max\{g^n_{1,2}(m^{+}_{1,\infty} + \varepsilon, k),\ g^n_{1,2}(m^{+}_{1,\infty} - \varepsilon, k)\}.\end{aligned}$$

However, it is easy to see that the single limits $\lim\limits_{n\to\infty} g^n_{1,2}(m^{+}_{1,\infty} \pm \varepsilon, k)$ and $\lim\limits_{\varepsilon\to 0} g^n_{1,2}(m^{+}_{1,\infty} \pm \varepsilon, k)$ exist and the convergence of $\lim\limits_{n\to\infty} g^n_{1,2}(m^{+}_{1,\infty} \pm \varepsilon, k)$ is uniform for all sufficiently small $\varepsilon > 0$. So the double limit and both iterated limits of $g^n_{1,2}(m^{+}_{1,\infty} \pm \varepsilon, k)$ exist and all three limits are equal. However, $\lim\limits_{\varepsilon\to 0}\lim\limits_{n\to\infty} g^n_{1,2}(m^{+}_{1,\infty} \pm \varepsilon, k) = \lim\limits_{\varepsilon\to 0} k_{1\pm\varepsilon} = k_{1,\infty}$. Taking the double limit on (2.8), we see that the double limit of $g^n_{1,2}(m^{+}_{1,n}, k)$ exists and equals to $k_{1,\infty}$. It is then easy to see that, for $(m, k) \in \mathbb{R}^2 - \{(x, y) : x = m^{-}_{i,\infty}\}$, $\lim\limits_{n\to\infty} G^n_1(m, k) = (m^{+}_{1,\infty}, \dfrac{m^{+}_{1,\infty}\bar{a}_1}{m^{+}_{1,\infty} - b})$. We thus complete the proof of theorem. $\square$

It then follows from Theorem 2.1 that for any $\ell = \ell_{(m,k)}$, $m \neq m^{-}_{i,\infty}$, $i = 1, 0, -1$, then $\ell_{i,i,\cdots,i,\cdots}$, i.e., applying the ith-dynamics on ℓ infinitely many times, are well-defined. The resulting images are denoted by ℓ_{i_∞}, where, for $i = 1, 0, -1$,

$$\ell_{i_\infty} = \ell_{(m',k')},\ (m', k') = A^{+}_i. \tag{2.9}$$

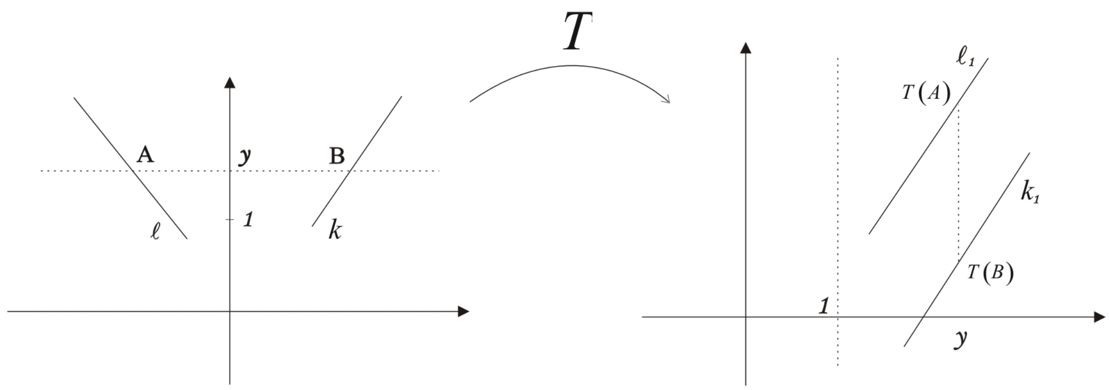

FIGURE 3.1

3. Boundary Influence on the Spatial Entropy

The following lemma is very useful in determining how the order of the line segments and half-lines $\ell_{i_1,i_2,\cdots,i_n}$ $i_j \in \{-1,0,1\}$, $j = 1,2,\cdots,n$, is. The proof is trivial and, thus, skipped.

Lemma 3.1. *Let $b > 0$. For fixed y, if $x_1 \geq x_2$, then the y-coordinate of $T(x_1,y)$ is no greater than that of $T(x_2,y)$.*

Remark 3.1. *Since our objective here is to study how the number of points in the intersection $T^n\ell_{(m_1,m_k)} \cap \ell_{(m_2,k_2)}$ grows as n increase, we may assume from here on by Remark 2.1 that the slopes of ℓ_1 and ℓ_{-1} are positive and that of ℓ_0 is negative.*

Using Remark 3.1, lemma 3.1 and the fact that T is one-to-one, we have the following principle. See Figure 3.1 for one special case.

