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Willmore 演進方程之等高集描述

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A level set formulation for Willmore flow

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執行機構：國立交通大學應用數學系
E-mail: yjhsu@math.nctu.edu.tw

摘要

假設 M 為 3 維歐氏空間上之緊緻曲面。此報告旨在等高集意義下導出對應的 Willmore 演進方程，進而嘗試定義出弱解。另一方面則找出廣義 Willmore 演進方程可能的幾何意義，希望藉此能得到較自然的描述。

關鍵詞 Willmore 演進方程、曲面、等高面

Abstract

In this report, we will find a level set formulation of the Willmore flow for compact surfaces in the 3-dimensional Euclidean space R^3. According to this formulation, we define the weak solution in the sense of viscosity. On the other hand, we find a possible interpretation for the generalized Willmore flow which decreases the Willmore energy in more natural way.

Keywords: Willmore flow, surface, level set

1. Introduction

Let M be a compact immersed surface in the 3-dimensional Euclidean space R^3. Let h_{ij} be the components of the second fundamental form of M, by $H=\sum h_{ii}$ the mean curvature. Let $\phi_{ij}=h_{ij}-\frac{H}{2}\delta_{ij}$ be the trace free tensor and $\Phi=\sum(\phi_{ij})^2$ the square length of ϕ_{ij}. Then the Willmore functional of X is given by

$$W(X)=\int_M \Phi,$$

where the integration is with respect to the area measure of M. This functional is preserved if we move M via conformal transformations of R^3.

The first variation formula of the Willmore functional is given by

$$(\int_M \Phi\}_t = \int_M (\Delta H+\Phi H)X_t\cdot e_3.$$

It follows that the L^2-gradient flow is $X_t=-(\Delta H+\Phi H)\,e_3$, so called the Willmore flow.

When we follow from the work of Droske and Rumpf [DR], they defined the viscosity solution for the mean curvature flow, the level set formulation for the Willmore flow is given by

$$u_t=2\,(\delta_{ij}-\frac{u_iu_j}{|\nabla u|^2})(\delta_{kl}-\frac{u_ku_l}{|\nabla u|^2})u_{ijkl}+lower\ order\ terms.$$

In this case, we can not define the viscosity solution because there has no maximum principle. The first work in this report is to find a way of defining the viscosity solution. In section 2, we find a level set formulation of the Willmore flow by introduced the weighted mean curvature, and define such a weak solution. In section 3, we observe two explicit examples which decrease the Willmore energy in a natural way, but they only satisfy the generalized Willmore flow. We try to give a possible geometric interpretation for these examples. There is a corresponding level set formulation, similar to section 2, if we can show that this interpretation works.

2. Definition of Weak solution

In this section we present briefly the procedure for defining a weak solution to the Willmore flow. We start with a formal derivation of a level set formulation for the Willmore flow $X_t = -(\Delta H + \Phi H) e_3$. The first three steps of this derivation follows essentially from the work of Droske and Rumpf [DR], but we use different notations.

Step 1. Suppose that $\phi = \phi(x,t)$ is a smooth function defined in $\Omega \subset R^3$ whose gradient does not vanish, and each level set of ϕ smoothly evolves according to the Willmore flow. Then $\phi_t = -|\tilde{\nabla}\phi| (\Delta H + \Phi H)$, where $\tilde{\nabla}$ is the gradient operator of R^3. Let θ be a smooth function with compact support in Ω. Then

$$\int_\Omega \frac{\phi_t}{|\tilde{\nabla}\phi|}\theta = -\int_\Omega (\Delta H + \Phi H)\,\theta.$$

