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Tall Block Models for Two-Phase flows in Fractured Media

Li-Ming Yeh
Department of Applied Mathematics, National Chiao Tung University
Hsinchu, 30050, Taiwan, R.O.C.
Email: liming@math.nctu.edu.tw

Two-phase, incompressible, immiscible flow in fractured media with tall matrix blocks is concerned. Suppose ϵ denotes horizontal size ratio of matrix blocks to whole medium, and suppose the horizontal widths of the fracture planes and matrix blocks are in same order. As ϵ goes to 0, microscopic model for the two-phase flow problem converges to 1) a dual-porosity model if permeability ratio of matrix blocks to fracture planes is of order ϵ^2; 2) a single-porosity model for fracture flow if the ratio is smaller than order ϵ^2; 3) another type of single-porosity model if the ratio is greater than order ϵ^2.

Keywords: dual-porosity model, fractured media

1. Introduction

Homogenization for two-phase, incompressible, immiscible flow in fractured media with tall matrix blocks is concerned. Within a fractured medium there is an interconnected system of fracture planes dividing the porous rock into a collection of matrix blocks. The fracture planes, while very thin, form paths of high permeability. Most of the fluids reside in matrix blocks, where they move very slow. Let ϵ be the horizontal size ratio of tall matrix blocks to the whole medium, and let the horizontal widths of the fracture planes and matrix blocks be in same order. In case permeability ratio of matrix blocks to fracture planes is of order ϵ^2, microscopic models for the two-phase flow problem converge to a dual-porosity model as ϵ tends to 0. For the macroscopic model, a fractured medium is regarded as a porous medium consisting of two superimposed continua, a continuous fracture system and a discontinuous system of matrix blocks. Matrix blocks play the role of a global source distributed over the entire medium. The immiscible two-phase flow is formulated by conservation of mass principles for each continuum plus sources from tall matrix blocks. This problem was also considered by formal asymptotic expansion in [8]. If the ratio is smaller than order ϵ^2, the microscopic models approach a single-porosity model for fracture flow. If the ratio is greater than order ϵ^2, then microscopic models tend to another type of single-porosity model. Our intention is to prove the convergence of the microscopic models.

Rest of the paper is organized as follows: In next section §2, we state microscopic model for two-phase flow in fractured media. Notation and assumption will be given in §3. Then in §4, we present our main results. Some known results needed for our main results will be recalled in §5. Proof of main result is in §6. In §6, we need to use the convergence of oil saturation in matrix blocks. The proof is lengthy and tedious, so we present it in last section §7.

2. Microscopic Model for Tall Matrix Blocks

Let $Y \equiv [0,1]^2$ be a cell consisting of a matrix block domain Y_m completely surrounded by a connected fracture domain Y_f. $\mathcal{X}_m(y)$ is the characteristic function of Y_m, extended Y-periodically to all of $\Re^2$. $\tilde{\Omega} \subset \Re^2$ contains two subdomains, $\tilde{\Omega}^\epsilon_f$ and $\tilde{\Omega}^\epsilon_m$. $\tilde{\Omega}^\epsilon_m \subset \{\tilde{x} \in \tilde{\Omega} | \mathcal{X}_m(\tilde{x}/\epsilon) = 1\}$, $\tilde{\Omega}^\epsilon_f = \tilde{\Omega} \setminus \tilde{\Omega}^\epsilon_m$. Let $\tilde{\Gamma}_\epsilon \equiv \partial\tilde{\Omega}^\epsilon_f \cap \partial\tilde{\Omega}^\epsilon_m \cap \tilde{\Omega}$. Boundary of $\tilde{\Omega}$ includes two parts $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ satisfying $\tilde{\Gamma}_1 \cup \tilde{\Gamma}_2 = \partial\tilde{\Omega}$ and $\tilde{\Gamma}^\circ_1 \cap \tilde{\Gamma}^\circ_2 = \emptyset$. Porous medium considered is a cylindrical aquifer $\Omega \equiv \tilde{\Omega} \times [0,H] \subset \Re^3$ and is assumed to be a two-connected domain with a periodic structure. It contains two subdomains, $\Omega^\epsilon_f \equiv \tilde{\Omega}^\epsilon_f \times [0,H]$ and $\Omega^\epsilon_m \equiv \tilde{\Omega}^\epsilon_m \times [0,H]$, representing the system of fracture planes and matrix blocks respectively. Let $\Gamma_\epsilon \equiv \tilde{\Gamma}_\epsilon \times [0,H]$ be that part of the interface of Ω^ϵ_m with Ω^ϵ_f that is interior to Ω. Both $\Gamma_1 \equiv \tilde{\Gamma}_1 \times [0,H]$ and $\Gamma_2 \equiv \tilde{\Gamma}_2 \times [0,H]$ are part of lateral boundary of Ω.

In fracture subdomain Ω^ϵ_f, porosity is denoted by Φ^ϵ, absolute permeability by K^ϵ, saturation of oil phase by S^ϵ, capillary pressure by $\Upsilon(S^\epsilon)$, relative permeability by $\Lambda_\alpha(S^\epsilon)$, phase pressure by P^ϵ_α, and a density-gravity term by G^ϵ_α for $\alpha = w, o$. $\phi^\epsilon, k^\epsilon, s^\epsilon, \upsilon(s^\epsilon), \lambda_\alpha(s^\epsilon), p^\epsilon_\alpha, g^\epsilon_\alpha$ for $\alpha = w, o$, in subdomain Ω^ϵ_m represent same quantities as those denoted by upper case symbol in fracture subdomain. Conservation of mass in each phase are written as, in Ω^ϵ_f, $t > 0$,

$$-\Phi^\epsilon \partial_t S^\epsilon - \nabla \cdot (K^\epsilon \Lambda_w(S^\epsilon) \nabla (P^\epsilon_w - G^\epsilon_w)) = 0, \tag{2.1}$$
$$\Phi^\epsilon \partial_t S^\epsilon - \nabla \cdot (K^\epsilon \Lambda_o(S^\epsilon) \nabla (P^\epsilon_o - G^\epsilon_o)) = 0, \tag{2.2}$$
$$\Upsilon(S^\epsilon) = P^\epsilon_o - P^\epsilon_w, \tag{2.3}$$

in Ω^ϵ_m, $t > 0$,

$$-\phi^\epsilon \partial_t s^\epsilon - \nabla \cdot \left(k^\epsilon \mathcal{I}^{2\varpi}_\epsilon \lambda_w(s^\epsilon) \nabla (p^\epsilon_w - G^\epsilon_w)\right) = 0, \tag{2.4}$$
$$\phi^\epsilon \partial_t s^\epsilon - \nabla \cdot \left(k^\epsilon \mathcal{I}^{2\varpi}_\epsilon \lambda_o(s^\epsilon) \nabla (p^\epsilon_o - G^\epsilon_o)\right) = 0, \tag{2.5}$$
$$\upsilon(s^\epsilon) = p^\epsilon_o - p^\epsilon_w, \tag{2.6}$$

where $\mathcal{I}^{\mathbf{d}}_\epsilon$ is a diagonal matrix defined by $\mathcal{I}^{\mathbf{d}}_\epsilon \equiv \begin{pmatrix} \epsilon^{\mathbf{d}} & 0 & 0 \\ 0 & \epsilon^{\mathbf{d}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Phase fluxes and pressures are required to be continuous on interface Γ_ϵ, $t > 0$, $\alpha = w, o$,

$$K^\epsilon \Lambda_\alpha(S^\epsilon) \nabla (P^\epsilon_\alpha - G^\epsilon_\alpha) \cdot \vec{\nu^\epsilon} = k^\epsilon \mathcal{I}^{2\varpi}_\epsilon \lambda_\alpha(s^\epsilon) \nabla (p^\epsilon_\alpha - G^\epsilon_\alpha) \cdot \vec{\nu^\epsilon}, \tag{2.7}$$
$$P^\epsilon_\alpha = p^\epsilon_\alpha, \tag{2.8}$$

where $\vec{\nu^\epsilon}$ is the unit vector normal to Γ_ϵ. Boundary conditions are, for $\alpha = w, o$,

$$K^\epsilon \Lambda_\alpha(S^\epsilon)\nabla(P^\epsilon_\alpha - G^\epsilon_\alpha)\cdot\vec{n} = 0 \qquad \text{on } \Gamma_1, \tag{2.9}$$

$$K^\epsilon \Lambda_\alpha(S^\epsilon)\partial_{x_3}(P^\epsilon_\alpha - G^\epsilon_\alpha)|_{x_3=0,H} = k^\epsilon \lambda_\alpha(s^\epsilon)\partial_{x_3}(p^\epsilon_\alpha - G^\epsilon_\alpha)|_{x_3=0,H} = 0, \tag{2.10}$$

$$P^\epsilon_\alpha = P_{b,\alpha} \qquad \text{on } \Gamma_2, \tag{2.11}$$

where $\vec{n}$ is the unit vector outer normal to Γ_1. Initial conditions are

$$S^\epsilon(0,x) = S^\epsilon_0(x) \quad \text{in } \Omega^\epsilon_f, \tag{2.12}$$

$$s^\epsilon(0,x) = s^\epsilon_0(x) \quad \text{in } \Omega^\epsilon_m. \tag{2.13}$$

3. Notation and Assumption

For any $x \in \Re^3$, $x = (\tilde{x}, x_3)$ where $\tilde{x} \in \Re^2$. $\tilde{\Omega}(2\epsilon) \equiv \{\tilde{x} \in \tilde{\Omega} : dist(\tilde{x}, \partial\tilde{\Omega}) > 2\epsilon\}$, $\tilde{\Omega}^\epsilon_m \equiv \{\tilde{x} : \tilde{x} \in \epsilon(Y_m + j) \subset \tilde{\Omega}(2\epsilon) \text{ for } j \in Z^2\}$, $\tilde{\Omega}^\epsilon_f \equiv \tilde{\Omega} \setminus \tilde{\Omega}^\epsilon_m$, and $\tilde{\Omega}^\epsilon \equiv \{z : z \in \epsilon(Y + j), \epsilon(Y_m + j) \subset \tilde{\Omega}(2\epsilon) \text{ for } j \in Z^2\}$. $\Omega^\epsilon \equiv \tilde{\Omega}^\epsilon \times [0,H]$, $\Omega^\epsilon_i \equiv \tilde{\Omega}^\epsilon_i \times [0,H]$, $Y^H_m \equiv Y_m \times [0,H]$, $\mathcal{Q} \equiv \Omega \times Y$, $\mathcal{Q}^\epsilon_m \equiv \Omega^\epsilon \times Y_m$, $\mathcal{Q}_i \equiv \Omega \times Y_i$, $i = f, m$. $\mathcal{B}^t \equiv (0,t) \times \mathcal{B}$ for $\mathcal{B} = Y^H_m, \Gamma_\epsilon, \mathcal{Q}, \mathcal{Q}^\epsilon_m, \Omega, \Omega^\epsilon_i, \mathcal{Q}_i$ $i = f, m$.

$\Re^+_0 \equiv \Re^+ \cup \{0\}$. Time difference is defined to be $\partial^h \psi(t) \equiv \frac{\psi(t+h) - \psi(t)}{h}$. For a set $\mathcal{B}$, $\mathcal{X}_\mathcal{B}$ is a characteristic function of $\mathcal{B}$. $\psi(t,x,y) \in L^r(\Omega^T; L^r_{per}(Y)), 1 < r < \infty$, coincides with a function in $L^r(\mathcal{Q}^T)$ extended by Y-periodicity in y to the whole of $\Re^2$. For $\mathcal{B} = Y_f, Y_m$, we define $L^r(\Omega^T; L^r_{per}(\mathcal{B})) \equiv \{\psi \in L^r(\Omega^T; L^r_{per}(Y)) : \psi(t,x,y) = 0 \text{ if } y \in Y \setminus \mathcal{B}\}$. $\mathcal{W}^{i,r}_0(\Omega) \equiv \{\psi \in W^{i,r}(\Omega) : \psi|_{\Gamma_2} = 0\}$ if $i \in \mathbf{N}$ and $r > 1$, $\mathcal{U} \equiv \mathcal{W}^{1,2}_0(\Omega)$, $\mathcal{U}_2 \equiv \mathcal{U} \times \mathcal{U}$, *dual* $X \equiv$ dual space of X, s_l (resp. $1 - s_r$) is residual matrix oil (resp. water) saturation. $L^{q,r}(\Omega^T) \equiv L^r(0,T; L^q(\Omega))$.