Proposition 3.1. *Suppose (2.5) holds. Let ℓ and k be lines or line segments or half lines, and $\ell \cap k = \emptyset$. If k is to the right of ℓ. Then so are k_i to ℓ_i, for $i = 1,-1$. However, ℓ_0 is to the right of k_0. Here k_i, ℓ_i, $i = 1,0,-1$ are defined in (2.3a).*

Note that the reverse of the ordering in k_0 and ℓ_0 is due to the fact that, in R_0, $F(y)$ has a negative slope. We next give the ordering of $\ell_{i_1,i_2,\cdots,i_n}$ in terms of their locations to each others. Moreover, we will show that all $\ell_{i_1,i_2,\cdots,i_n}$, except 6 possibly line segments/ half lines, lie in an N-shaped tube whose boundaries are given in Figure 3.2, where ℓ_{i_∞}, $i = 1,0,-1$, are defined in (2.9) and $\ell_{i_\infty,j}$, $j = 1,0,-1$, means the j-th dynamics is applied to ℓ_{i_∞}.

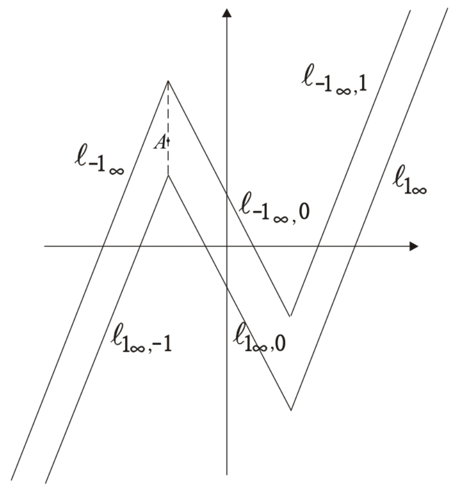

FIGURE 3.2

Proposition 3.2. *Suppose (2.5) holds. Then the following holds true.*

(1) *Let $\{i_m\}_{m=1}^n$ and $\{j_m\}_{m=1}^n$ be two distinct finite sequences, i_m and $j_m \in \{1,0,-1\}$, $m = 1,2,\cdots,n$. Suppose k is the first index such that $i_\ell = j_\ell$ for all $\ell \geq k$. Then $\ell_{i_1,i_2,\cdots,i_n}$ is to the right of $\ell_{j_1,j_2,\cdots,j_n}$ provided that the following hold.*
   (a) *$i_k = 1$ or -1, and $i_{k-1} > j_{k-1}$.*
   (b) *$i_k = 0$ and $i_{k-1} < j_{k-1}$.*

(2) *For any straight ℓ and $n \in \mathbb{N}$, $\ell_{i_1,i_2,\cdots,i_n}$, $i_j \in \{-1,0,1\}$, $j = 1,2,\cdots,n$, lie in the N-shaped tube, see Figure 3.2, except possibly for those $\ell_{1,\cdots,1,i_n}$, $\ell_{-1,\cdots,-1,i_n}$, $i_n = 1,0,-1$. Here the boundaries of the N-shaped tube are ℓ_{1_∞}, $\ell_{1_\infty,0}$, $\ell_{1_\infty,-1}$, ℓ_{-1_∞}, $\ell_{-1_\infty,0}$ and $\ell_{-1_\infty,1}$.*

Proof. The assertions for (1) follows directly from Proposition 3.1. To see (2), let $\ell_{i_1,i_2,\cdots,i_{n-1},1} \neq \ell_{1,1,\cdots,1}$. Then there exists $1 \leq j \leq n-1$ such that $i_j \neq 1$ and so $\ell_{i_1,i_2,\cdots,i_{j-1},i_j} \in R_{-1}$ or R_0. Thus, $\ell_{i_1,i_2,\cdots,i_j}$ is to the left of ℓ_{1_∞}. Therefore, it follows from Proposition 3.1 that $\ell_{i_1,i_2,\cdots,i_j,i_{j+1},\cdots,i_{n-1},1}$ is to the left of $\ell_{1_\infty,1,\cdots,1} = \ell_{1_\infty}$. Hence, all $\ell_{i_1,i_2,\cdots,i_{n-1},1}$, $i_j \in \{1,0,-1\}$, $1 \leqslant j \leqslant n-1$, are to the left of ℓ_{1_∞} except possibly $\ell_{1,1,\cdots,1}$. The proof for the other parts of (2) is similar. $\square$

We note that the boundary points of the N-shaped tube are ω-limit points $\omega(\ell_1;T)$ of ℓ_1 for T. That is, if $B \in \omega(\ell_1;T)$, then there exists an $A \in \ell_1$, and a sequence $\{n_k\}_{k=1}^\infty$, $n_k \in \mathbb{N}$, such that $T^{n_k}(A) \to B$ as $k \to \infty$.

