行政院國家科學委員會專題研究計畫　成果報告

針對一 impedance crack 的逆散射問題

計畫類別：個別型計畫
計畫編號：NSC93-2119-M-009-004-
執行期間：93 年 11 月 01 日至 94 年 07 月 31 日
執行單位：國立交通大學應用數學系(所)

計畫主持人：李國明

報告類型：精簡報告
報告附件：出席國際會議研究心得報告及發表論文
處理方式：本計畫可公開查詢

中　華　民　國 94 年 10 月 24 日

Fréchet differentiability of integral operators in inverse acoustic scattering

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Abstract

Using integral equation methods to solve the time-harmonic acoustic scattering problem with impedance boundary conditions for a crack, it is possible to reduce the solution of the scattering problem to the solution of a system of integral equations of the second kind. In this research, we will establish the Fréchet differentiability of the Far Field operator w.r.t. the crack of the inverse scattering problem.

Contents

[*]Research supported by the NSC, Taiwan

1 Introduction

The inverse problem of recovering the geometry and the physical properties of a scatterer from the knowledge of the far field pattern of a scattered field is of fundamental importance for example in non-destructive testing or in medical imaging. In this research we consider as a model the time-harmonic impedance scattering problem from a crack. We shall use the boundary integral equation method for solving the scattering problem. The scattering problem in the unbounded domain is thus converted into a system of integral equations. For the reconstuction of the shape of the crack, we will therefore need the Fréchet derivative of the far field operator which maps the unknown crack to the far field pattern of the scattered field. The aim of this reseach is to establish the Fréchet differentiability of the far field operator with respect to the crack. Since the study of inverse problems always need a solid knowledge of the coressponding direct problems, we will include a brief discussion of the direct impedance problem in section 2. In the section 3 we will define the inverse problem. The far field pattern of the inverse problem will also be defined. The concept of Fréchet differentiability will be introduced in the follwing section. In the very last section of this paper we will prove the Fréchet differentiability of this far field operator.

2 Direct Impedance Problem

Let an open arc $\Gamma \subset \mathbb{R}^2$ be given. We assume that Γ is regular, i.e.

$$\Gamma := \{z(s) : s \in [-1, 1], z \in C^3(-1, 1) \text{ and } |z'(s)| \neq 0, \forall s \in [-1, 1]\}$$

We denote the end points of the crack with x^*_{-1}, x^*_1 respectively. The left hand side and the right hand side of the crack are written by Γ_+ and Γ_- respectively. The outwards normal to Γ_+ is denoted by ν. Further we set $\Gamma_0 := \Gamma \setminus \{x^*_{-1}, x^*_1\}$

We can now define the direct scattering problem.

Definition 1. (DP)
Given two functions $f_\pm \in C^{0,\alpha}(\Gamma)$, find a solution $u \in C^2(\mathbb{R}^2 \setminus \Gamma) \cap C(\mathbb{R}^2 \setminus \Gamma_0)$ to the Helmholtz equation

$$\Delta u + k^2 u = 0, \quad in\ \mathbb{R}^2 \setminus \Gamma, k > 0. \tag{1}$$

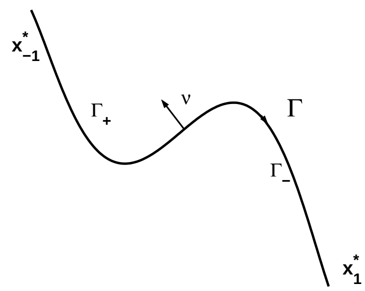

Figure 1: Open Arc in $\mathbb{R}^2$

which satisfies the impedance boundary conditions(IBC)

$$\frac{\partial u_\pm}{\partial \nu} \pm ik\lambda u_\pm = f_\pm \qquad on\ \Gamma_0 \tag{2}$$

with an impedance function $\lambda \in C^{0,\alpha}(\Gamma), \lambda \geq 0$
and the Sommerfeld radiation condition(SRC)