If $\Upsilon : [0,1) \to \Re^+_0$ (resp. $\upsilon : [s_l, s_r) \to \Re^+_0$) is onto and strictly increasing, Υ^{-1} (resp. υ^{-1}) denotes the inverse function of Υ (resp. υ). Then we define $\mathcal{J} : [s_l, s_r) \to [0,1)$ by $\mathcal{J}(z) \equiv \Upsilon^{-1}(\upsilon(z))$, and denote by $\mathcal{J}^{-1}$ the inverse function of $\mathcal{J}$.

$P_{b,c} \equiv P_{b,o} - P_{b,w}$, $S_b \equiv \Upsilon^{-1}(P_{b,c})$, $\Lambda \equiv \Lambda_w + \Lambda_o$, $\lambda \equiv \lambda_w + \lambda_o$,

$$\begin{cases} \mathcal{R}(z) \equiv \int_0^z \frac{\Lambda_w \Lambda_o}{\Lambda} \frac{d\Upsilon}{dS}(\xi) d\xi & \text{for } z \in [0,1), \\ \mathcal{A}(z) \equiv \int_0^z \sqrt{\frac{\Lambda_w \Lambda_o}{\Lambda}(\Upsilon^{-1}(\xi))} d\xi & \text{for } z \in [0,\infty), \\ \mathcal{M}(z) \equiv \int_{s_l}^z \frac{\lambda_w \lambda_o}{\lambda} \frac{d\upsilon}{ds}(\xi) d\xi & \text{for } z \in [s_l, s_r). \end{cases} \tag{3.1}$$

$\vartheta \in (0, 1/8)$ is a number such that $\mathcal{R}'$ is increasing (resp. decreasing) in $(0,\vartheta)$ (resp. $(1-\vartheta, 1)$).

Next let us assume the following conditions: For $\alpha = w, o$,

A1. $\Gamma_2 \neq \emptyset$, $Y_m \subset \Re^2$ is a bounded smooth domain, and $\Omega \subset \Re^3$ is open, bounded, and connected with Lipschitz boundary,

A2. $K^\epsilon, G^\epsilon_\alpha(x_3) \in W^{1,\infty}(\Omega)$, $\partial_t P_{b,\alpha} \in L^2(0,T; H^1(\Omega))$, $P_{b,\alpha} \in C(0,T; C^{1,\mathbf{d}_1}(\Omega))$, $S^\epsilon_0, s^\epsilon_0 \in H^1(\Omega) \cap C^{0,\mathbf{d}_2}(\overline{\Omega})$ for $\mathbf{d}_1, \mathbf{d}_2 \in (0,1)$, A3. $K^\epsilon, k^\epsilon, \Lambda, \lambda \in [\mathbf{d}_3, \mathbf{d}_4]$, $S_b, S_0^\epsilon, \mathcal{J}(s_0^\epsilon) \in (\mathbf{d}_5, 1-\mathbf{d}_5)$ and $\mathbf{d}_5 \in (0,1)$,

A4. $\phi^\epsilon = \phi(\frac{x}{\epsilon})$, $k^\epsilon = k(\frac{x}{\epsilon})$, where ϕ, k are smooth Y-periodic functions,

A5. Λ_w, λ_w (resp. Λ_o, λ_o) : $[0,1] \to [0,1]$ are continuous and decreasing (resp. increasing), $\Lambda_w(1-z) \propto z^{\mathbf{d}_6}, \Lambda_o(z) \propto z^{\mathbf{d}_7}$ for $z \in (0,\vartheta)$, $\frac{\Lambda_\alpha}{\Lambda}(\mathcal{J}(z)) = \frac{\lambda_\alpha}{\lambda}(z)$,

A6. $\Upsilon : [0,1) \to \Re_0^+$ $(\upsilon : [s_l, s_r) \to \Re_0^+)$ is onto, increasing, and a locally Lipschitz continuous function, and $\inf\limits_{z\in[0,1)} \frac{d\Upsilon}{dS}(z) > 0$, $\frac{\frac{d\Upsilon}{dS}(\mathcal{J}(z))}{\frac{dv}{ds}(z)}, \Phi^\epsilon, \phi^\epsilon \in [\mathbf{d}_8, \mathbf{d}_9]$ for $z \in [s_l, s_r]$, $\frac{\mathbf{d}_9}{\mathbf{d}_8} \sim 1$,

A7. $\Lambda_o^{3/2}(z) \leq \int_z^{2z} (\mathcal{A}(\Upsilon(2z)) - \mathcal{A}(\Upsilon(\xi)))d\xi$ for $z \in (0,\vartheta)$ and
$\Lambda_w^{3/2}(1-z) \leq \int_{1-2z}^{1-z} (\mathcal{A}(\Upsilon(\xi)) - \mathcal{A}(\Upsilon(1-2z)))d\xi$ for $z \in (0,\vartheta)$,

A8. $|\Lambda_\alpha(z_1) - \Lambda_\alpha(z_2)| \leq \mathbf{d}_{10}\sqrt{(\mathcal{R}(z_1) - \mathcal{R}(z_2))(z_1 - z_2)}$ for any $z_1, z_2 \in [0,1]$,

A9. $\max\limits_{z\in[0,1]} |\Lambda(z) - 1| + \max\limits_{z\in[s_l,s_r]} |\lambda(z) - 1| \leq \mathbf{d}_{11}$ ($\mathbf{d}_{11}$ only depends on $\Omega, K^\epsilon, k^\epsilon$),

A10. $\Lambda_o\Lambda_w(z) \leq \mathbf{d}_{12} z|1-z|\sqrt{\mathcal{R}'(z)}$, $\mathcal{R}'(z) \propto z^{\mathbf{m}}|1-z|^{\mathbf{m}_1}$ for $z \in (0,\vartheta) \cup (1-\vartheta, 1)$ and $\mathbf{m}, \mathbf{m}_1 > 1$,

where $\mathbf{m}, \mathbf{m}_1$, $\mathbf{d}_i, i = 1, \cdots, 12$ are positive constants.

Remark 3.1 *From A1, Ω_f^ϵ is an open, bounded, and connected domain with Lipschitz boundary. In A2, the density-gravity terms $G_w^\epsilon, G_o^\epsilon$ are functions depending on x_3 variable. Initial and boundary saturations are away from two end points* 0 *and* 1 *(see A3). A5 implies that relative permeability Λ_w (resp. λ_w) in the neighbor of end point* 1 *has similar properties as Λ_o (resp. λ_o) in the neighbor of end point* 0*. Relative phase mobilities in fractures and matrix blocks behave similar. A6 requires that fracture capillary pressure increases as fast as capillary pressure of matrix blocks. Usually, derivative of capillary pressure $\Upsilon'(z)$ (resp. $\upsilon'(z)$) tends to infinity as $z \to 0$ or* 1 *(resp. s_l or s_r). A10 allows parabolic equations considered are degenerate at end points* 0 *and* 1*, a characteristic of a porous medium equation. Indeed, it also implies $\mathcal{R}' \in L^\infty(0,1)$. A7-8,10 are the restrictions on relative permeability and capillary pressure in fractures. Indeed, if $\mathbf{d}_6, \mathbf{d}_7$ (see A5) are large enough (depending on capillary pressure), A7-8,10 hold. One may also note that because of A5-10, Λ_o and $\mathcal{R}'$ at the end point* 0 *have similar properties as Λ_w and $\mathcal{R}'$ at the end point* 1*.*

4. Main Result

In this section, we present the limit models of (2.1–2.13) as $\epsilon \to 0$. Roughly speaking, the limit models are fracture flow equations plus interior sources from matrix blocks. The source terms depend on how fast the matrix permeability tends to 0 as $\epsilon \to 0$. For $0 < \varpi < 1$ case, matrix permeability tends to 0 very slow and saturation variation in fracture system and in matrix blocks is almost simultaneous.

So the limit model is a single-porosity model with sources from matrix blocks. For $\varpi = 1$ case, saturation variation in fracture system and in matrix blocks is not simultaneous and the limit model is a dual-porosity model. In this case, domain acts as a porous medium consisting of two superimposed continua, a continuous fracture system Ω and a discontinuous system of matrix blocks $\mathcal{Q}_m$. Primary flow occurs in fracture system Ω, and each point $x \in \Omega$ is associated with a matrix block Y_m. Flow in matrix blocks plays the role of a global source in the whole fracture system. The model includes two systems of equations, one for flow in fracture system and the other for flow in matrix block system. The two systems are coupled through nonlinear sources. For $1 < \varpi$ case, matrix permeability tends to 0 so fast that matrix blocks play no roles in the limit model. The limit model is a single-porosity model containing only fracture flow equations without matrix sources.

4.1. *For* $\varpi = 1$ *case*

Let $\Omega \subset \Re^3$ be a fractured medium. Equations for fracture flow are, for $x \in \Omega,\ t > 0$,

$$-\Phi\partial_t S - \nabla \cdot (K\Lambda_w(S)\nabla(P_w - G_w)) = q_w, \tag{4.1}$$

$$\Phi\partial_t S - \nabla \cdot (K\Lambda_o(S)\nabla(P_o - G_o)) = q_o, \tag{4.2}$$

$$\Upsilon(S) = P_o - P_w. \tag{4.3}$$

Φ is porosity, K is permeability field, S is oil saturation, $\Upsilon(S)$ is capillary pressure curve, Λ_α $(\alpha = w, o)$ is relative permeability curve of α-phase, P_α denotes phase pressure, G_α is a function depending on density, gravity, and position, and q_α is the matrix-fracture source.

Above each point $x \in \Omega$ is suspended topologically a matrix block $Y_m \subset \Re^2$. Equations for flow in a matrix block are, for $x \in \Omega,\ y \in Y_m,\ t > 0$,

$$-\phi\partial_t s - \partial_{y,x_3} \cdot (k\lambda_w(s)\partial_{y,x_3}(p_w - G_w)) = 0, \tag{4.4}$$

$$\phi\partial_t s - \partial_{y,x_3} \cdot (k\lambda_o(s)\partial_{y,x_3}(p_o - G_o)) = 0, \tag{4.5}$$

$$\upsilon(s) = p_o - p_w. \tag{4.6}$$

Here functions s, p_w, p_0 are defined in space domain $\mathcal{Q}_m$ and $\partial_{y,x_3} = (\partial_{y_1}, \partial_{y_2}, \partial_{x_3})$. Each lower case symbol denotes the quantity on Y_m corresponding to that denoted by an upper case symbol in the fracture system equations.