Step 2. Substituting the following equations

$$\Delta H = \tilde{\Delta} H - \frac{\tilde{\phi}_A}{|\tilde{\nabla}\phi|}\frac{\tilde{\phi}_B}{|\tilde{\nabla}\phi|}\tilde{H}_{AB} + \frac{\tilde{\phi}_A}{|\tilde{\nabla}\phi|}\tilde{H}_A H,$$

$$|\tilde{\nabla}\phi| {h_{ij}}^2 = {\tilde{\phi}_{AB}}^2 - \frac{2}{|\tilde{\nabla}\phi|^2}(\tilde{\phi}_A\tilde{\phi}_{AB})^2 + \frac{1}{|\tilde{\nabla}\phi|^4}(\tilde{\phi}_A\tilde{\phi}_B\tilde{\phi}_{AB})$$

into the last equation of step 1 and applying the divergence formula we have

$$\int_\Omega \frac{\phi_t}{|\tilde{\nabla}\phi|}\theta = \int_\Omega (\frac{1}{|\tilde{\nabla}\phi|}P_{AB}(|\tilde{\nabla}\phi|H)_B + \frac{1}{2|\tilde{\nabla}\phi|^3}(|\tilde{\nabla}\phi|H)^2\tilde{\phi}_A)\,\tilde{\theta}_A,$$

where

$$1 \le i, j, k \le 2,\ 1 \le A, B, C \le 3,$$

$$P_{AB} = \delta_{AB} - \frac{\tilde{\phi}_A}{|\tilde{\nabla}\phi|}\frac{\tilde{\phi}_B}{|\tilde{\nabla}\phi|}.$$

Step3. Let $w = |\tilde{\nabla}\phi| H$ be the weighted mean curvature and ψ a smooth function with compact support in Ω. Then the level set formulation of Willmore flow in integral form is given by

Given ϕ_0 in Ω, find $(\phi(x,t), w(x,t))$ with $\phi(0) = \phi_0$ such that

$$\int_\Omega \frac{\phi_t}{|\tilde{\nabla}\phi|}\theta = \int_\Omega (\frac{1}{|\tilde{\nabla}\phi|}P_{AB}\tilde{w}_B + \frac{1}{2|\tilde{\nabla}\phi|^3}w^2\tilde{\phi}_A)\,\tilde{\theta}_A$$

and

$$\int_\Omega \frac{w}{|\tilde{\nabla}\phi|}\psi = \int_\Omega (\frac{1}{|\tilde{\nabla}\phi|}\tilde{\nabla}\phi\cdot\tilde{\nabla}\psi)$$

for all $\theta,\ \psi \in C_c^\infty(\Omega),\ t > 0$.

Step 4. To find the level set formulation of Willmore flow in differential form, we need

$$(\frac{\tilde{\phi}_A}{|\tilde{\nabla}\phi|})_B = \frac{1}{|\tilde{\nabla}\phi|}P_{AC}\tilde{\phi}_{CB},$$

$$(P_{AB})_C = -\frac{1}{|\tilde{\nabla}\phi|}(P_{AD}\tilde{\phi}_{DC}\frac{\tilde{\phi}_B}{|\tilde{\nabla}\phi|} + P_{BD}\tilde{\phi}_{DC}\frac{\tilde{\phi}_A}{|\tilde{\nabla}\phi|}),$$

$$(\frac{w}{|\tilde{\nabla}\phi|})A = \frac{1}{|\tilde{\nabla}\phi|}(w_A - \tilde{\phi}_{BA}\frac{\tilde{\phi}_B}{|\tilde{\nabla}\phi|}\frac{w}{|\tilde{\nabla}\phi|}).$$

It follows from the divergence formula that
$w = -P_{AB}\tilde{\phi}_{AB}$,

$$\phi_t = -P_{AB}\tilde{w}_{AB} + 2\,(P_{AD}\tilde{\phi}_{DB} + P_{BD}\tilde{\phi}_{DA} - w\delta_{AB})\frac{\tilde{\phi}_A}{|\tilde{\nabla}\phi|}\frac{\tilde{w}_B}{|\tilde{\nabla}\phi|}$$
$$+ (\frac{w}{|\tilde{\nabla}\phi|})^2(\tilde{\phi}_{AB}\frac{\tilde{\phi}_A}{|\tilde{\nabla}\phi|}\frac{\tilde{\phi}_B}{|\tilde{\nabla}\phi|} + \frac{1}{2}w).$$