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u}{\partial \nu} - iku\right) = 0, \qquad r := |x| \tag{3}$$

uniformly for all directions $\hat{x} := \frac{x}{|x|}$

In the definition 1, the limits

$$u_\pm(x) := \lim_{h\to 0} u(x \pm h\nu(x)), \quad x \in \Gamma,$$

and

$$\frac{\partial u_\pm(x)}{\partial \nu} := \lim_{h\to 0} \nu(x)\mathrm{grad}u(x \pm h\nu(x)), \quad x \in \Gamma_0$$

are required to exist in the sense of local uniform convergence.

For $0 < \alpha < 1$ we define the function space

$$C^{1,\alpha}_{0,loc}(\Gamma) := C^{1,\alpha}_{loc}(\Gamma) \cap \{\varphi \in C(\Gamma) | \varphi(z^*_{-1}) = \varphi(z^*_1) = 0, \varphi' \in L^1(\Gamma)\}$$

For the direct problem we have proved the following theorem which ensures the unique solvability.

Theorem 1. *The direct Impedance problem (1) has a unique solution given by*

$$u^s(x) := \int_\Gamma \frac{\partial \Phi(x,y)}{\partial \nu(y)} \varphi_1(y) ds(y) + \int_\Gamma \Phi(x,y)\varphi_2(y) ds(y), \quad x \in \mathbb{R}^2 \setminus \Gamma. \quad (4)$$

where $\varphi_1 \in C^{1,\alpha}_{0,loc}(\Gamma), \varphi_2 \in C^{0,\alpha}_{loc}(\Gamma) \cap C(\Gamma)$ *are the (unique) solution to the following system :*

$$\begin{cases} 2\left(\frac{\partial}{\partial \nu(x)} \int_\Gamma \frac{\partial \Phi(x,y)}{\partial \nu(y)} \varphi_1(y) ds(y) + \int_\Gamma \frac{\partial \Phi(x,y)}{\partial \nu(x)} \varphi_2(y)\right) + ik\lambda(x)\varphi_1(x) \\ = f_-(x) + f_+(x) \\ \varphi_2(x) - 2ik\lambda(x)\left(\int_\Gamma \frac{\partial \Phi(x,y)}{\partial \nu(y)} \varphi_1(y) ds(y) + \int_\Gamma \Phi(x,y)\varphi_2(y) ds(y)\right) \\ = f_-(x) - f_+(x) \end{cases} \quad (5)$$

The idea of the proof is as follows: We first proved a special version of the Green's theorem. The uniqueness of the solution to the direct problems was then settled by this Green's theorem, the Rellich Lemma and the Sommerfeld radiation condition. After answering the question of uniqueness of the problem, we defined a solution ansatz by the combined single layer and double layer approach. The solvability of the boundary value problem was then converted to the solvability of the induced system of boundary integral equations which solvability can be determined by the Riesz theory in the operator form.

For a better presentation we would like to transform our system of integral equations (5) into operator form. For this purpose, we define the following integral operators:

$$(S\varphi)(x) := 2\int_\Gamma \Phi(x,y)\varphi(y) ds(y)$$

$$(K\varphi)(x) := 2\int_\Gamma \frac{\partial \Phi(x,y)}{\partial \nu(y)} \varphi(y) ds(y)$$

$$(K'\varphi)(x) := 2\int_\Gamma \frac{\partial \Phi(x,y)}{\partial \nu(x)} \varphi(y) ds(y)$$

$$(T_0\varphi)(x) := 2\frac{\partial}{\partial \nu(x)} \int_\Gamma \frac{\partial \Phi(x,y)}{\partial \nu(y)} \varphi(y) ds(y)$$