To ensure that each $\ell_{i_1,i_2,\cdots,i_n}$ is nonempty, we need the following lemma.

Lemma 3.2. *Let*

$$\min\{a_1, a_{-1}\} \geq -a_0 > 1 + 2b, \text{ and } \bar{a}_0 \text{ is sufficiently small.} \tag{3.1}$$

Then the y-coordinate $(\ell_{-1_\infty,0} \cap \ell_{-1_\infty,1})_y$ of $(\ell_{-1_\infty,0} \cap \ell_{-1_\infty,1})$ is less than -1, and $(\ell_{1_\infty,-1} \cap \ell_{1_\infty,0})_y > 1$.

Proof. We illustrate only $(\ell_{1_\infty,-1} \cap \ell_{1_\infty,0})_y > 1$. The other assertion is similarly obtained. Note that the equation of the line ℓ_{1_∞} is $y = m^+_{1,\infty}x + k_{1,\infty}$. Letting $y = -1$, we have that $x = \frac{-k_{1,\infty}-1}{m^+_{1,\infty}}$. Clearly, $(\ell_{1_\infty,-1} \cap \ell_{1_\infty,0})_y=$

$$\text{the } y\text{-coordinate of } T(\frac{-k_{1,\infty} - 1}{m^+_{1,\infty}}, -1) = -a_0 + \bar{a}_0 + \frac{b(k_{1,\infty} + 1)}{m^+_{1,\infty}} =: t. \tag{3.2}$$

Now, taking $\bar{a}_0 = 0$, we have

$$\frac{-k_{1,\infty} - 1}{m^+_{1,\infty}} < \frac{-k_{1,\infty}}{m^+_{1,\infty}} = \frac{a_1 - a_0}{m^+_{1,\infty} - b} \leq \frac{2a_1}{m^+_{1,\infty} - b} \leq \frac{2(1 + 2b)}{1 + \sqrt{1 + 4b^2}} \leq 2. \tag{3.3}$$

The fact that $\frac{2a_1}{m^+_{1,\infty}-b}$ is decreasing in a_1 has been used to justify the above inequalities. Thus, for $\bar{a}_0 = 0$, we have $t \geq -a_0 - 2b > 1$. We just completed the proof the lemma. □

Remark 3.2. *Since our objective is to study the spatial entropy of the system, without loss of generality, we may assume, via Proposition 3.2 and Lemma 3.2, that all $\ell_{i_1,i_2,\cdots,i_n}$, $i_j \in \{1, 0, -1\}$, $1 \leq j \leq n$, are nonempty provided that (3.1) holds.*

We next give stronger conditions on a_i, $i = 1, 0, -1$ so as to ensure that for any $j \in \mathbb{N}$, $\ell_{i_1,i_2,\cdots,i_j,\cdots,i_{n+j}}$, where $i_j = \cdots = i_{j+n} = 1$ or $i_j = \cdots = i_{j+n} = -1$ become unbounded as n grows larger for any j.

Lemma 3.3. *Suppose*

$$\frac{1}{2}\min\{a_1, a_{-1}\} \geq -a_0 \geq 3 + 2b \text{ and that } \bar{a}_0 \text{ is sufficiently small.} \tag{3.4}$$

Let A , see Figure 3.2, be any point in the line segment for which its both endpoints are $\ell_{-1_\infty} \cap \ell_{-1_\infty,0}$ and $\ell_{1_\infty,-1} \cap \ell_{1_\infty,0}$ (resp., $\ell_{1_\infty} \cap \ell_{1_\infty,0}$ and $\ell_{-1_\infty,0} \cap \ell_{-1_\infty,1}$). Then the limit of both coordinates of $T^n(A)$ approaches to $+\infty$ (resp., $-\infty$).