The matrix-fracture sources are given by, for $x \in \Omega$, $t > 0$,

$$q_\alpha = \frac{-1}{|Y_m|}\int_{Y_m} (\sigma_\alpha\phi\partial_t s - \partial_{x_3}(k\lambda_\alpha(s)\partial_{x_3}(p_\alpha - G_\alpha)))\, dy, \tag{4.7}$$

where $\sigma_w = -1$, $\sigma_o = 1$, and $|Y_m|$ is the volume of Y_m. Boundary conditions are, for $t > 0$, $\alpha = w, o$,

$$K\Lambda_\alpha(S)\nabla(P_\alpha - G_\alpha) \cdot \vec{n} = 0 \qquad \text{for } x \in \Gamma_1, \tag{4.8}$$

$$K\Lambda_\alpha(S)\partial_{x_3}(P_\alpha - G_\alpha)|_{x_3=0,H} = k\lambda_\alpha(s)\partial_{x_3}(p_\alpha - G_\alpha)|_{x_3=0,H} = 0, \tag{4.9}$$
$$P_\alpha = P_{b,\alpha} \qquad \text{for } x \in \Gamma_2, \tag{4.10}$$

where $\vec{n}$ is the unit vector outward normal to Γ_1. On interface, pressures are continuous, that is, for $t > 0$, $x \in \Omega$, $y \in \partial Y_m$, $\alpha = w, o$,

$$p_\alpha(t, x, y) = P_\alpha(t, x). \tag{4.11}$$

Initial conditions are

$$S(0, x) = S_0(x) \qquad \text{for } x \in \Omega, \tag{4.12}$$
$$s(0, x, y) = s_0(x) \qquad \text{for } x \in \Omega, \quad y \in Y_m. \tag{4.13}$$

Theorem 4.1 *Under* $A1-10$*, a subsequence of solutions of the microscopic models (2.1–2.13) converges in two-scale sense to a solution of (4.1–4.13) (see next section for the definition of convergence in two-scale sense).*

4.2. *For* $0 < \varpi < 1$ *case*

Equations are, for $x \in \Omega,\ t > 0$,

$$-\Phi\partial_t S - \nabla \cdot (K\Lambda_w(S)\nabla(P_w - G_w)) = q_w, \tag{4.14}$$
$$\Phi\partial_t S - \nabla \cdot (K\Lambda_o(S)\nabla(P_o - G_o)) = q_o, \tag{4.15}$$
$$\Upsilon(S) = P_o - P_w = \upsilon(s). \tag{4.16}$$

Φ, K, S, $\Upsilon(S)$, $\upsilon(s)$, Λ_α, P_α, G_α, and q_α ($\alpha = w, o$) are the same quantities as those in $\varpi = 1$. The matrix-fracture sources are given by, for $x \in \Omega$, $t > 0$,

$$q_\alpha = \frac{-1}{|Y_m|}\int_{Y_m} \left(\sigma_\alpha \phi \partial_t s - \partial_{x_3}\left(k\lambda_\alpha(s)\partial_{x_3}(P_\alpha - G_\alpha)\right)\right) dy, \tag{4.17}$$

where $\sigma_w = -1$, $\sigma_o = 1$, and $|Y_m|$ is the volume of Y_m. Boundary conditions are, for $t > 0$, $\alpha = w, o$,

$$K\Lambda_\alpha(S)\nabla(P_\alpha - G_\alpha) \cdot \vec{n} = 0 \qquad \text{for } x \in \Gamma_1, \tag{4.18}$$
$$K\Lambda_\alpha(S)\partial_{x_3}(P_\alpha - G_\alpha)|_{x_3=0,H} = 0, \tag{4.19}$$
$$P_\alpha = P_{b,\alpha} \qquad \text{for } x \in \Gamma_2, \tag{4.20}$$

where $\vec{n}$ is the unit vector outward normal to Γ_1. Initial condition is

$$S(0, x) = S_0(x) \qquad \text{for } x \in \Omega. \tag{4.21}$$

Theorem 4.2 *Under* $A1-10$*, a subsequence of solutions of the microscopic models (2.1–2.13) converges in two-scale sense to a solution of (4.14–4.21) (see next section for the definition of convergence in two-scale sense).*

4.3. *For $\varpi > 1$ case*

Equations are, for $x \in \Omega,\ t > 0$,

$$-\Phi\partial_t S - \nabla \cdot (K\Lambda_w(S)\nabla(P_w - G_w)) = 0, \tag{4.22}$$

$$\Phi\partial_t S - \nabla \cdot (K\Lambda_o(S)\nabla(P_o - G_o)) = 0, \tag{4.23}$$

$$\Upsilon(S) = P_o - P_w. \tag{4.24}$$

Φ, K, S, $\Upsilon(S)$, $\upsilon(s)$, Λ_α, P_α, G_α, and q_α ($\alpha = w, o$) are the same quantities as those in $\varpi = 1$. Boundary conditions are, for $t > 0$, $\alpha = w, o$,

$$K\Lambda_\alpha(S)\nabla(P_\alpha - G_\alpha) \cdot \vec{n} = 0 \qquad \text{for } x \in \Gamma_1, \tag{4.25}$$

$$K\Lambda_\alpha(S)\partial_{x_3}(P_\alpha - G_\alpha)|_{x_3=0,H} = 0, \tag{4.26}$$

$$P_\alpha = P_{b,\alpha} \qquad \text{for } x \in \Gamma_2, \tag{4.27}$$

where $\vec{n}$ is the unit vector outward normal to Γ_1. Initial condition is

$$S(0, x) = S_0(x) \qquad \text{for } x \in \Omega. \tag{4.28}$$

Theorem 4.3 *Under $A1-10$, a subsequence of solutions of the microscopic models (2.1–2.13) converges in two-scale sense to a solution of (4.22–4.28) (see next section for the definition of convergence in two-scale sense).*

5. Some Known Results

Lemma 5.1 [1] *Let $1 \le r < \infty$ and A1 hold. There is a constant $\mathbf{d}_{13}(Y_f, r)$ and a linear continuous extension operator $\Pi_\epsilon : W^{1,r}(\Omega_f^\epsilon) \cap L^\infty(\Omega_f^\epsilon) \to W^{1,r}(\Omega) \cap L^\infty(\Omega)$ such that if $\varphi \in W^{1,r}(\Omega_f^\epsilon) \cap L^\infty(\Omega_f^\epsilon)$ and $\mathbf{d}_{14} \le \varphi \le \mathbf{d}_{15}$, then*

$$\begin{cases} \Pi_\epsilon\varphi = \varphi \quad \text{in } \Omega_f^\epsilon \text{ almost everywhere}, \\ \|\Pi_\epsilon\varphi\|_{W^{1,r}(\Omega)} \le \mathbf{d}_{13}\|\varphi\|_{W^{1,r}(\Omega_f^\epsilon)}, \\ \mathbf{d}_{14} \le \Pi_\epsilon\varphi \le \mathbf{d}_{15}. \end{cases}$$

Definition 5.1 *For a given $\epsilon > 0$ and $1 \le r < \infty$, we define a dilation operator "–" mapping a measurable function $\varphi \in L^r(\Omega_m^{\epsilon,T})$ to a measurable function $\overline{\varphi} \in L^r(\mathcal{Q}_m^T)$ by, for $(t, \tilde{x}, x_3, y) \in \mathcal{Q}_m^T$,*

$$\overline{\varphi}(t, \tilde{x}, x_3, y) \equiv \begin{cases} \varphi(t, \ell^\epsilon(\tilde{x}) + \epsilon y, x_3) & \text{if } (\ell^\epsilon(\tilde{x}) + \epsilon y, x_3) \in \Omega_m^\epsilon, \\ 0 & \text{elsewhere}, \end{cases}$$

where $\ell^\epsilon(\tilde{x}) \equiv \epsilon j$ if $\tilde{x} \in \epsilon(Y + j)$, $j \in Z^2$, denoting the lattice translation point of ϵ-cell domain containing $\tilde{x}$.

Definition 5.2 *A sequence of functions φ^ϵ in $L^r(\Omega^T), 1 < r < \infty$, is said to two-scale converge to φ in $L^r(\Omega^T; L^r_{per}(Y))$ if, for any function $\psi \in C_0^\infty(\Omega^T; C^\infty_{per}(Y))$, we have*

$$\lim_{\epsilon \to 0} \int_{\Omega^T} \varphi^\epsilon(t, x)\psi(t, x, \tilde{x}/\epsilon)dxdt = \int_{\mathcal{Q}^T} \varphi(t, x, y)\psi(t, x, y)dydxdt,$$

denoted by $\varphi^\epsilon \xrightarrow{2} \varphi \in L^r(\Omega^T; L^r_{per}(Y))$*. Besides* $\lim_{\epsilon\to 0}\|\varphi^\epsilon\|_{L^r(\Omega^T)} = \|\varphi\|_{L^r(\mathcal{Q}^T)}$*,* φ^ϵ *is said to two-scale converge to* φ *in* $L^r(\Omega^T; L^r_{per}(Y))$ *strongly, and denoted by* $\varphi^\epsilon \xrightarrow{2} \varphi \in L^r(\Omega^T; L^r_{per}(Y))$ *strongly.*

6. Proof of Main Result

A1-10 are assumed from now on. Let us derive a weak formulation of (2.1–2.6). Multiplying (2.1) and (2.4) by η as well as (2.2) and (2.5) by ζ, integrating over Ω^T, and employing boundary conditions (2.7) and (2.9), we obtain

$$\begin{aligned}
&-\int_{\Omega_f^{\epsilon,T}} \Phi^\epsilon \partial_t S^\epsilon \eta + \int_{\Omega_f^{\epsilon,T}} K^\epsilon \Lambda_w(S^\epsilon)\nabla(P_w^\epsilon - G_w^\epsilon)\nabla\eta \\
&\qquad -\int_{\Omega_m^{\epsilon,T}} \phi^\epsilon \partial_t s^\epsilon \eta + \int_{\Omega_m^{\epsilon,T}} k^\epsilon \mathcal{I}_\epsilon^{2\varpi}\lambda_w(s^\epsilon)\nabla(p_w^\epsilon - G_w^\epsilon)\nabla\eta = 0, \qquad (6.1)\\
&\int_{\Omega_f^{\epsilon,T}} \Phi^\epsilon \partial_t S^\epsilon \zeta + \int_{\Omega_f^{\epsilon,T}} K^\epsilon \Lambda_o(S^\epsilon)\nabla(P_o^\epsilon - G_o^\epsilon)\nabla\zeta \\
&\qquad +\int_{\Omega_m^{\epsilon,T}} \phi^\epsilon \partial_t s^\epsilon \zeta + \int_{\Omega_m^{\epsilon,T}} k^\epsilon \mathcal{I}_\epsilon^{2\varpi}\lambda_o(s^\epsilon)\nabla(p_o^\epsilon - G_o^\epsilon)\nabla\zeta = 0, \qquad (6.2)
\end{aligned}$$

for smooth functions $\eta, \zeta \in L^2(0,T;\mathcal{U})$. Next we define global pressure [11] as

$$\begin{cases}
P^\epsilon \equiv \frac{1}{2}\left(P_o^\epsilon + P_w^\epsilon + \int_0^{\Upsilon(S^\epsilon)}\left(\frac{\Lambda_o}{\Lambda}(\Upsilon^{-1}(\xi)) - \frac{\Lambda_w}{\Lambda}(\Upsilon^{-1}(\xi))\right)d\xi\right), \\
p^\epsilon \equiv \frac{1}{2}\left(p_o^\epsilon + p_w^\epsilon + \int_0^{\upsilon(s^\epsilon)}\left(\frac{\lambda_o}{\lambda}(\upsilon^{-1}(\xi)) - \frac{\lambda_w}{\lambda}(\upsilon^{-1}(\xi))\right)d\xi\right),
\end{cases} \qquad (6.3)$$

P_b is defined as P^ϵ in $(6.3)_1$ except replacing $P_o^\epsilon, P_w^\epsilon, \Upsilon(S^\epsilon)$ by $P_{b,o}^\epsilon, P_{b,w}^\epsilon, P_{b,c}$ respectively. Then $\nabla P^\epsilon = \frac{\Lambda_w}{\Lambda}(S^\epsilon)\nabla P_w^\epsilon + \frac{\Lambda_o}{\Lambda}(S^\epsilon)\nabla P_o^\epsilon$ and $\nabla p^\epsilon = \frac{\lambda_w}{\lambda}(s^\epsilon)\nabla p_w^\epsilon + \frac{\lambda_o}{\lambda}(s^\epsilon)\nabla p_o^\epsilon$ by (2.3) and (2.6). (6.2) can be rewritten as

$$\begin{aligned}
&\int_{\Omega_f^{\epsilon,T}} \Phi^\epsilon \partial_t S^\epsilon \zeta + \int_{\Omega_f^{\epsilon,T}} K^\epsilon\big(\Lambda_o(S^\epsilon)\nabla(P^\epsilon - G_o^\epsilon) + \nabla\mathcal{R}(S^\epsilon)\big)\nabla\zeta \\
&\quad +\int_{\Omega_m^{\epsilon,T}} \phi^\epsilon \partial_t s^\epsilon \zeta + \int_{\Omega_m^{\epsilon,T}} k^\epsilon \mathcal{I}_\epsilon^{2\varpi}\big(\lambda_o(s^\epsilon)\nabla(p^\epsilon - G_o^\epsilon) + \nabla\mathcal{M}(s^\epsilon)\big)\nabla\zeta = 0. \qquad (6.4)
\end{aligned}$$

See §3 for $\mathcal{R}, \mathcal{M}$. Summing (6.1) and (6.2), we obtain, for $\eta \in L^2(0,T;\mathcal{U})$,

$$\begin{aligned}
&\int_{\Omega_f^{\epsilon,T}} K^\epsilon\big(\Lambda(S^\epsilon)\nabla(P^\epsilon - G_o^\epsilon) - \Lambda_w(S^\epsilon)\nabla(G_w^\epsilon - G_o^\epsilon)\big)\nabla\eta \\
&\quad +\int_{\Omega_m^{\epsilon,T}} k^\epsilon \mathcal{I}_\epsilon^{2\varpi}\big(\lambda(s^\epsilon)\nabla(p^\epsilon - G_o^\epsilon) - \lambda_w(s^\epsilon)\nabla(G_w^\epsilon - G_o^\epsilon)\big)\nabla\eta = 0. \qquad (6.5)
\end{aligned}$$

For $\zeta \in L^2(0,T;\mathcal{U}) \cap H^1(\Omega^T), \zeta(T) = 0$,

$$\int_{\Omega_f^{\epsilon,T}} \Phi^\epsilon \partial_t S^\epsilon \zeta + \Phi^\epsilon(S^\epsilon - S_0^\epsilon)\partial_t\zeta = -\int_{\Omega_m^{\epsilon,T}} \phi^\epsilon \partial_t s^\epsilon \zeta + \phi^\epsilon(s^\epsilon - s_0^\epsilon)\partial_t\zeta. \qquad (6.6)$$

(6.1–6.6), (2.3), (2.6), (2.8), (2.11) form a weak formulation of (2.1–2.13).