Using the Mau's Identity we can split the hypersingular operator T_0 into two easier parts, namely

$$T_0\varphi = \frac{\partial}{\partial\vartheta} S \frac{\partial\varphi}{\partial\vartheta} + k^2 < \nu, S\varphi\nu >,$$

where ϑ is the unit tangent vector. The system (5) now becomes

$$\begin{cases} \frac{\partial}{\partial\vartheta} S \frac{\partial\varphi_1}{\partial\vartheta} + k^2 < \nu, S\varphi_1\nu > + ik\lambda\varphi_1 + K'\varphi_2 = f_- + f_+ \\ ik\lambda K\varphi_1 + ik\lambda S\varphi_2 - \varphi_2 = f_+ - f_- \end{cases} \tag{6}$$

We can write the system in the following operator equation

$$W\psi = g \tag{7}$$

3 Inverse Scattering Problem

Definition 2. (IP)
Determine the scatterer Γ if the far field pattern $u_\infty(\cdot, d)$ is known for all incident directions d and for one wave number $k > 0$.

A mathematical formulation of this inverse problem would be the solving of the following far field equation

$$F(\Gamma) = u_\infty \tag{8}$$

which will be described in details.

In order to define the far field operator for the inverse impedance scattering problem, we will first describe how the far field pattern is obtained from the direct scattering problem. In the direct scattering problem we have the incident field u^i as input data. Since that we are dealing with a boundary value problem, the incident field is restrict on the boundary and serves as the right-hand side. Using the solution ansatz means that we have to solve a system of integral equations for the two densities φ_1, φ_2. After solving the system, we use the definition of the ansatz to contruct the far field pattern, since there is a one-to-one correspondence between the scattered field and the far field pattern. In summary, we have the following diagram:

$$F : u^i \xrightarrow{R} u^i|_\Gamma \xrightarrow{W^{-1}} (\varphi_1, \varphi_2)^t \xrightarrow{Q} u^s \xrightarrow{P} u_\infty \tag{9}$$

Using the parameterization $\Gamma = \{z(t) : t \in [-1, 1]\}$, we have the following parameterized form for the operators appeared above:

$$R(u^i)(z(t)) := \begin{pmatrix} -2ik < n(z(t)), d > u^i(z(t), d) \\ -2ik\lambda(z(t))u^i(z(t), d) \end{pmatrix} \tag{10}$$

$$\begin{aligned} Q((\varphi_1, \varphi_2)^t)(z(t)) := \quad & \int_{-1}^{1} \frac{\partial \Phi(z(t), z(\tau))}{\partial n(z(\tau))} \varphi_1(z(\tau))|z'(\tau)|d\tau \\ + \quad & \int_{-1}^{1} \Phi(z(t), z(\tau))\varphi_2(z(\tau))|z'(\tau)|d\tau \end{aligned} \tag{11}$$

$$\begin{aligned} P(\varphi_1, \varphi_2)(\hat{x}) := \quad & \frac{1-i}{4}\sqrt{\frac{k}{\pi}} \int_{-1}^{1} < n(z(\tau)), \hat{x} > e^{-ik<\hat{x}, z(\tau)>} \varphi_1(z(\tau))d\tau \\ + \quad & \frac{1+i}{4\sqrt{k\pi}} \int_{-1}^{1} e^{-ik<\hat{x}, z(\tau)>} \varphi_2(z(\tau))d\tau, \end{aligned} \tag{12}$$

We have the representation for the scattered field

$$u^s = QW^{-1}Ru^i. \tag{13}$$

For the far field pattern of the inverse problem, we have the following operator form

$$u_\infty = PQW^{-1}Ru^i. \tag{14}$$

The far field operator of our inverse problem is defined by

$$F := PQW^{-1}R \tag{15}$$

Because of the nonlinearity of the far field operator F, Newton method is used to linearize the equation 8. Therefore, the derivative of F is needed.