Proof. We first note that T has a fixed point $B = (\frac{a_1-a_0-\bar{a}_0}{a_1-1-b}, \frac{a_1-a_0-\bar{a}_0}{a_1-1-b})$ in $\mathbb{R}^1$ for which its stable (resp., unstable) direction is $(1, \frac{a_1-\sqrt{a_1^2-4b}}{2})$. (resp., $(1, \frac{a_1+\sqrt{a_1^2-4b}}{2})$). Since $(\ell_{-1_\infty} \cap \ell_{-1_\infty,0})_y > (\ell_{1_\infty,-1} \cap \ell_{1_\infty,0})_y > 1$, via lemma 3.2, it suffices to show that $T^n(\ell_{1_\infty,-1} \cap \ell_{1_\infty,0}) \to (+\infty, +\infty)$ as $n \to \infty$. To this end, we need to show that $T(\ell_{1_\infty,-1} \cap \ell_{1_\infty,0}) = T(-1,t)$, t as given in (3.2), lies on the upper half of the stable line

$$(y - \frac{a_1 - a_0 - \bar{a}_0}{a_1 - 1 - b}) = m^-_{1,\infty}(x - \frac{a_1 - a_0 - \bar{a}_0}{a_1 - 1 - b}),$$

or, equivalently,

$$F(t) + b - \frac{a_1 - a_0 - \bar{a}_0}{a_1 - 1 - b} - m^-_{1,\infty} t + m^-_{1,\infty} \frac{a_1 - a_0 - \bar{a}_0}{a_1 - 1 - b} =: h(\bar{a}_0) > 0.$$

Now,

$$h(0) = a_1 t + a_0 - a_1 - \frac{a_1 - a_0}{a_1 - 1 - b} + b - m^-_{1,\infty} t + m^-_{1,\infty} \frac{a_1 - a_0}{a_1 - 1 - b}. \tag{3.5}$$

We also have that with $\bar{a}_0 = 0$,

$$\begin{aligned} b - m^-_{1,\infty} t = b - \frac{2b}{a_1 + \sqrt{a_1^2 + 4b}} t \geq b - \frac{2b}{a_1 + \sqrt{a_1^2 + 4b}}(-a_0) \\ \geq b - \frac{b(-a_0)}{a_1} > 0, \end{aligned} \tag{3.6}$$

and

$$\frac{a_1 - a_0}{a_1 - 1 - b} \leq a_1 - a_0. \tag{3.7}$$

It then follows from (3.3), (3.5), (3.6) and (3.7) that

$$h(0) > a_1(-a_0 - 2b) + 2(a_0 - a_1) = a_1(-a_0 - 2b - 2) + 2a_0 \geq a_1 + 2a_0 \geq 0$$

We thus complete the proof of the lemma. $\square$

The first main results of the paper are stated in the following.

Theorem 3.1. *Let (3.4) hold.* (1) *If* $a_1 > a_{-1}$, *(resp.,* $a_1 < a_{-1}$*), let* $\ell_{(2)}$ *be a line satisfying the following* (i) $(\ell_{(2)} \cap \ell_{(\infty,1)})_y \leq (\ell_{1_\infty} \cap \ell_{1_\infty,0})_y$ (resp., $(\ell_{(2)} \cap \ell_{(\infty,-1)})_y \geq (\ell_{-1_\infty} \cap \ell_{-1_\infty,0})_y$. (ii) $m^+_{-1,\infty} \leq m \leq m^+_{1,\infty}$ (resp., $m^+_{1,\infty} \leq m \leq m^+_{-1,\infty}$), where m is the slope of $\ell_{(2)}$, then $h_{\ell_{(1)},\ell_{(2)}}(T) = 0$; otherwise, $h_{\ell_{(1)},\ell_{(2)}}(T) = \ln 3$. (2) If $a_1 = a_{-1}$, let $\ell_{(2)}$ be a line with the slope $m = a_1$ and the y-intercept $\bar{y}$ of k satisfying $\bar{y} \geq k_{-1,\infty}$ or $\bar{y} \leq k_{1,\infty}$, then $h_{\ell_{(1)},\ell_{(2)}}(T) = 0$; otherwise, $h_{\ell_{(1)},\ell_{(2)}}(T) = \ln 3$. (3) $h_{\ell_{(1)},\ell_{(2)}}(T)$ is independent of the choice of $\ell_{(1)}$.

Proof. Let $a_1 > a_{-1}$. We will break down $\ell_{(2)} = k$ into three cases.

(a) $(\ell_{-1_\infty,0} \cap \ell_{-1_\infty,1})_y > (k \cap \ell_{(\infty,1)})_y > (\ell_{1_\infty} \cap \ell_{1_\infty,0})_y$. See Figure 3.3.
(b) $(k \cap \ell_{(\infty,1)})_y \geq (\ell_{-1_\infty,0} \cap \ell_{-1_\infty,1})_y$.
(c) $(k \cap \ell_{(\infty,1)})_y \leq (\ell_{1_\infty} \cap \ell_{1_\infty,0})_y$ and $m > m^+_{1,\infty}$ or $m < m^+_{-1,\infty}$.