Next we consider a regularized problem. Let $\mathbf{v}$ be a small number satisfying $0 < \mathbf{v} < \frac{\mathbf{d}_5}{4}$. Extend Λ_α ($\alpha = w, o$) constantly and continuously to $\Re$ and define $\Lambda_{\alpha,\mathbf{v}}, \Lambda_{\mathbf{v}}, \lambda_{\alpha,\mathbf{v}}, \lambda_{\mathbf{v}}$ as

$$\begin{cases} \Lambda_{\alpha,\mathbf{v}}(z) \equiv \Lambda_\alpha\big(0.5(\frac{z-\mathbf{v}}{0.5-\mathbf{v}})\big), & \Lambda_{\mathbf{v}} \equiv \Lambda_{w,\mathbf{v}} + \Lambda_{o,\mathbf{v}}, \\ \lambda_{\mathbf{v}}(z) \equiv \Lambda_{\mathbf{v}}(\mathcal{J}(z)), & \lambda_{\alpha,\mathbf{v}}(z) = \Lambda_{\alpha,\mathbf{v}}(\mathcal{J}(z)). \end{cases} \tag{6.7}$$

By A2-3, there exist smooth functions $S^\epsilon_{0,\mathbf{v}}, S_{b,\mathbf{v}}, s^\epsilon_{0,\mathbf{v}}$ such that

$$S^\epsilon_{0,\mathbf{v}}, S_{b,\mathbf{v}}, \mathcal{J}(s^\epsilon_{0,\mathbf{v}}) \in (\mathbf{d}_5/2, 1-\mathbf{d}_5/2), \quad S^\epsilon_{0,\mathbf{v}}|_{\Gamma_2} = S_{b,\mathbf{v}}|_{\Gamma_2}(t=0), \tag{6.8}$$

$$\begin{cases} S^\epsilon_{0,\mathbf{v}}, S_{b,\mathbf{v}}, s^\epsilon_{0,\mathbf{v}} \to S^\epsilon_0, S_b, s^\epsilon_0 & \text{in } L^2(0,T;H^1(\Omega)), \\ \partial_t \Upsilon(S_{b,\mathbf{v}}) \to \partial_t(P_{b,o} - P_{b,w}) & \text{in } L^1(\Omega^T), \end{cases} \quad \text{as } \mathbf{v} \to 0. \tag{6.9}$$

The regularized problem is: Find $\{S^\epsilon_{\mathbf{v}} \mathcal{X}_{\Omega^\epsilon_f} + s^\epsilon_{\mathbf{v}} \mathcal{X}_{\Omega^\epsilon_m}, P^\epsilon_{\mathbf{v}} \mathcal{X}_{\Omega^\epsilon_f} + p^\epsilon_{\mathbf{v}} \mathcal{X}_{\Omega^\epsilon_m}\}$ satisfying

$$\Phi^\epsilon \partial_t S^\epsilon_{\mathbf{v}} \mathcal{X}_{\Omega^\epsilon_f} + \phi^\epsilon \partial_t s^\epsilon_{\mathbf{v}} \mathcal{X}_{\Omega^\epsilon_m} \in dual\ L^2(0,T;\mathcal{U}), \tag{6.10}$$

$$\mathbf{v} \le S^\epsilon_{\mathbf{v}} \mathcal{X}_{\Omega^\epsilon_f} + \mathcal{J}(s^\epsilon_{\mathbf{v}}) \mathcal{X}_{\Omega^\epsilon_m} \le 1 - \mathbf{v}, \tag{6.11}$$

$$\mathcal{R}(S^\epsilon_{\mathbf{v}} \mathcal{X}_{\Omega^\epsilon_f} + \mathcal{J}(s^\epsilon_{\mathbf{v}}) \mathcal{X}_{\Omega^\epsilon_m}) - \mathcal{R}(S^\epsilon_b),\ P^\epsilon_{\mathbf{v}} \mathcal{X}_{\Omega^\epsilon_f} + p^\epsilon_{\mathbf{v}} \mathcal{X}_{\Omega^\epsilon_m} - P_b \in L^2(0,T;\mathcal{U}), \tag{6.12}$$

$$\begin{aligned} &\int_{\Omega^{\epsilon,T}_f} \Phi^\epsilon \partial_t S^\epsilon_{\mathbf{v}} \zeta + \int_{\Omega^{\epsilon,T}_f} K^\epsilon \big(\Lambda_{o,\mathbf{v}}(S^\epsilon_{\mathbf{v}}) \nabla (P^\epsilon_{\mathbf{v}} - G^\epsilon_o) + \nabla \mathcal{R}(S^\epsilon_{\mathbf{v}})\big) \nabla \zeta \\ &\quad + \int_{\Omega^{\epsilon,T}_m} \phi^\epsilon \partial_t s^\epsilon_{\mathbf{v}} \zeta + \int_{\Omega^{\epsilon,T}_m} k^\epsilon \mathcal{I}^{2\varpi}_\epsilon \big(\lambda_{o,\mathbf{v}}(s^\epsilon_{\mathbf{v}}) \nabla (p^\epsilon_{\mathbf{v}} - G^\epsilon_o) + \nabla \mathcal{M}(s^\epsilon_{\mathbf{v}})\big) \nabla \zeta = 0, \end{aligned} \tag{6.13}$$

$$\begin{aligned} &\int_{\Omega^{\epsilon,T}_f} K^\epsilon \big(\Lambda_{\mathbf{v}}(S^\epsilon_{\mathbf{v}}) \nabla (P^\epsilon_{\mathbf{v}} - G^\epsilon_o) - \Lambda_{w,\mathbf{v}}(S^\epsilon_{\mathbf{v}}) \nabla (G^\epsilon_w - G^\epsilon_o)\big) \nabla \eta \\ &\quad + \int_{\Omega^{\epsilon,T}_m} k^\epsilon \mathcal{I}^{2\varpi}_\epsilon \big(\lambda_{\mathbf{v}}(s^\epsilon_{\mathbf{v}}) \nabla (p^\epsilon_{\mathbf{v}} - G^\epsilon_o) - \lambda_{w,\mathbf{v}}(s^\epsilon_{\mathbf{v}}) \nabla (G^\epsilon_w - G^\epsilon_o)\big) \nabla \eta = 0, \end{aligned} \tag{6.14}$$

$$S^\epsilon_{\mathbf{v}} \mathcal{X}_{\Omega^\epsilon_f}(0,x) + s^\epsilon_{\mathbf{v}} \mathcal{X}_{\Omega^\epsilon_m}(0,x) = S^\epsilon_{0,\mathbf{v}} \mathcal{X}_{\Omega^\epsilon_f} + s^\epsilon_{0,\mathbf{v}} \mathcal{X}_{\Omega^\epsilon_m}, \tag{6.15}$$

for any $\zeta, \eta \in L^2(0,T;\mathcal{U})$. It is easy to see that (6.13) is a nondegenerate (depending on $\mathbf{v}$) parabolic equation, and (6.13–6.14) imply, if $S^\epsilon_{w,\mathbf{v}} \equiv 1 - S^\epsilon_{\mathbf{v}}$,

$$\begin{aligned} 0 = &\int_{\Omega^{\epsilon,T}_f} \Phi^\epsilon \partial_t S^\epsilon_{w,\mathbf{v}} \zeta + K^\epsilon \big(\Lambda_{w,\mathbf{v}}(1 - S^\epsilon_{w,\mathbf{v}}) \nabla (P^\epsilon_{\mathbf{v}} - G^\epsilon_w) - \nabla \mathcal{R}(1 - S^\epsilon_{w,\mathbf{v}})\big) \nabla \zeta \\ + &\int_{\Omega^{\epsilon,T}_m} \phi^\epsilon \partial_t s^\epsilon_{w,\mathbf{v}} \zeta + k^\epsilon \mathcal{I}^{2\varpi}_\epsilon \left(\lambda_{w,\mathbf{v}}(1 - s^\epsilon_{w,\mathbf{v}}) \nabla (p^\epsilon_{\mathbf{v}} - G^\epsilon_w) - \nabla \mathcal{M}(1 - s^\epsilon_{w,\mathbf{v}})\right) \nabla \zeta. \end{aligned} \tag{6.16}$$

By [4, 5, 6, 9, 12, 20, 22, 29], it is known

Lemma 6.1 *Under (6.8–6.9), there exist functions $S^\epsilon_{\mathbf{v}}, P^\epsilon_{\mathbf{v}}$ in Ω^ϵ_f and $s^\epsilon_{\mathbf{v}}, p^\epsilon_{\mathbf{v}}$ in Ω^ϵ_m satisfying (6.10–6.15) for each $\mathbf{v}, \epsilon$ as well as there exist functions $S^\epsilon, P^\epsilon, P^\epsilon_\alpha$ in Ω^ϵ_f and $s^\epsilon, p^\epsilon, p^\epsilon_\alpha$ in Ω^ϵ_m for $\alpha = w, o$ satisfying (6.1–6.6), (2.3), and (2.6–2.11). $\check{\mathcal{S}}^\epsilon_{\mathbf{v}} \equiv S^\epsilon_{\mathbf{v}} \mathcal{X}_{\Omega^\epsilon_f} + \mathcal{J}(s^\epsilon_{\mathbf{v}}) \mathcal{X}_{\Omega^\epsilon_m}$ is in $L^2(0,T;H^1(\Omega))$ and is Hölder continuous in $\overline{\Omega}^T$,* *and, as* $\mathbf{v} \to 0$,

$$\begin{cases} \check{\mathcal{S}}^\epsilon_{\mathbf{v}} \to \check{\mathcal{S}}^\epsilon \equiv S^\epsilon \mathcal{X}_{\Omega^\epsilon_f} + \mathcal{J}(s^\epsilon)\mathcal{X}_{\Omega^\epsilon_m} & \textit{pointwise}, \\ \mathcal{R}(\check{\mathcal{S}}^\epsilon_{\mathbf{v}}), P^\epsilon_{\mathbf{v}}\mathcal{X}_{\Omega^\epsilon_f} + p^\epsilon_{\mathbf{v}}\mathcal{X}_{\Omega^\epsilon_m} \to \mathcal{R}(\check{\mathcal{S}}^\epsilon), P^\epsilon \mathcal{X}_{\Omega^\epsilon_f} + p^\epsilon \mathcal{X}_{\Omega^\epsilon_m} & \textit{in } L^2(0,T;H^1(\Omega)). \end{cases}$$

Moreover, $0 < S^\epsilon < 1$, $s_l < s^\epsilon < s_r$, *and*

$$\begin{aligned} \sum_{\alpha=w,o} &\Big(\|\sqrt{\Lambda_\alpha(S^\epsilon)}\nabla P^\epsilon_\alpha\|_{L^2(\Omega^{\epsilon,T}_f)} + \|\mathcal{I}^\varpi_\epsilon \sqrt{\lambda_\alpha(s^\epsilon)}\nabla p^\epsilon_\alpha\|_{L^2(\Omega^{\epsilon,T}_m)} \Big) \\ &+\| \ |\nabla P^\epsilon| + |\nabla \mathcal{R}(S^\epsilon)| + |\nabla \mathcal{A}^\epsilon| \ \|_{L^2(\Omega^{\epsilon,T}_f)} \\ &+\| \ |\mathcal{I}^\varpi_\epsilon \nabla p^\epsilon| + |\mathcal{I}^\varpi_\epsilon \nabla \mathcal{M}(s^\epsilon)| + |\mathcal{I}^\varpi_\epsilon \nabla \mathcal{A}^\epsilon| \ \|_{L^2(\Omega^{\epsilon,T}_m)} \le c, \end{aligned}$$

where $\mathcal{A}^\epsilon \equiv \begin{cases} \mathcal{A}(\Upsilon(S^\epsilon)) & \textit{if } x \in \Omega^\epsilon_f, \\ \mathcal{A}(\upsilon(s^\epsilon)) & \textit{if } x \in \Omega^\epsilon_m. \end{cases}$ *and* c *is a constant independent of* ϵ.