4 Fréchet Derivative

Definition 3. *Let X, Y be normed spaces. $U \subset X$ is open.*
An operator $F : U \to Y$ is called Fréchet differentiable *at $z \in U$, if there exists a bounded linear mapping $F'(z) : X \to Y$ with*

$$\|F(z + h) - F(z) - F'(z)h\| = o(\|h\|)$$

for $\|h\| \to 0$.
If F *is Fréchet differentiable at every point* $z \in U$, *then the mapping* $F' : U \to L(X,Y), \quad z \mapsto F'(z)$ *is called the* Fréchet derivative *of* F.

Notation :

$$F^{(n)}(z;h) := (F^{(n)}(z))(h,\dots,h), \qquad n \in \mathbb{N}$$

The proof of the following lemma is straight forward.

Lemma 2. *It holds:*

1. $F_1 : z \mapsto z(t) - z(\tau) \Rightarrow F_1'(z;h) = h(t) - h(\tau)$
2. $F_2 : z \mapsto \|z(t) - z(\tau)\| \Rightarrow F_2'(z;h) = \frac{<z(t)-z(\tau),h(t)-h(\tau)>}{\|z(t)-z(\tau)\|}$
3. $F_3 : z \mapsto z'(t) \Rightarrow F_3'(z;h) = h'(t)$
4. $F_4 : z \mapsto \|z'(t)\| \Rightarrow F_4'(z;h) = \frac{<z'(t),h'(t)>}{\|z'(t)\|}$
5. $F_5 : z \mapsto < z, v > \Rightarrow F_5'(z;h) = < h, v >$
6. $F_6 : z \mapsto \frac{<z'(t),v>}{\|z('t)\|} \Rightarrow F_6'(z;h) = \frac{<h'(t),v>}{\|z'(t)\|} - \frac{<z'(t),h'(t)><z'(t),v>}{\|z'(t)\|^3}$
7. $(F^{-1})' = -F^{-1} \circ F' \circ F^{-1}$, *if* F' *exists.*

5 Fréchet Differentiability of the Far Field Operator

In this section, we will prove the Fréchet differentiability of the far field operator by showing the Fréchet differentiability of the operators appeared in section 3 in a systematic way. We recall the diagram for the definition of the far field operator:

$$F : u^i \xrightarrow{R} u^i|_\Gamma \xrightarrow{W^{-1}} (\varphi_1, \varphi_2)^t \xrightarrow{Q} u^s \xrightarrow{P} u_\infty$$

Since the operator P is not a function of Γ, we only have to differentiate $QW^{-1}R$. Using chain rule and Lemma 2 we obtain

$$u_\gamma^{s\,\prime}(\cdot;h) = Q_\gamma'(W^{-1}Ru^i;h) - QW^{-1}W_\gamma'(W^{-1}Ru^i;h) + QW^{-1}R_\gamma'(u^i;h) \quad (16)$$

Before proving the Fréchet differentiability of the far field operator, we first develop a criterium for testing the Fréchet differentiability of the integral operators appeared in the definition of the far field operator. Assume that the sets $G_1 \subset \mathbb{R}^n, G_2 \subset \mathbb{R}^m$, are measurable. We define the following two integral operators

$$\begin{aligned} A(z) \quad &: \quad C(G_2) \to C(G_1) \\ (A(z)\varphi)(t) \quad &:= \quad \int_{G_2} f(t,\tau,z)\varphi(\tau)d\tau \end{aligned} \tag{17}$$

$$\begin{aligned} A^{(1)}(z;h) \quad &: \quad C(G_2) \to C(G_1) \\ (A^{(1)}(z;h)\varphi)(t) \quad &:= \quad \int_{G_2} \frac{\partial f}{\partial z}(t,\tau,z;h)\varphi(\tau)d\tau \end{aligned} \tag{18}$$

Theorem 3. *Let*

$$D := \begin{cases} G_1 \times G_2, & \text{if } m \neq n \\ \{(t,\tau) \in G_1 \times G_2 : t \neq \tau\}, & \text{if } m = n \end{cases}$$

Assume $f : D \times U \to \mathbb{C}$ satisfies the following conditions :