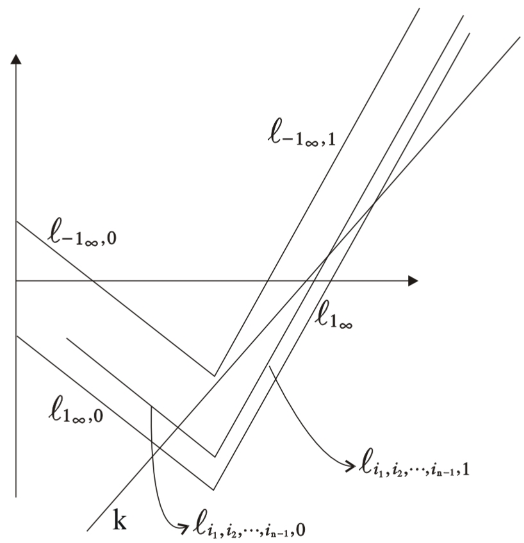

FIGURE 3.3

To prove case (a), we note, via Proposition 2.1-(iii) and Proposition 3.1, that for N sufficiently large, $\ell_{0,i_2,\cdots,i_N}$, where $i_2 = i_3 = \cdots = i_N = 1$, we have that

$$(k \cap \ell_{(\infty,1)})_y > (\ell_{0,i_2,\cdots,i_N} \cap \ell_{(\infty,1)})_y \text{ for any } \ell. \tag{3.8}$$

In particular, the natural number N is independent of the choice of ℓ. We also note that ℓ_0 has a negative slope. Therefore, for any $n \geq N$, by identifying $\ell = \ell_{j_1,j_2,\cdots,j_{n-N}}$, we see that

$$\begin{aligned} &\ell_{j_1,j_2,\cdots,j_{n-N+1},\cdots,j_n}, \text{ where } j_k \in \{1,0,-1\}, \text{ for } 1 \leq k \leq n-N, \\ &j_{n-N+1} = 0 \text{ and } j_{n-N+2} = \cdots = j_n = 1, \text{ satisfying (3.8).} \end{aligned} \tag{3.9}$$

Hence, k must intersects with $\ell_{j_1,j_2,\cdots,j_{n-1},0}$, where $j_1,\cdots,j_{n-1}$ are given as in (3.9), see Figure 3.3. Hence, for any $n \geq N$, the number $\mathcal{N}(n,\ell_{(1)},k,T)$ of intersections of $T^n\ell_{(1)} \cap k$ satisfies

$$3^{n-N} \leq \mathcal{N}(n,\ell_{(1)},k,T) \leq 3^n.$$

Thus, $h_{\ell_{(1)},\ell_{(2)}}(T) = \ln 3$. If $a_1 > a_{-1}$ and case (b) or case (c) holds, then k must intersect with ℓ_{-1_∞} and $\ell_{1_\infty,-1}$ or $\ell_{-1_\infty,0}$ and $\ell_{1_\infty,0}$ or $\ell_{-1_\infty,1}$ and ℓ_{1_∞}. Upon using Lemma 3.3, we conclude, in any of three cases, that $h_{\ell_1,\ell_2}(T) = \ln 3$. The proof for $a_1 < a_{-1}$ is similar and thus omitted. We have just completed the proof of the first part of the theorem. The second part of the theorem is obvious and thus omitted. The last part of the theorem is a direct consequence of the first and the second assertions of the theorem. $\square$

4. **Stable Mosaic Solutions and Their Corresponding Boundary Influence**

In this section, we will show that $h^M(T) = h^M_D(T) = h^M_N(T) = \ln 2$ in some reasonable parameters range. To this end, we first give a sufficient condition on $a_i, b,\ i = 1, 0, -1$ so that the mosaic solutions are stable. In real applications, r and l are "small" positive constants. To have $a_1, a_{-1}, -a_0$ and b are all positive, we shall assume that $\beta > 0$, $a > 1$, and $\alpha > 0$.

Lemma 4.1. *If*

$$a_i > 1 + b,\ \ i = 1, -1, \tag{4.1}$$

then the stability condition (1.6) for the mosaic solutions of (1.1) is satisfied.

Proof. Dividing β on both side of (1.6) yields that

$$\min\{\frac{1}{\beta r}, \frac{1}{\beta\ell}\} > \frac{a}{\beta} + \frac{\alpha}{\beta} + 1.$$

Using (1.4d), we get

$$a_i + \frac{a}{\beta} > \frac{a}{\beta} + b + 1,\ \ i = 1, -1.$$

Thus, the assertion of the lemma holds as asserted. $\square$

To show that $h^M_N(T) = \ln 2$, we need the following lemma.