Lemma 6.2 *For any* β, τ *satisfying* $2 \le \beta_0 \le \beta - 2 \in \mathbf{N}$, $\frac{\mathbf{d}_5}{\beta_0} \le \vartheta$, *and* $\tau \le T$, *the following inequality holds:*

$$\sup_{t \le \tau} \big|\{x \in \Omega : \check{\mathcal{S}}^\epsilon(t) \le \mu \textit{ or } 1-\mu \le \check{\mathcal{S}}^\epsilon(t)\}\big| \le \frac{c_0|c_0\tau|^{\beta-\beta_0}}{(\beta-\beta_0)^{(\beta-\beta_0)\mathbf{f}_\beta}}, \tag{6.17}$$

where $\mu \equiv \frac{\mathbf{d}_5}{2^\beta}$, $\lim_{\beta\to\infty} \mathbf{f}_\beta = 1$, *and* c_0 *is a constant independent of* $\tau, \beta, \epsilon, \mu$.

Proof: Let us define $\mathcal{L}_\mu, \mathcal{K}_\mu, \widehat{\mathcal{K}_\mu}$ as

$$\begin{cases} \mathcal{L}_\mu(z) \equiv \begin{cases} 1 & \text{if } \mu \le z \le 2\mu, \\ 0 & \text{elsewhere}, \end{cases} \\ \mathcal{K}_\mu(z) \equiv \int^z_{\mathcal{A}(\Upsilon(2\mu))} \mathcal{L}_\mu(\Upsilon^{-1}(\mathcal{A}^{-1}(\xi)))d\xi & \text{for } z \in [0, \mathcal{A}(\infty)), \\ \widehat{\mathcal{K}_\mu}(z) \equiv \int^z_{\mathcal{A}(\Upsilon(2\mu))} (\mathcal{L}_\mu \frac{\Lambda_o}{\Lambda}) \circ (\Upsilon^{-1}(\mathcal{A}^{-1}(\xi)))d\xi & \text{for } z \in [0, \mathcal{A}(\infty)). \end{cases}$$

By $2\mu \le \frac{\mathbf{d}_5}{2}$ and A2-3,5, we take $\zeta = \mathcal{K}_\mu(\mathcal{A}^\epsilon) \in L^2(0,T;\mathcal{U})$ in (6.4) and $\eta = \widehat{\mathcal{K}_\mu}(\mathcal{A}^\epsilon) \in L^2(0,T;\mathcal{U})$ in (6.5) to obtain

$$\begin{aligned} \int_{\Omega^{\epsilon,\tau}_f} &\Phi^\epsilon \mathcal{K}_\mu(\mathcal{A}^\epsilon)\partial_t S^\epsilon + \int_{\Omega^{\epsilon,\tau}_f} K^\epsilon \Lambda_o(S^\epsilon)\mathcal{L}_\mu(S^\epsilon)\nabla\Upsilon(S^\epsilon)\nabla\mathcal{A}^\epsilon \\ &+ \int_{\Omega^{\epsilon,\tau}_m} \phi^\epsilon \mathcal{K}_\mu(\mathcal{A}^\epsilon)\partial_t s^\epsilon + \int_{\Omega^{\epsilon,\tau}_m} k^\epsilon \mathcal{I}^{2\varpi}_\epsilon \Lambda_o(\mathbf{u}^\epsilon)\mathcal{L}_\mu(\mathbf{u}^\epsilon)\nabla\upsilon(s^\epsilon)\nabla\mathcal{A}^\epsilon \\ \le c_1 &\Big(\int_{\Omega^{\epsilon,\tau}_f} K^\epsilon \Lambda_o(S^\epsilon)\mathcal{L}_\mu(S^\epsilon)|\partial_{x_3}\mathcal{A}^\epsilon| + \int_{\Omega^{\epsilon,\tau}_m} k^\epsilon \Lambda_o(\mathbf{u}^\epsilon)\mathcal{L}_\mu(\mathbf{u}^\epsilon)|\partial_{x_3}\mathcal{A}^\epsilon| \Big), \end{aligned} \tag{6.18}$$

where $\mathbf{u}^\epsilon \equiv \mathcal{J}(s^\epsilon)$ and constant c_1 is independent of ϵ, μ. Suppose

$$\int \mathcal{K}_\mu(\mathcal{A}^\epsilon)\big(\Phi^\epsilon \partial_t S^\epsilon \mathcal{X}_{\Omega^{\epsilon,\tau}_f} + \phi^\epsilon \partial_t s^\epsilon \mathcal{X}_{\Omega^{\epsilon,\tau}_m}\big) \ge 0, \tag{6.19}$$

(6.18–6.19) imply

$$\int_{\Omega_f^{\epsilon,\tau}} K^\epsilon \Lambda_o(S^\epsilon)\mathcal{L}_\mu(S^\epsilon)|\partial_{x_3}\mathcal{A}^\epsilon| + \int_{\Omega_m^{\epsilon,\tau}} k^\epsilon \Lambda_o(\mathbf{u}^\epsilon)\mathcal{L}_\mu(\mathbf{u}^\epsilon)|\partial_{x_3}\mathcal{A}^\epsilon|$$
$$\leq c_2\Big(\int_{\Omega_f^{\epsilon,\tau}} K^\epsilon \Lambda_o^{3/2}\mathcal{L}_\mu(S^\epsilon)\Big)^{\frac{1}{2}}\Big(\int_{\Omega_f^{\epsilon,\tau}} K^\epsilon \Lambda_o(S^\epsilon)\mathcal{L}_\mu(S^\epsilon)\partial_{x_3}\Upsilon(S^\epsilon)\partial_{x_3}\mathcal{A}^\epsilon\Big)^{\frac{1}{2}}$$
$$+c_2\Big(\int_{\Omega_m^{\epsilon,\tau}} k^\epsilon \Lambda_o^{3/2}\mathcal{L}_\mu(\mathbf{u}^\epsilon)\Big)^{\frac{1}{2}}\Big(\int_{\Omega_m^{\epsilon,\tau}} k^\epsilon \Lambda_o(\mathbf{u}^\epsilon)\mathcal{L}_\mu(\mathbf{u}^\epsilon)\partial_{x_3}\upsilon(s^\epsilon)\partial_{x_3}\mathcal{A}^\epsilon\Big)^{\frac{1}{2}}, \qquad (6.20)$$

where constant c_2 is independent of ϵ, μ. A3 and (6.18–6.20) imply

$$\int \mathcal{K}_\mu(\mathcal{A}^\epsilon)\big(\Phi^\epsilon\partial_t S^\epsilon \mathcal{X}_{\Omega_f^{\epsilon,\tau}} + \phi^\epsilon \partial_t s^\epsilon \mathcal{X}_{\Omega_m^{\epsilon,\tau}}\big) \leq c_3\int_{\Omega^\tau} \Lambda_o^{3/2}\mathcal{L}_\mu(\check{\mathcal{S}}^\epsilon). \qquad (6.21)$$

Let us define

$$\mathcal{Z}(S^\epsilon, s^\epsilon, \mu) \equiv \begin{cases} \Phi^\epsilon \int_{2\mu}^{S^\epsilon} \mathcal{K}_\mu(\mathcal{A}(\Upsilon(\xi)))d\xi & \text{in } \Omega_f^\epsilon, \\ \phi^\epsilon \int_{\mathcal{J}^{-1}(2\mu)}^{s^\epsilon} \mathcal{K}_\mu(\mathcal{A}(\upsilon(\xi)))d\xi & \text{in } \Omega_m^\epsilon. \end{cases}$$

(6.21) implies

$$\int_{\Omega^\tau} \partial_t \mathcal{Z}(S^\epsilon, s^\epsilon, \mu) \leq c_4\int_{\Omega^\tau} \Lambda_o^{3/2}\mathcal{L}_\mu(\check{\mathcal{S}}^\epsilon). \qquad (6.22)$$

(6.22) and A6-7 yield that, if $0 \leq t_1 \leq t_2 \leq T$,

$$\int_{t_1}^{t_2}\int_\Omega \partial_t \mathcal{Z}(S^\epsilon, s^\epsilon, \mu) \leq c_4\int_{t_1}^{t_2}\int_\Omega \mathcal{Z}(S^\epsilon, s^\epsilon, 2\mu), \qquad (6.23)$$

where c_4 is independent of t_1, t_2, μ, ϵ. Define

$$\mathcal{F}^\epsilon(\tau, \mu) \equiv \frac{1}{\Lambda_o(\mu)^{3/2}} \sup_{t\leq\tau} \int_\Omega \mathcal{Z}(S^\epsilon, s^\epsilon, \mu).$$

A5 and (6.23) imply that, for $0 \leq t_1 \leq t_2 \leq T$,

$$\mathcal{F}^\epsilon(t_2, \mu) - \mathcal{F}^\epsilon(t_1, \mu) \leq c_5(t_2 - t_1)\mathcal{F}^\epsilon(t_2, 2\mu),$$

where c_5 is independent of t_1, t_2, μ, ϵ. By induction and A3, one obtains, for $j \in \mathbf{N}$, $jh \leq T$,

$$\mathcal{F}^\epsilon(jh, \frac{\mathbf{d}_5}{2^\beta}) \leq (\beta - \beta_0 + 1)^{j-1}|c_5 h|^{\beta-\beta_0}\mathcal{F}^\epsilon(jh, \frac{\mathbf{d}_5}{2^{\beta_0}}). \qquad (6.24)$$

If $j = \frac{\beta-\beta_0}{\log(\beta-\beta_0)}$ and $\tau = jh$ in (6.24), then

$$\mathcal{F}^\epsilon(\tau, \frac{\mathbf{d}_5}{2^\beta}) \leq \frac{|c_5\tau|^{\beta-\beta_0}}{(\beta-\beta_0)^{(\beta-\beta_0)\mathbf{f}_\beta}}\mathcal{F}^\epsilon(\tau, \frac{\mathbf{d}_5}{2^{\beta_0}}), \qquad (6.25)$$

where $\mathbf{f}_\beta \to 1$ as $\beta \to \infty$. Define $\mathcal{B}(t) \equiv \{x \in \Omega : \check{\mathcal{S}}^\epsilon(t,x) \le \frac{\mathbf{d}_5}{2^\beta}\}$. (6.25) implies

$$\sup_{t\le\tau} \int \mathcal{X}_{\mathcal{B}(t)} \le c_6 \mathcal{F}^\epsilon(\tau, \frac{\mathbf{d}_5}{2^\beta}) \le \frac{c_6 |c_5 \tau|^{\beta-\beta_0}}{(\beta-\beta_0)^{(\beta-\beta_0)\mathbf{f}_\beta}} \mathcal{F}^\epsilon(\tau, \frac{\mathbf{d}_5}{2^{\beta_0}}),$$

where constant c_6 is independent of $\tau, \beta, \epsilon, \mu$. So proof of first part of (6.17) is completed. The other part can be proved in a similar way, so we skip it. ∎

Lemma 6.3 *If $r \in (1,2)$, $\|P_\alpha^\epsilon\|_{L^r(0,T;W^{1,r}(\Omega_f^\epsilon))} + \|\mathcal{I}_\epsilon^\varpi \nabla p_\alpha^\epsilon\|_{L^r(\Omega_m^{\epsilon,T})} \le c$, where $\alpha = w, o$ and c is a constant independent of ϵ. Moreover, if $\varpi \le 1$, then $\|p_\alpha^\epsilon\|_{L^r(\Omega_m^{\epsilon,T})} \le c$.*