1. *For all $(t,\tau) \in D$, $f(t,\tau,\cdot)$ is two times continuous Fréchet differentiable on U.*

2. *For all $z \in U$ and $h \in X$, the operators $A(z), A^{(1)}(z;h)$ are well-defined.*

3. *For every $z_0 \in U$, there exist $g_m : D \to \mathbb{R}, m = 1,2$ with*

   $$\left| \frac{\partial^m f}{\partial z^m}(t,\tau,z;h) \right| \leq g_m(t,\tau)\|h\|^m \quad (t,\tau) \in D, h \in z, z \in B(z_0)$$

   where

   $$\int_{G_2} g_m(t,\tau)d\tau \leq C_m$$

   for all $t \in G_1, m = 1,2$ and $C_m > 0$.

Then A is for every $x \in U$ Fréchet differentiable with the derivative $A'(z;h) = A^{(1)}(z;h)$.

Proof. Let $z_0 \in U$ be arbitrary.
The mapping $h \mapsto A^{(1)}(z_0; h)$ is linear ($\frac{\partial f}{\partial z}(t, \tau z_0; h)$ is linear w.r.t. h) and bounded according to

$$\sup_{\|h\|=1} \sup_{\|\varphi\|=1} \| \int_{G_2} \frac{\partial}{\partial z} f(\cdot, \tau, z_0; h)\varphi(\tau) d\tau \|_{\infty, G_1} \leq \| \int_{G_2} g_1(\cdot, \tau) d\tau \|_{\infty, G_1} \leq C_1$$

For all $\varphi \in C(G_2)$ and small $h \in X$, it follows

$$\begin{aligned}
& \|A(z_0 + h)\varphi - A(z_0)\varphi - A^{(1)}(z_0; h)\varphi\|_{\infty, G_1} \\
= \quad & \| \int_{G_2} [f(\cdot, \tau, z_0 + h) - f(\cdot, \tau, z_0) - \frac{\partial f}{\partial z}(\cdot, \tau, z_0; h)]\varphi(\tau) d\tau \|_{\infty, G_1} \\
\overset{Taylor}{=} \quad & \| \int_{G_2} \int_0^1 (1 - s) \frac{\partial^2 f}{\partial z^2}(\cdot, \tau, z_0 + sh; h) ds \varphi(\tau) d\tau \|_{\infty, G_1} \\
\leq \quad & \| \int_{G_2} \int_0^1 |1 - s| g_2(\cdot, \tau) \|h\|^2 ds |\varphi(\tau)| d\tau \|_{\infty, G_1} \\
\leq \quad & \| \int_{G_2} g_2(\cdot, \tau) \|h\|^2 \|\varphi\|_{\infty, G_2} d\tau \|_{\infty, G_1} \\
\leq \quad & C_2 \|\varphi\|_{\infty, G_2} \|h\|^2
\end{aligned}$$

□

Corollary 4. *Assume the function f satisfies the conditions of Theorem 3. Then the mapping*

$$\begin{aligned}
B : U & \rightarrow F(G_1) \\
z & \mapsto \int_{G_2} f(\cdot, \tau, z) d\tau
\end{aligned}$$

is Fréchet differentiable. Furthermore, the derivative is given by

$$B'(z; h) = \int_{G_2} \frac{\partial f}{\partial z}(\cdot, \tau, z; h) d\tau$$

for all $z \in U$

Proof. $B(z) = A(z) 1_{G_2}$ □

Now we can prove our main result

Theorem 5. *The operator F is Fréchet differentiable.*

Proof. Because of similiarity of the structure of the integral operators, we prove here only the Fréchet differentiability of the operator S and K. That is, we have to deal with integral operators of the form

$$(A(z)\varphi)(t) := \int_{-1}^{1} f(t,\tau,z)\varphi(\tau)d\tau$$

with the kernel functions

$$f_S(t,\tau,z) = \frac{i}{2}H_0^{(1)}(k\|z(t)-z(\tau)\|)\|z'(\tau)\|$$

$$f_K(t,\tau,z) = \frac{-ik}{2}H_0^{(1)\prime}(k\|z(t)-z(\tau)\|)\frac{< z(t)-z(\tau), \begin{pmatrix} z_2'(\tau) \\ -z_1'(\tau) \end{pmatrix} >}{\|z(t)-z(\tau)\|}$$

We have to show that f_S, f_K satisfy the assumptions of theorem 3. Let's consider a small neighborhood $B_r(z_0)$ of z_0.