Lemma 4.2. *Suppose*

$$-a_0 > 2 + 2b,\ \ \min\{a_1, a_{-1}\} > 4 + b \text{ and } \bar{a}_0 \text{ is sufficiently small.} \tag{4.2}$$

Then $T(-1, t) = T(\ell_{1_\infty,-1} \cap \ell_{1_\infty,0})$ (resp., $T(\ell_{-1_\infty,0} \cap \ell_{-1_\infty,1})$) stays above (resp., below) the line $y = x$, where t is given as in (3.2).

Proof. We only illustrate the proof of the first assertion of the lemma. To this end, we first note that (4.2) implies (3.1). We then need to show that

$$F(t) + b > t. \tag{4.3}$$

Letting $\bar{a}_0 = 0$, (4.3) becomes

$$(a_1 - 1)t + a_0 - a_1 + b := L > 0. \tag{4.4}$$

Using (3.3), we see, via (4.2) that

$$L \geq (a_1 - 1)(-a_0 - 2b) + a_0 - a_1 + b = (-a_0 - 1 - 2b)(a_1 - 2) - 2 - b > 0.$$

We just complete the proof of the theorem. $\square$

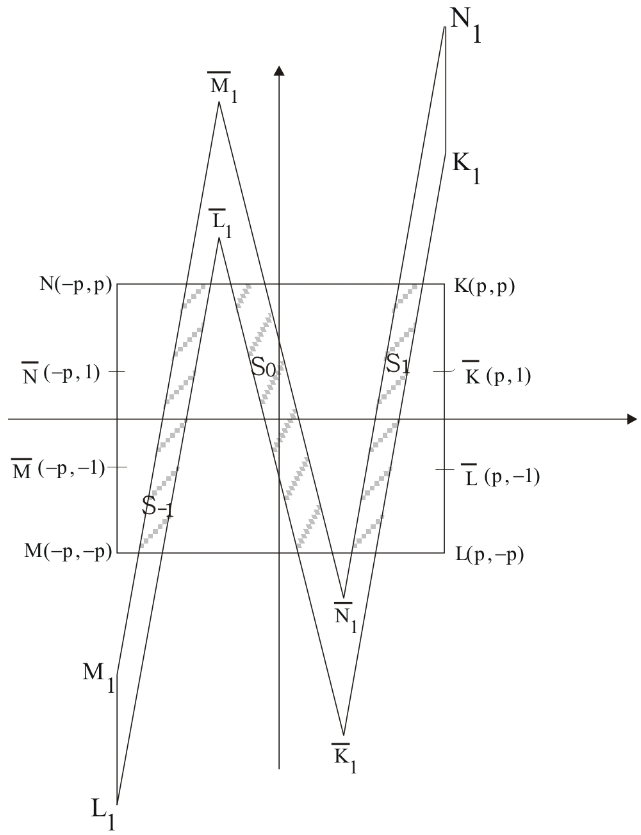

FIGURE 4.1. Here we denote by $K_1 = T(K)$. We use similar notations to denote points under the first iteration of T.

Let S be a square defined as

$$S = \{(x, y) \in \mathbb{R}^2 : |x| \leq p,\ |y| \leq p\}, \text{ where}$$

$p > 1$. Then $T(S) \cap S = S_{-1} \cup S_0 \cup S_1$. See Figure 4.1.

Inductively, we see that $T^n(S) \cap S$ consists of 3^n nested pieces of $S_{i_1,i_2,\cdots,i_n}$, $i_j = 1, 0, -1$, $j = 1, 2, \cdots, n$. Likewise, backward iterations: $T^{-n}(S) \cap S$ will produce 3^n nested pieces of $\bar{S}_{i_1,i_2,\cdots,i_n}$, $i_j = 1, 0, -1$, $j = 1, 2, \cdots, n$ with each piece $\bar{S}_{i_1,i_2,\cdots,i_n}$ crossing the east and west side of the rectangle S. Let A be a point in $\mathbb{R}^2$. $T(A)$ is denoted by A_1. Let $K = (p, p)$, $\bar{L} = (p, -1)$, $\bar{N} = (-p, 1)$ and $M = (-p, -p)$. If

$$K_1 \text{ and } \bar{L}_1 \text{ stay above } y = p, \tag{4.5a}$$

and

$$\bar{N}_1 \text{ and } M_1 \text{ stay below } y = -p, \tag{4.5b}$$

then each of $S_{i_1,i_2,\cdots,i_n}$ is nonempty. The following lemma gives a sufficient condition on the parameters a_i, $i = 1, 0, -1$ and b so that (4.5) holds.