Proof: We define, for $2 \le \beta_0 \in \mathbf{N}$,

$$\begin{cases} \mathcal{B}_{1+\beta_0} \equiv \{(t,x) \in \Omega_f^{\epsilon,T} : \ \frac{\mathbf{d}_5}{2^{2+\beta_0}} \le S^\epsilon\}, \\ \mathcal{B}_\beta \equiv \{(t,x) \in \Omega_f^{\epsilon,T} : \ \frac{\mathbf{d}_5}{2^{\beta+1}} \le S^\epsilon < \frac{\mathbf{d}_5}{2^\beta}\} \quad \text{if } 2+\beta_0 \le \beta \in \mathbf{N}. \end{cases}$$

A5, Lemmas 6.1-6.2, and Hölder inequality imply

$$\begin{aligned} \|\nabla P_o^\epsilon\|^r_{L^r(\Omega_f^{\epsilon,T})} &\le \|\sqrt{\Lambda_o(S^\epsilon)}\ \nabla P_o^\epsilon\|^r_{L^2(\Omega_f^{\epsilon,T})} \|\Lambda_o^{-1}(S^\epsilon)\|^{r/2}_{L^{r/(2-r)}(\Omega_f^{\epsilon,T})} \\ &\le c_1 \Big(\int_{\Omega_f^{\epsilon,T}} |\Lambda_o(S^\epsilon)|^{\frac{-r}{2-r}} \sum_{\beta=1+\beta_0}^{\infty} \mathcal{X}_{\mathcal{B}_\beta} \Big)^{\frac{2-r}{2}} \le c_2 \ (\text{indep. of } \epsilon). \end{aligned} \tag{6.26}$$

Similar argument will give $\|\nabla P_w^\epsilon\|_{L^r(\Omega_f^{\epsilon,T})} + \sum_{\alpha=w,o} \|\mathcal{I}_\epsilon^\varpi \nabla p_\alpha^\epsilon\|_{L^r(\Omega_m^{\epsilon,T})} \le c$. By boundary condition A2, $\|P_\alpha^\epsilon\|_{L^r(\Omega_f^{\epsilon,T})} \le c$, $\alpha = w,o$. By Lemma 5.1, (2.8), and $\varpi \le 1$, $\|p_\alpha^\epsilon - \Pi_\epsilon P_\alpha^\epsilon\|_{L^r(\Omega_m^{\epsilon,T})} \le \|\epsilon \partial_{x_1}(p_\alpha^\epsilon - \Pi_\epsilon P_\alpha^\epsilon)\|_{L^r(\Omega_m^{\epsilon,T})} \le c$. So $\|p_\alpha^\epsilon\|_{L^r(\Omega_m^{\epsilon,T})}$ is bounded. ∎

Lemma 6.4 *For $r \in [1,\infty)$ and sufficiently small δ,*

$$\|\delta^2 \partial^{-\delta} S^\epsilon\ \partial^{-\delta} \mathcal{A}^\epsilon\|_{L^r((\delta,T)\times\Omega_f^\epsilon)} + \|\delta^2 \partial^{-\delta} s^\epsilon\ \partial^{-\delta} \mathcal{A}^\epsilon\|_{L^r((\delta,T)\times\Omega_m^\epsilon)} \le c\delta^{1/r}, \tag{6.27}$$

where c is independent of ϵ, δ. See §4 for notation $\partial^{-\delta}$.

Proof: Note $\zeta(t,x) \equiv \int_{max(t,\delta)}^{min(t+\delta,T)} \delta\ \partial^{-\delta}\big(\mathcal{A}^\epsilon - \mathcal{A}(P_{b,c})\big)(\tau,x) d\tau \in L^2(0,T;\mathcal{U})$ by A2-3 and Lemma 6.1. Take ζ above in (6.2) to get, by Fubini's theorem, A2, and Lemma 6.1,

$$\begin{aligned} &\int_\delta^T\!\!\int_{\Omega_f^\epsilon} \Phi^\epsilon \delta^2 \partial^{-\delta} S^\epsilon\ \partial^{-\delta} \mathcal{A}^\epsilon(\tau,x) + \int_\delta^T\!\!\int_{\Omega_m^\epsilon} \phi^\epsilon \delta^2 \partial^{-\delta} s^\epsilon \partial^{-\delta} \mathcal{A}^\epsilon(\tau,x) \\ &\quad = \int_{\Omega_f^{\epsilon,T}} \Phi^\epsilon \partial_t S^\epsilon(t,x)\zeta + \int_{\Omega_m^{\epsilon,T}} \phi^\epsilon \partial_t s^\epsilon(t,x)\zeta \\ &\qquad + \int_\delta^T\!\!\int_{\Omega_f^\epsilon} \Phi^\epsilon \delta^2 \partial^{-\delta} S^\epsilon\ \partial^{-\delta} \mathcal{A}(P_{b,c}) + \int_\delta^T\!\!\int_{\Omega_m^\epsilon} \phi^\epsilon \delta^2 \partial^{-\delta} s^\epsilon \partial^{-\delta} \mathcal{A}(P_{b,c}) \le c\delta, \end{aligned}$$

where c is independent of ϵ, δ. So we prove (6.27) for $r=1$ case. (6.27) for $r>1$ case follows directly because $\mathcal{A}^\epsilon, \check{\mathcal{S}}^\epsilon$ are bounded and (6.27) for $r=1$ holds. ∎

Lemma 6.5 *A subsequence of* $\Pi_\epsilon(\mathcal{A}^\epsilon|_{\Omega_f^\epsilon})$ *converges to* $\mathcal{A}^*$ *in* $L^2(\Omega^T)$ *and pointwise.*

Proof: This is due to A6,10, Lemmas 5.1, 6.1-6.4, and compactness principle. ■

Lemma 6.6 $\overline{s^\epsilon}, \overline{p^\epsilon}, \overline{p^\epsilon_\alpha}$ $(\alpha = w, o)$ *satisfy, for almost all* $x \in \tilde{\Omega}^\epsilon$,

$$\phi\partial_t\overline{s^\epsilon} - \partial_{y,x_3}\cdot\left(k\mathcal{I}_\epsilon^{2\varpi-2}\left(\partial_{y,x_3}\mathcal{M}(\overline{s^\epsilon}) + \lambda_o(\overline{s^\epsilon})\partial_{y,x_3}(\overline{p^\epsilon} - \overline{G^\epsilon_o})\right)\right) = 0, \tag{6.28}$$

$$\partial_{y,x_3}\cdot\left(k\mathcal{I}_\epsilon^{2\varpi-2}\left(\lambda(\overline{s^\epsilon})\partial_{y,x_3}\overline{p^\epsilon} - \sum\lambda_\alpha(\overline{s^\epsilon})\partial_{y,x_3}\overline{G^\epsilon_\alpha}\right)\right) = 0, \tag{6.29}$$

$$-\phi\partial_t\overline{s^\epsilon} - \partial_{y,x_3}\cdot\left(k\mathcal{I}_\epsilon^{2\varpi-2}\lambda_w(\overline{s^\epsilon})\partial_{y,x_3}(\overline{p^\epsilon_w} - \overline{G^\epsilon_w})\right) = 0, \tag{6.30}$$

$$\phi\partial_t\overline{s^\epsilon} - \partial_{y,x_3}\cdot\left(k\mathcal{I}_\epsilon^{2\varpi-2}\lambda_o(\overline{s^\epsilon})\partial_{y,x_3}(\overline{p^\epsilon_o} - \overline{G^\epsilon_o})\right) = 0, \tag{6.31}$$

in $L^2(0,T;H^{-1}(Y_m^H))$.

Proof: Let $\hat{\zeta} \in L^2(0,T;C_0^\infty(Y_m^H))$. For $x \in \Omega, y \in \Re^2$, we define

$$\check{\zeta}(t,x,y) \equiv \begin{cases} \hat{\zeta}(t, \frac{y-\ell^\epsilon(\tilde{x})}{\epsilon}, x_3) & \text{for } y \in \epsilon Y_m + \ell^\epsilon(\tilde{x}), \\ 0 & \text{elsewhere.} \end{cases}$$

Then we plug $\zeta(t,x) \equiv \mathcal{X}_{\epsilon(Y_m+j)}(\tilde{x})\check{\zeta}(t,x,\tilde{x})$ for $j \in Z^2$ into (6.4). Since *supp* $\zeta \subset (0,T)\times\epsilon(Y_m+j)\times[0,H]$,

$$\int_0^T\int_0^H\int_{\epsilon(Y_m+j)} \phi^\epsilon\partial_t s^\epsilon\zeta + k^\epsilon\mathcal{I}_\epsilon^{2\varpi}\left(\lambda_o(s^\epsilon)\nabla(p^\epsilon - G^\epsilon_o) + \nabla\mathcal{M}(s^\epsilon)\right)\nabla\zeta = 0.$$

Since $\tilde{x} \in \epsilon(Y_m+j)$, $\ell^\epsilon(\tilde{x}) = \epsilon j$. Changing variable $y = \frac{\tilde{x}-\ell^\epsilon(\tilde{x})}{\epsilon}$ gives

$$\int_0^T\int_{Y_m^H} \phi\partial_t\overline{s^\epsilon}\hat{\zeta} + k\mathcal{I}_\epsilon^{2\varpi-2}\left(\partial_{y,x_3}\mathcal{M}(\overline{s^\epsilon}) + \lambda_o(\overline{s^\epsilon})\partial_{y,x_3}(\overline{p^\epsilon} - \overline{G^\epsilon_o})\right)\partial_{y,x_3}\hat{\zeta} = 0, \tag{6.32}$$

for almost all $\tilde{x} \in \epsilon(Y_m+j), j \in Z^2$. Actually, by Definition 5.1, (6.32) holds for $\tilde{x} \in \tilde{\Omega}^\epsilon$, i.e., (6.28). (6.29–6.31) can be proved in a similar way. ■

Remark 6.2 *By Lemmas 5.1, 6.5, if we define* $S^\epsilon \equiv \Upsilon^{-1}(\mathcal{A}^{-1}(\Pi_\epsilon(\mathcal{A}^\epsilon|_{\Omega_f^\epsilon})))$ *and* $S \equiv \begin{cases} \Upsilon^{-1}(\mathcal{A}^{-1}(\mathcal{A}^*)) & \text{if } \mathcal{A}^* < \mathcal{A}(\infty), \\ 1 & \text{if } \mathcal{A}^* = \mathcal{A}(\infty), \end{cases}$ *then* $0 \le S^\epsilon, S \le 1$.

Lemma 6.7 *There is a* $r \in (1,2)$ *and a subsequence of* $\{S^\epsilon, s^\epsilon, S_0^\epsilon, s_0^\epsilon, \phi^\epsilon, k^\epsilon, P^\epsilon_\alpha, p^\epsilon_\alpha, \alpha = w, o\}$ *such that, as* $\epsilon \to 0$,

$$\begin{cases} \mathcal{X}_{\Omega_f^\epsilon}P^\epsilon_\alpha \overset{2}{\rightharpoonup} \mathcal{X}_{Y_f}(y)P_\alpha(t,x) \text{ where } P_\alpha \in L^r(0,T;W^{1,r}(\Omega)), P_\alpha = P_{b,\alpha} \text{ in } \Gamma_2, \\ \mathcal{X}_{\Omega_f^\epsilon}\nabla P^\epsilon_\alpha \overset{2}{\rightharpoonup} \mathcal{X}_{Y_f}(y)(\nabla P_\alpha + \partial_y P_{\alpha,1}(t,x,y)) \text{ where } P_{\alpha,1} \in L^r(\Omega^T;L^r_{per}(Y_f)), \\ \mathcal{X}_{\Omega_f^\epsilon}S_0^\epsilon \overset{2}{\rightharpoonup} S_0 \in L^2(\Omega;L^2_{per}(Y_f)), \\ S^\epsilon \to S \text{ strongly in } L^2(\Omega^T) \text{ and pointwise}, \\ \mathcal{X}_{\Omega_f^\epsilon}S^\epsilon \overset{2}{\to} \mathcal{X}_{Y_f}(y)S(t,x) \text{ strongly}, \\ \mathcal{X}_{\Omega_m^\epsilon}s_0^\epsilon \overset{2}{\to} s_0 \in L^2(\Omega;L^2_{per}(Y_m)) \text{ strongly}, \\ \overline{p^\epsilon_\alpha} \rightharpoonup p_\alpha \quad \text{weakly in } L^r(\Omega^T;W^{1,r}(Y_m)). \end{cases}$$

Proof: By Lemma 5.1 and Lemma 6.3, $\Pi_\epsilon P_\alpha^\epsilon$ is bounded in $L^r(0,T;W^{1,r}(\Omega))$. So a subsequence of $\Pi_\epsilon P_\alpha^\epsilon$ converges weakly to limit $P_\alpha \in L^r(0,T;W^{1,r}(\Omega))$. Since $\Pi_\epsilon P_\alpha^\epsilon = P_{b,\alpha}$ in Γ_2, $P_\alpha = P_{b,\alpha}$ in Γ_2. Rest of proof are due to A2-4,6,10, Lemmas 6.1, 6.3, 6.5, and [3]. ■

Lemma 6.8 *$\overline{s^\epsilon}$ converges to s in $L^2(\mathcal{Q}_m^T)$ if $0 < \varpi \le 1$.*

Proof of this lemma is lengthy, and will be postponed untill the last five sections.