- Set $f_S = pq$, where $p(t,\tau,z) := \frac{i}{2}H_0^{(1)}(k\|z(t)-z(\tau)\|)$, $q(t,\tau,z) := \|z'(\tau)\|$

  $\Rightarrow \exists C_{z_0} > 0 : |q(t,\tau,z)| \leq C_{z_0}$ uniformly in $B_r(z_0)$, and

  $$|H_0^1(k\|z(t)-z(\tau)\|)| \leq \frac{C_{z_0}}{\|z(t)-z(\tau)\|^\alpha}, \quad \alpha \in (0,1)$$

  Lemma 2 gives

  $$\frac{\partial}{\partial z}q(t,\tau,z;h) = \frac{< z'(t), h'(t) >}{\|z'(t)\|}$$

  $$\frac{\partial^2}{\partial^2 z}q(t,\tau,z;h) = \frac{< h'(t), v >}{\|z'(t)\|} - \frac{< z'(t), h'(t) >< z'(t), v >}{\|z'(t)\|^3}$$

  We also have the estimates:

  $$|\frac{\partial}{\partial z}q(t,\tau,z;h)| \leq \|h\|, \quad |\frac{\partial^2}{\partial^2 z}q(t,\tau,z;h)| \leq \frac{2\|h\|^2}{c_{z_0}}$$

For p we have :

$$\frac{\partial}{\partial z}p(t,\tau,z;h) = \frac{ik}{2}H_0^{(1)\prime}(k\|z(t)-z(\tau)\|)\frac{< z(t)-z(\tau), h(t)-h(\tau) >}{\|z(t)-z(\tau)\|}$$

$$\begin{aligned}\frac{\partial^2}{\partial z^2}p(t,\tau,z;h) &= \frac{ik^2}{2}H_0^{(1)\prime\prime}\frac{< z(t)-z(\tau), h(t)-h(\tau) >^2}{\|z(t)-z(\tau)\|^2} \\ &+\frac{ik}{2}\left(\frac{\|h(t)-h(\tau)\|^2}{\|z(t)-z(\tau)\|} - \frac{< z(t)-z(\tau), h(t)-h(\tau) >^2}{\|z(t)-z(\tau)\|^3}\right)\end{aligned}$$

and the estimates

$$|\frac{\partial}{\partial z}p(t,\tau,z;h)| \leq \frac{|k|C_{z_0}}{c_{z_0}}\|h\|, \quad |\frac{\partial^2}{\partial z^2}p(t,\tau,z;h)| \leq \left(\frac{|k|^2C_{z_0}}{c_{z_0}^2} + \frac{2|k|C_{z_0}}{c_{z_0}^2}\right)\|h\|^2$$

- Set $f_K = pq$, where

$$\begin{aligned}p(t,\tau,z) &= \frac{-ik}{2}H_0^{(1)\prime}(k\|z(t)-z(\tau)\|)\|z(t)-z(\tau)\| \\ q(t,\tau,z) &= \frac{< z(t)-z(\tau), \begin{pmatrix} z_2'(\tau) \\ -z_1'(\tau)\end{pmatrix} >}{\|z(t)-z(\tau)\|^2}\end{aligned}$$