Lemma 4.3. *Suppose*

$$(-a_0-1-b)(\min\{a_1, a_{-1}\}-2(1+b)) > 2(1+b)^2 \text{ and } \bar{a}_0 \text{ is sufficiently small} . \quad (4.6)$$

Then there exists a $p > 1$ such that the following holds.

$$F(p) - bp > p, \quad (4.7a)$$

$$F(1) + bp < -p, \quad (4.7b)$$

$$F(-1) - bp > p, \quad (4.7c)$$

and

$$F(-p) + bp < -p. \quad (4.7d)$$

.

Proof. Equations (4.7) are equivalent to

$$\min\{\frac{-a_0+\bar{a}_0}{1+b}, \frac{-a_0-\bar{a}_0}{1+b}\} > p > \max\{\frac{a_1-a_0-\bar{a}_0}{a_1-1-b}, \frac{a_{-1}-a_0+\bar{a}_0}{a_{-1}-1-b}\}. \quad (4.8)$$

Letting $\bar{a}_0 = 0$, (4.8) reduces to

$$\frac{-a_0}{1+b} > p > \max\{\frac{a_1-a_0}{a_1-1-b}, \frac{a_{-1}-a_0}{a_{-1}-1-b}\}. \quad (4.9)$$

Clearly, $-\frac{a_0}{1+b} > 1$. Thus, if

$$\frac{-a_0}{1+b} > \max\{\frac{a_1-a_0}{a_1-1-b}, \frac{a_{-1}-a_0}{a_{-1}-1-b}\}, \quad (4.10)$$

then there exists a $p > 1$ such that (4.9) holds. However, (4.6) implies (4.10). The proof of the lemma is thus complete. □

Remark 4.1. *Note that $T^{-1}(x, y) = (\frac{1}{b}F(x) - \frac{y}{b}, x)$. Replacing a_i, $i = 1, 0, -1$, by $\frac{a_i}{b}$, b by $\frac{1}{b}$ and $\bar{a}_0$ by $\frac{\bar{a}_0}{b}$, we see (4.6) is invariant. That is $(-\frac{a_0}{b} - 1 - b)(\min\{\frac{a_1}{b}\frac{a_{-1}}{b}\} - 2(1 + \frac{1}{b})) > 2(1 + \frac{1}{b})^2$ and $\frac{\bar{a}_0}{b}$ is sufficiently small if and only if (4.6) holds. Thus, (4.6) is not only to ensure that each of $S_{i_1,i_2,\cdots,1_n}$ is nonempty but also that each of $\bar{S}_{i_1,i_2,\cdots,1_n}$ is nonempty.*

To show that T has a Smale-Horseshoe, we need to show that each of $S_{i_1,i_2,\cdots,i_n}$, and $\bar{S}_{i_1,i_2,\cdots,i_n}$ shrinks to a line segment as $n \to \infty$. To this end, we first need the following notations. Let the slope and intercept pair of two straight lines ℓ and $\bar{\ell}$ are (m, k) and $(m, \bar{k})$, respectively. Here $k \neq \bar{k}$. Let the slope and intercept pair of $\ell_{i_1,i_2,\cdots,i_n}$ and $\bar{\ell}_{i_1,i_2,\cdots,i_n}$, $i_1 = i_2 = \cdots = i_n \in \{1,0,-1\}$, are, respectively, $(m_{n,i}, k_{n,i})$ and $(m_{n,i}, \bar{k}_{n,i})$. Define

$$d_{0,i} = |k - \bar{k}|,\ i = 1, 0, -1, \tag{4.11a}$$

and

$$d_{n,i} = |k_{n,i} - \bar{k}_{n,i}|,\ i = 1, 0, -1. \tag{4.11b}$$

Lemma 4.4. *Let $m > m^+_{i,\infty}$, $i = 1, -1$, and $m < -m^+_{0,\infty}$, respectively. Suppose*

$$a_i \geq 1 + 2b,\ i = 1, -1,\ \textit{and}\ -a_0 \geq 1 + 2b,\ \textit{respectively.} \tag{4.12}$$

Then, respectively, $d_{n+1,i} \leq \frac{1}{2} d_{n,i}$, $i = 1, 0, -1$, for all $n \in \mathbb{N}$. Moreover, if $-\infty < m < m^+_{i,\infty}$, $i = 1, -1$, and $m^-_{0,\infty} < m < \infty$, then $d_{n+1,i} \leq \frac{1}{2} d_{n,i}$ for $i = 1, 0, -1$, respectively, for all $n \geq 3$.