Lemma 6.9 *If $\varpi = 1$, then $p_o - p_w = \upsilon(s)$, $P_o - P_w = \Upsilon(S)$, and $p_\alpha(t,x,y) = P_\alpha(t,x)$ for $x \in \Omega, y \in \partial Y_m$, $\alpha = w, o$. If $\varpi < 1$, then $\upsilon(s) = P_o - P_w = \Upsilon(S)$ and $p_\alpha(t,x,y) = P_\alpha(t,x)$ for $x \in \Omega, y \in Y_m$, $\alpha = w, o$.*

Proof: First we consider $\varpi = 1$ case. Note $0 \le S < 1$, $s_l \le s < s_r$ by Egoroff's theorem [25] and Lemmas 6.1-6.2, 6.7-6.8. Since $\overline{p_o^\epsilon} - \overline{p_w^\epsilon} = \upsilon(\overline{s^\epsilon})$, we get $p_o - p_w = \upsilon(s)$ by Lemmas 6.7-6.8. Similarly, one can derive $P_o - P_w = \Upsilon(S)$. By Lemmas 5.1, 6.3 and (2.8), $\overline{(\Pi_\epsilon P_\alpha^\epsilon)|_{\Omega_m^\epsilon}} - \overline{p_\alpha^\epsilon} \in L^r(\Omega^T; W_0^{1,r}(Y_m))$ for $1 < r < 2$. So, a subsequence of $\overline{(\Pi_\epsilon P_\alpha^\epsilon)|_{\Omega_m^\epsilon}} - \overline{p_\alpha^\epsilon}$ converges weakly to $\mathcal{X}_{Y_m}(y)P_\alpha(t,x) - p_\alpha \in L^r(\Omega^T; W_0^{1,r}(Y_m))$ by Lemma 6.7. So, $p_\alpha(t,x,y) = P_\alpha(t,x)$ for $y \in \partial Y_m$. Results for $\varpi < 1$ case can be obtained by similar argument as above, so we skip it. ■

Now we consider the limit model of (2.1–2.13) as $\epsilon \to 0$. Plug into (6.1) and (6.6) a test function $\eta = \hat\zeta(t,x) + \epsilon\hat\eta(t,x,\frac{\tilde x}{\epsilon})$ where $\hat\zeta \in C_0^\infty(\Omega^T), \hat\eta \in C_0^\infty(\Omega^T; C_{per}^\infty(Y))$ to obtain

$$
\begin{aligned}
0 = &\int_{\Omega_f^{\epsilon,T}} \Phi^\epsilon S^\epsilon(\partial_t\hat\zeta + \epsilon\partial_t\hat\eta) + K^\epsilon\Lambda_w(S^\epsilon)\nabla(P_w^\epsilon - G_w^\epsilon)(\nabla\hat\zeta + \epsilon\partial_x\hat\eta + \partial_y\hat\eta) \\
&+ \int_{\Omega_m^{\epsilon,T}} \phi^\epsilon s^\epsilon(\partial_t\hat\zeta + \epsilon\partial_t\hat\eta) + k^\epsilon\mathcal{I}_\epsilon^{2\varpi}\lambda_w(s^\epsilon)\nabla(p_w^\epsilon - G_w^\epsilon)(\nabla\hat\zeta + \epsilon\partial_x\hat\eta + \partial_y\hat\eta) \\
&+ \int_{\Omega_f^\epsilon} \Phi^\epsilon S_0^\epsilon(\hat\zeta + \epsilon\hat\eta)(0) + \int_{\Omega_m^\epsilon} \phi^\epsilon s_0^\epsilon(\hat\zeta + \epsilon\hat\eta)(0).
\end{aligned}
$$

By A2 and Lemma 6.7, $K^\epsilon\Lambda_w(S^\epsilon)$ converges to $K^*\Lambda_w(S)$ in $L^r(\Omega^T), r < \infty$ strongly. Passing to two-scale limit, we get, by A2-4, Lemmas 6.3-6.9, Theorem 2.28 of [2], Theorem 1.8 of [3], and [8, 10],

$$
\begin{aligned}
&\int_{\mathcal{Q}_f^T} \Phi^* S\partial_t\hat\zeta + K^*\Lambda_w(S)(\nabla P_w + \partial_y P_{w,1} - \nabla G_w)(\nabla\hat\zeta + \partial_y\hat\eta) \\
&\qquad = -\int_{\mathcal{Q}_m^T} \phi s\partial_t\hat\zeta + \mathcal{F}_w^*\partial_{x_3}\hat\zeta - \int_{\mathcal{Q}_f} \Phi^* S_0\hat\zeta(0) - \int_{\mathcal{Q}_m} \phi s_0\hat\zeta(0),
\end{aligned}
$$

where

$$
\mathcal{F}_w^* \equiv \begin{cases} k\lambda_w(s)\partial_{x_3}(P_w - G_w) & \text{if } 0 < \varpi < 1, \\ k\lambda_w(s)\partial_{x_3}(p_w - G_w) & \text{if } \varpi = 1, \\ \text{An } L^2 \text{ function} & \text{if } \varpi > 1. \end{cases} \tag{6.33}
$$

Apply Green's theorem in t variable to get

$$
-\int_{\mathcal{Q}_f^T} \Phi^*\partial_t S\ \hat\zeta - K^*\Lambda_w(S)(\nabla P_w + \partial_y P_{w,1} - \nabla G_w)(\nabla\hat\zeta + \partial_y\hat\eta)
$$

$$= \int_{\mathcal{Q}_m^T} \phi\partial_t s\ \hat{\zeta} - \mathcal{F}_w^* \partial_{x_3}\hat{\zeta} + \int_{\mathcal{Q}_f} \Phi^*(S(0)-S_0)\hat{\zeta}(0) + \int_{\mathcal{Q}_m} \phi(s(0)-s_0)\hat{\zeta}(0).$$

So we have, in Ω^T,

$$(S(0)-S_0)\int_{Y_f} \Phi^* dy + \int_{Y_m} \phi(s(0)-s_0)dy = 0, \tag{6.34}$$

and the choice of $\hat{\eta}=0$ gives, in Ω^T,

$$\begin{aligned}\int_{Y_f} \Phi^* dy \partial_t S + \nabla \int_{Y_f} K^*\Lambda_w(S)(\nabla P_w + \partial_y P_{w,1} - \nabla G_w)&\\ = -\int_{Y_m} (\phi\partial_t s + \partial_{x_3}\mathcal{F}_w^*)\, dy.&\end{aligned} \tag{6.35}$$

The choice of $\hat{\zeta}=0$ gives, by A2-3 and Lemma 6.7,

$$\begin{cases} \partial_y^2 P_{w,1} = 0 & \text{in } \mathcal{Q}_f,\\ (\partial_{\tilde{x}} P_w + \partial_y P_{w,1})\cdot \vec{\nu_y} = 0 & \text{on } \partial Y_m, \end{cases} \tag{6.36}$$

where $\vec{\nu_y}$ is the unit vector outward normal to ∂Y_m. Let $\vec{e_j}$ be the unit vector in jth direction. We denote by Ξ the tensor whose (i,j) component is $\partial\varphi_j/\partial y_i$, where φ_j is a periodic solution in Y of the auxiliary problem

$$\begin{cases} \Delta_y \varphi_j = 0 & \text{in } Y_f,\\ \partial_y\varphi_j\cdot\vec{\nu_y} = -\vec{e_j}\cdot\vec{\nu_y} & \text{on } \partial Y_m. \end{cases}$$

$P_{w,1}$ of (6.36) is given by the product $P_{w,1} = \sum_j \varphi_j(y)\partial_{x_j} P_w$. So (6.35) becomes

$$\Phi\partial_t S + \nabla\cdot(K\Lambda_w(S)\nabla(P_w - G_w)) = \frac{-1}{|Y_m|}\int_{Y_m} (\phi\partial_t s + \partial_{x_3}\mathcal{F}_w^*)dy, \tag{6.37}$$

where $\Phi \equiv \frac{1}{|Y_m|}\int_{Y_f}\Phi^* dy$ and K is a diagonal matrix satisfying

$$K_{11} = K_{22} = \frac{K^*}{|Y_m|}\int_{Y_f}(I + \Xi(y))dy, \quad K_{33} = \frac{|Y_f|K^*}{|Y_m|}.$$

Proceeding as the proof of (6.37), we obtain, by (6.2),

$$-\Phi\partial_t S + \nabla\cdot(K\Lambda_o(S)\nabla(P_o - G_o)) = \frac{1}{|Y_m|}\int_{Y_m} (\phi\partial_t s - \partial_{x_3}\mathcal{F}_o^*)dy, \tag{6.38}$$

where

$$\mathcal{F}_o^* \equiv \begin{cases} k\lambda_o(s)\partial_{x_3}(P_o - G_o) & \text{if } 0 < \varpi < 1,\\ k\lambda_o(s)\partial_{x_3}(p_o - G_o) & \text{if } \varpi = 1,\\ \text{An } L^2 \text{ function} & \text{if } \varpi > 1. \end{cases} \tag{6.39}$$

Matrix sources for $0 < \varpi < 1$ case is clear from (6.33), (6.39), and Lemma 6.9. Next we consider the matrix source terms for $\varpi \geq 1$ cases.

6.1. *For* $\varpi = 1$ *case*

By (6.30) of Lemma 6.6, we have, for any $\eta \in L^2(\Omega^T; H_0^1(Y_m))$,

$$\int_{\mathcal{Q}_m^{\epsilon,T}} \phi \partial_t \overline{s^\epsilon} \eta + \int_{\mathcal{Q}_m^{\epsilon,T}} k\lambda_o(\overline{s^\epsilon}) \partial_{y,x_3}(\overline{p_o^\epsilon} - \overline{G_o^\epsilon}) \partial_{y,x_3} \eta = 0.$$

As $\epsilon \to 0$, by Lemmas 6.7-6.8, one obtains

$$\int_{\mathcal{Q}_m^T} \phi \partial_t s \ \eta + \int_{\mathcal{Q}_m^T} k\lambda_o(s) \partial_{y,x_3}(p_o - G_o) \partial_{y,x_3} \eta = 0. \tag{6.40}$$

In a similar way, we obtain, by (6.31),

$$\int_{\mathcal{Q}_m^T} \phi \partial_t s \ \eta - \int_{\mathcal{Q}_m^T} k\lambda_w(s) \partial_{y,x_3}(p_w - G_w) \partial_{y,x_3} \eta = 0. \tag{6.41}$$

By (6.37–6.41) and Lemmas 6.7-6.9, it is easy to show Theorem 4.1.

6.2. *For* $\varpi > 1$ *case*

By (6.30) of Lemma 6.6, we have, for any $\eta \in L^2(\Omega^T; H_0^1(Y_m))$,

$$\int_{\mathcal{Q}_m^{\epsilon,T}} \phi \partial_t \overline{s^\epsilon} \eta + \int_{\mathcal{Q}_m^{\epsilon,T}} k\mathcal{I}_\epsilon^{2\varpi-2} \lambda_o(\overline{s^\epsilon}) \partial_{y,x_3}(\overline{p_o^\epsilon} - \overline{G_o^\epsilon}) \partial_{y,x_3} \eta = 0.$$

As $\epsilon \to 0$, by Lemmas 6.7-6.8, one obtains

$$\int_{\mathcal{Q}_m^T} \phi \partial_t s \ \eta + \int_{\mathcal{Q}_m^T} \mathcal{F}_o^* \partial_{x_3} \eta = 0.$$

So we get $\phi \partial_t s - \partial_{x_3} \mathcal{F}_o^* = 0$. In a similar way, we obtain $\phi \partial_t s + \partial_{x_3} \mathcal{F}_w^* = 0$. Therefore we prove Theorem 4.3.