$\Rightarrow D_{z_0} > 0 : |q(t,\tau,z)| \leq D_{z_0}$ We have

$$\begin{aligned}\frac{\partial}{\partial z}q(t,\tau,z;h) &= \frac{< z(t)-z(\tau), \begin{pmatrix} h_2'(\tau) \\ -h_1'(\tau)\end{pmatrix} > + < h(t)-h(\tau), \begin{pmatrix} z_2'(\tau) \\ -z_1'(\tau)\end{pmatrix} >}{\|z(t)-z(\tau)\|^2} \\ &-\frac{2 < z(t)-z(\tau), \begin{pmatrix} z_2'(\tau) \\ -z_1'(\tau)\end{pmatrix} >< z(t)-z(\tau), h(t)-h(\tau) >}{\|z(t)-z(\tau)\|^4}\end{aligned}$$

$$|\frac{\partial}{\partial z}q(t,\tau,z;h)| \leq D_{z_0}\|h\|$$

$$
\begin{aligned}
\frac{\partial^2}{\partial z^2}q(t,\tau,z;h) &= \frac{2 < h(t)-h(\tau), \begin{pmatrix} h_2'(\tau) \\ -h_1'(\tau) \end{pmatrix} >}{\|z(t)-z(\tau)\|^2} \\
&\quad -\frac{2(< z(t)-z(\tau), \begin{pmatrix} h_2'(\tau) \\ -h_1'(\tau) \end{pmatrix} > + < h(t)-h(\tau), \begin{pmatrix} z_2'(\tau) \\ -z_1'(\tau) \end{pmatrix} >)}{\|z(t)-z(\tau)\|^2} \\
&\quad \times \frac{< z(t)-z(\tau), h(t)-h(\tau) >}{\|z(t)-z(\tau)\|^2} \\
&\quad -\frac{2 < z(t)-z(\tau), \begin{pmatrix} z_2'(\tau) \\ -z_1'(\tau) \end{pmatrix} > \|h(t)-h(\tau)\|^2}{\|z(t)-z(\tau)\|^4} \\
&\quad +\frac{8 < z(t)-z(\tau), \begin{pmatrix} z_2'(\tau) \\ -z_1'(\tau) \end{pmatrix} >< z(t)-z(\tau), h(t)-h(\tau) >^2}{\|z(t)-z(\tau)\|^6} \\
&\quad |\frac{\partial^2}{\partial z^2}q(t,\tau,z;h)| \leq D_{z_0}\|h\|^2
\end{aligned}
$$

For the function p we have

$$
\begin{aligned}
\frac{\partial}{\partial z}p(t,\tau,z;h) &= \frac{-ik^2}{2}H_0^{(1)''} < z(t)-z(\tau), h(t)-h(\tau) > \\
&\quad +\frac{-ik}{2}H_0^{(1)'}\frac{< z(t)-z(\tau), h(t)-h(\tau) >}{\|z(t)-z(\tau)\|} \\
&\quad |\frac{\partial}{\partial z}p(t,\tau,z;h)| \leq D_{z_0}\|h\|
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2}{\partial z^2}p(t,\tau,z;h) &= \frac{-ik^3}{2}H_0^{(1)'''}\frac{< z(t)-z(\tau), h(t)-h(\tau) >^2}{\|z(t)-z(\tau)\|} \\
&\quad -\frac{ik^2}{2}H_0^{(1)''}\left(\|h(t)-h(\tau)\|^2 + \frac{< z(t)-z(\tau), h(t)-h(\tau) >^2}{\|z(t)-z(\tau)\|^2}\right) \\
&\quad -\frac{ik}{2}H_0^{(1)'}\left(\frac{\|h(t)-h(\tau)\|^2}{\|z(t)-z(\tau)\|} - \frac{< z(t)-z(\tau), h(t)-h(\tau) >^2}{\|z(t)-z(\tau)\|^3}\right) \\
&\quad |\frac{\partial^2}{\partial z^2}p(t,\tau,z;h)| \leq D_{z_0}\|h\|^2
\end{aligned}
$$

$\square$