Proof. We illustrate only the case $i = 1$. Using (2.4), we see that

$$d_{n+1,1} = \frac{b}{m_{n,1}} d_{n,1} \leq \frac{b}{m^+_{1,\infty}} d_{n,1}$$

The inequality above is justify by the fact that $m_{n,1} > m^+_{1,\infty}$ for all $n \in \mathbb{N}$, see Figure 2.1. However,

$$\frac{b}{m^+_{1,\infty}} = \frac{2b}{a_1 + \sqrt{a_1^2 - 4b}} \leq \frac{2b}{1 + 2b + \sqrt{1 + 4b^2}} = \frac{2}{\frac{1}{b} + 2 + \sqrt{\frac{1}{b^2} + 4}} \leq \frac{1}{2},$$

we thus complete the proof of the first part the lemma. Upon using Figure 2.1, we see immediately that the remaining assertions of the lemma holds true. □

Remark 4.2. *In the case of T^{-1}, if one reverses the role of x and y in the slope and intercept pair (m, k), then the assertions of Lemma 4.4 hold provided that (4.12) is replaced by*

$$\frac{a_1}{b} \geq 1 + \frac{2}{b}\ \textit{or equivalentlly}\ a_i \geq 2 + b,\ i = 1, -1\ \textit{and}\ -a_0 \geq 2 + b.$$

Thus, if

$$\min\{a_1, a_{-1}, a_0\} \geq \max\{1 + 2b, 2 + b\}, \tag{4.13}$$

then the size of each of $S_{i_1,i_2,\cdots,i_n}$ or $\bar{S}_{i_1,i_2,\cdots,i_n}$ shrinks by a factor no greater than $\frac{1}{2}$ as one applies the i-th dynamics on them.

We are now in the position to state the second main results of the paper.

Theorem 4.1. (i) *Suppose (3.1) holds. Then* $h_D^M(T) = \ln 2$. (ii) *Suppose (4.2) holds. Then* $h_N^M(T) = \ln 2$. (iii) *Suppose*

$$-a_0 > 2(1+b) \text{ and } \min\{a_1, a_{-1}\} > 4(1+b). \tag{4.14}$$

Then $h^M(T) = h_N^M(T) = h_D^M(T) = \ln 2$.

Proof. Suppose (3.1) holds. Then the mosaic solutions under consideration are all stable. The first assertion of the theorem follows from Lemma 3.2. Suppose (4.2) holds. Let $\Gamma_n =$ the number of intersections points of $\ell_{i_1,i_2,\cdots,i_n}$, $i_j \in \{1,-1\}$, $1 \le j \le n$, and the line $y = x$. We see, via Lemma 4.2, that $2^n - 4 \le \Gamma_n \le 2^n$. Thus, $h_N^M(T) = \ln 2$. To prove (iii), we first note that if (4.14) holds, then (4.2), and (4.6) and (4.13) are satisfied. Applying Lemmas 4.3 and 4.4, we see that $\bigcap_{n=-\infty}^{\infty} T^n(S) \cap S =: \Lambda$ is a cantor set of infinite points. Let $\Lambda_2 = \Lambda \cap \{(x,y) \in \mathbb{R}^2 : |x|, |y| > 1\}$, and Σ_2 be the space the two sided sequences of 1's and -1's. Define the itinerary map $i : \Lambda_2 \to \Sigma_2$

$$i(P) = (\cdots\cdots s_{-2}s_{-1}s_0s_1s_2\cdots\cdots),$$

where

$$p \in \Lambda_2 \text{ and } s_j = k \text{ if and only if } T^j(P) \in S_k.$$

Impose a metric on Σ_2 by defining

$$d[(s_i)_{i=-\infty}^{\infty}, (t_i)_{i=-\infty}^{\infty}] = \sum_{i=-\infty}^{\infty} \frac{|s_i - t_i|}{2^{|i|}}.$$

Define the shift map σ by

$$\sigma((s_i)_{i=-\infty}^{\infty}) = (t_i)_{i=-\infty}^{\infty}, \text{ where } t_i = s_{i+1}.$$

It is then not difficult to show, see e.g., [Devaney, [10]] and [Robinson, [14]], that the dynamics of T on the invariant set Λ_2 is conjugate to the shift map on Σ_2. Note that any trajectory of T in Λ_2 is a bounded, stable mosaic steady state of (1.1). We just complete the proof of the theorem. □

We conclude this paper with the following remarks.

(1) If (3.4) holds, then (4.14) is also satisfied. Consequently, all assertions in Theorem 3.1 hold true for the spatial mosaic entropy $h_{\ell_1,\ell_2}^M(T)$.
(2) In the language of CNNs, condition (4.14) means that the slopes r and l of the output function f are chosen to be small and so is the bias term z. The self-interaction weight a has to be relatively strong.

(3) It is also of interest to see if our techniques developed here can be applied to the cases when $F(y)$ is a cubic polynomial, such as those in p.163 of Afraimovich and Hsu [2003] or a quadratic map for which the resulting T is a Henon map.

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