Rest of this work is to prove Lemma 6.8.

7. Convergence of $\overline{s^\epsilon}$

Remark 7.3 *Define*

$$\begin{cases} \mathcal{G}^\epsilon \equiv \begin{cases} \upsilon^{-1}(\mathcal{A}^{-1}(\Pi_\epsilon(\mathcal{A}^\epsilon|_{\Omega_f^\epsilon}))) & \text{if } \Pi_\epsilon \mathcal{A}^\epsilon < \mathcal{A}(\infty), \\ s_r & \text{if } \Pi_\epsilon \mathcal{A}^\epsilon = \mathcal{A}(\infty), \end{cases} \\ \mathcal{G} \equiv \begin{cases} \upsilon^{-1}(\mathcal{A}^{-1}(\mathcal{A}^*)) & \text{if } \mathcal{A}^* < \mathcal{A}(\infty), \\ s_r & \text{if } \mathcal{A}^* = \mathcal{A}(\infty). \end{cases} \end{cases}$$

See Lemma 6.5 for $\mathcal{A}^*$. By Lemma 6.7, A1,3, Theorem 2.28 of [2], and [3, 8, 10], it is easy to see that

$$\begin{cases} \|\mathcal{M}(\mathcal{G}^\epsilon)\|_{L^2(0,T;H^1(\Omega))} \ \text{are bounded independently of } \epsilon, \\ \mathcal{M}(\overline{\mathcal{G}^\epsilon|_{\Omega_m^\epsilon}}) \to \mathcal{M}(\mathcal{G}) \quad \text{strongly in } L^2(\mathcal{Q}_m^T), \\ \mathcal{M}(\overline{\mathcal{G}^\epsilon|_{\Omega_m^\epsilon}}) - \mathcal{M}(\overline{s^\epsilon}) \in L^2(\Omega^T; H_0^1(Y_m)). \end{cases} \tag{7.1}$$

Assume that $\overline{s^{\epsilon_i}}, \overline{p^{\epsilon_i}}, i = 1, 2$ are two solutions of (6.28–6.29), and ζ, η are smooth functions satisfying

$$\zeta(T) = 0,\ \zeta|_{\partial Y_m \times [0,H]} = \eta|_{\partial Y_m \times [0,H]} = \partial_{x_3}\zeta|_{x_3 \in \{0,H\}} = \partial_{x_3}\eta|_{x_3 \in \{0,H\}} = 0. \quad (7.2)$$

Let $x \in \Omega^{\epsilon_1} \cap \Omega^{\epsilon_2}$. By subtracting one solution from the other and integration by parts, we obtain

$$\int_{Y_m^{H,T}} (\overline{s^{\epsilon_1}} - \overline{s^{\epsilon_2}})\big(\phi\partial_t\zeta + \mathcal{F}_1\partial_{y,x_3}(k\mathcal{I}_\epsilon^{2\varpi-2}\partial_{y,x_3}\zeta) - \mathcal{F}_2\partial_{y,x_3}\zeta - \mathcal{F}_3\partial_{y,x_3}\eta\big)$$
$$+ \int_{Y_m^{H,T}} (\overline{p^{\epsilon_1}} - \overline{p^{\epsilon_2}})\big(\partial_{y,x_3} k\mathcal{I}_\epsilon^{2\varpi-2}(\lambda(\overline{s^{\epsilon_1}})\partial_{y,x_3}\eta + \lambda_o(\overline{s^{\epsilon_1}})\partial_{y,x_3}\zeta)\big) = \mathcal{F}_4 + \mathcal{F}_5, \quad (7.3)$$

where

$$\mathcal{F}_1 \equiv \mu + \begin{cases} \frac{\mathcal{M}(\overline{s^{\epsilon_1}}) - \mathcal{M}(\overline{s^{\epsilon_2}})}{\overline{s^{\epsilon_1}} - \overline{s^{\epsilon_2}}} & \text{if } \overline{s^{\epsilon_1}} \neq \overline{s^{\epsilon_2}}, \\ 0 & \text{otherwise}, \end{cases} \quad (7.4)$$

$$\mathcal{F}_2 \equiv \begin{cases} \frac{k(\lambda_o(\overline{s^{\epsilon_1}}) - \lambda_o(\overline{s^{\epsilon_2}}))\mathcal{I}_\epsilon^{2\varpi-2}\partial_{y,x_3}(\overline{p^{\epsilon_2}} - \overline{G_o^{\epsilon_2}})}{\overline{s^{\epsilon_1}} - \overline{s^{\epsilon_2}}} & \text{if } \overline{s^{\epsilon_1}} \neq \overline{s^{\epsilon_2}}, \\ 0 & \text{otherwise}, \end{cases} \quad (7.5)$$

$$\mathcal{F}_3 \equiv \begin{cases} \sum_\alpha \frac{k(\lambda_\alpha(\overline{s^{\epsilon_1}}) - \lambda_\alpha(\overline{s^{\epsilon_2}}))\mathcal{I}_\epsilon^{2\varpi-2}\partial_{y,x_3}(\overline{p^{\epsilon_2}} - \overline{G_\alpha^{\epsilon_2}})}{\overline{s^{\epsilon_1}} - \overline{s^{\epsilon_2}}} & \text{if } \overline{s^{\epsilon_1}} \neq \overline{s^{\epsilon_2}}, \\ 0 & \text{otherwise}, \end{cases} \quad (7.6)$$

$$\mathcal{F}_4 \equiv \mu \int_{Y_m^{H,T}} (\overline{s^{\epsilon_1}} - \overline{s^{\epsilon_2}})\partial_{y,x_3}(k\mathcal{I}_\epsilon^{2\varpi-2}\partial_{y,x_3}\zeta), \quad (7.7)$$

$$\mathcal{F}_5 \equiv \int_{Y_m^{H,T}} \epsilon^{2\varpi-2}\partial_y\left(\left(\mathcal{M}(\overline{\mathcal{G}^{\epsilon_1}|_{\Omega_m^{\epsilon_1}}}) - \mathcal{M}(\overline{\mathcal{G}^{\epsilon_2}|_{\Omega_m^{\epsilon_2}}})\right)k\partial_y\zeta\right)$$
$$+ \int_{Y_m^{H,T}} \epsilon^{2\varpi-2}\partial_y\left((\overline{\Pi_{\epsilon_1}P^{\epsilon_1}|_{\Omega_m^{\epsilon_1}}} - \overline{\Pi_{\epsilon_2}P^{\epsilon_2}|_{\Omega_m^{\epsilon_2}}})\big(k\lambda(\overline{s^{\epsilon_1}})\partial_y\eta + k\lambda_o(\overline{s^{\epsilon_1}})\partial_y\zeta\big)\right)$$
$$- \sum_{\alpha \in \{w,o\}} \int_{Y_m^{H,T}} k\lambda_\alpha(\overline{s^{\epsilon_1}})\partial_{x_3}(\overline{G_\alpha^{\epsilon_1}} - \overline{G_\alpha^{\epsilon_2}})\partial_{x_3}\eta$$
$$- \int_{Y_m^{H,T}} k\lambda_o(\overline{s^{\epsilon_1}})\partial_{x_3}(\overline{G_o^{\epsilon_1}} - \overline{G_o^{\epsilon_2}})\partial_{x_3}\zeta - \int_{Y_m^H} (\overline{s_0^{\epsilon_1}} - \overline{s_0^{\epsilon_2}})\phi\zeta(0). \quad (7.8)$$

Define $\widetilde{\mathcal{U}}_1 \equiv \{\zeta : \zeta \in H^1(Y_m^{H,T}) \cap L^\infty(0,T; H^1(Y_m^H)), \zeta|_{\partial Y_m \times [0,H]} = \partial_{x_3}\zeta|_{x_3=0,H} = \zeta(0) = 0\}$. We consider the following auxiliary problem for fixed μ:

Lemma 7.1 *Let $\mathcal{F}_2, \mathcal{F}_3 \in L^\infty(Y_m^{H,T})$ and $0 < \mathbf{d}_{18} < \mathcal{F}_1 < \mathbf{d}_{19} < \infty$. For $(f_1, f_2) \in L^2(Y_m^{H,T}) \times L^2(Y_m^{H,T})$, there is a unique $(\zeta, \eta) \in \widetilde{\mathcal{U}}_1 \times L^2(0,T; H^1(Y_m^H))$ such that*

$$-\phi\partial_t\zeta + \mathcal{F}_1\partial_{y,x_3}(k\mathcal{I}_\epsilon^{2\varpi-2}\partial_{y,x_3}\zeta) - \mathcal{F}_2\partial_{y,x_3}\zeta - \mathcal{F}_3\partial_{y,x_3}\eta = f_1, \quad (7.9)$$
$$\partial_{y,x_3}\big(k\mathcal{I}_\epsilon^{2\varpi-2}(\lambda\partial_{y,x_3}\eta + \lambda_o\partial_{y,x_3}\zeta)\big) = f_2. \quad (7.10)$$

Moreover,

$$\sup_{\tau \le T} \|\mathcal{I}_\epsilon^{\varpi-1}\partial_{y,x_3}\zeta(\tau)\|_{L^2(Y_m^H)} + \||\mathcal{I}_\epsilon^{\varpi-1}\partial_{y,x_3}\eta| + \mathbf{d}_{18}^{1/2}|\partial_{y,x_3}(k\mathcal{I}_\epsilon^{2\varpi-2}\partial_{y,x_3}\zeta)|\|_{L^2(Y_m^{H,T})}$$
$$\le c\left(\mathbf{d}_{19}, \|(|\mathcal{F}_2| + |\mathcal{F}_3|)/\mathcal{F}_1^{1/2}\|_{L^\infty(Y_m^{H,T})}\right)\|\ |f_1|/\mathcal{F}_1^{1/2} + |f_2|\ \|_{L^2(Y_m^{H,T})}. \quad (7.11)$$

Proof: This is proved by following the argument of Lemma 5.1 [29]. ■

Finally we give the proof of Lemma 6.8.

Proof: For $x \in \Omega^{\epsilon_1} \cap \Omega^{\epsilon_2}$, we take $f_1 = \mathcal{M}(\overline{s^{\epsilon_1}}) - \mathcal{M}(\overline{s^{\epsilon_2}})$ in (7.9) and $f_2 = \overline{p^{\epsilon_1}} - \overline{p^{\epsilon_2}}$ in (7.10) to obtain solution (ζ^μ, η^μ) for each μ by (7.4–7.6), Remark 7.3, and Lemma 7.1. After substitution $t \to T - t$ for the solution (ζ^μ, η^μ), we plug it into (7.3) to obtain

$$\int_{Y_m^{H,T}} (\overline{s^{\epsilon_1}} - \overline{s^{\epsilon_2}})(\mathcal{M}(\overline{s^{\epsilon_1}}) - \mathcal{M}(\overline{s^{\epsilon_2}})) + \int_{Y_m^{H,T}} |\overline{p^{\epsilon_1}} - \overline{p^{\epsilon_2}}|^2 = \mathcal{F}_4 + \mathcal{F}_5. \qquad (7.12)$$

By Lemmas 6.1, 7.1 and [13, 15, 26, 27], we see 1) $\mathcal{F}_4$ is bounded by $c\sqrt{\mu}$, where c is a constant independent of $\mu, \epsilon_1, \epsilon_2$; and 2) For fixed μ, $\mathcal{F}_5$ converges to 0 as ϵ_1, ϵ_2 tend to 0. So it is not difficult to show that $\mathcal{M}(\overline{s^{\epsilon_2}})$ is a Cauchy sequence in $L^2(\mathcal{Q}_m^T)$, which implies $\overline{s^{\epsilon_2}}$ is a Cauchy sequence in $L^2(\mathcal{Q}_m^T)$ as well. ■

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