Since the second term of this equation can not compare by the maximum principle, we need modify it when define weak solution. From now on, for simplification, all derivatives are in the sense of R^3, and $1 \le i, j, k \le 3$. For given functions

$$\phi, w \in C(R^3 \times [0,\infty)) \cap L^\infty (R^3 \times [0,\infty)),$$
$$\overline{\phi}, \hat{\phi}, \overline{w}, \hat{w} \in C^\infty (R^3 \times [0,\infty)),$$
$$\textit{If} \ \text{at} \ (x_0, t_0) \in R^3 \times [0,\infty),$$
$$\phi - \hat{\phi} \text{ has a local maximum}, \ \phi(x_0, t_0) = \hat{\phi}(x_0, t_0),$$
$$\phi - \overline{\phi} \text{ has a local minimum}, \ \phi(x_0, t_0) = \overline{\phi}(x_0, t_0),$$
$$w - \hat{w} \text{ has a local maximum}, \ \hat{w}(x_0, t_0) = -P_{ij}\overline{\phi}_{ij}(x_0, t_0),$$
$$\text{if } |\nabla\phi| \ne 0; \ \hat{w}(x_0, t_0) = -(\delta_{ij} - \xi_i \xi_j)\overline{\phi}_{ij}(x_0, t_0)$$
$$\text{for some } \xi \in R^3, |\xi| = 1, \text{if } |\nabla\phi|(x_0, t_0) = 0,$$
$$w - \overline{w} \text{ has a local minimum}, \ \overline{w}(x_0, t_0) = -P_{ij}\hat{\phi}_{ij}(x_0, t_0),$$
$$\text{if } |\nabla\phi| \ne 0; \ \overline{w}(x_0, t_0) = -(\delta_{ij} - \xi_i \xi_j)\hat{\phi}_{ij}(x_0, t_0)$$
$$\text{for some } \xi \in R^3, |\xi| = 1, \text{if } |\nabla\phi|(x_0, t_0) = 0,$$

Here we use notations

$$\phi_i(x_0, t_0) := \hat{\phi}_i(x_0, t_0) = \overline{\phi}_i(x_0, t_0),$$
$$|\nabla\phi|(x_0, t_0) := |\nabla\hat{\phi}|(x_0, t_0) = |\nabla\overline{\phi}|(x_0, t_0),$$
$$w_i(x_0, t_0) := \hat{w}_i(x_0, t_0) = \overline{w}_i(x_0, t_0).$$

Since

$$\phi - \hat{\phi}, \ w - \hat{w} \ \text{ has a local maximum at } (x_0, t_0),$$
$$\phi - \overline{\phi}, \ w - \overline{w} \text{ has a local minimumat } (x_0, t_0),$$

the above notations make sense.

Now we define our weak supersolution in terms of pointwise behavior as follows:

$$(\phi, w) \text{ is a weak supersolution if at } (x_0, t_0)$$
$$\phi_t \ge -P_{ij}\overline{w}_{ij} - \overline{\phi}_{ij}\frac{w_i}{|\nabla\phi|}\frac{w_j}{|\nabla\phi|} - \overline{\phi}_{ij}\frac{\phi_i}{|\nabla\phi|}\frac{\phi_j}{|\nabla\phi|}\left(\frac{\phi_k}{|\nabla\phi|} + \frac{w_k}{|\nabla\phi|}\right)^2$$
$$-\overline{w}\left(\frac{\phi_k}{|\nabla\phi|} + \frac{w_k}{|\nabla\phi|}\right)^2 + \hat{\phi}_{ij}\left(\frac{\phi_i}{|\nabla\phi|} + \frac{w_i}{|\nabla\phi|}\right)\left(\frac{\phi_j}{|\nabla\phi|} + \frac{w_j}{|\nabla\phi|}\right)$$
$$+\hat{\phi}_{ij}\frac{\phi_i}{|\nabla\phi|}\frac{\phi_j}{|\nabla\phi|}\left(\frac{w_k}{|\nabla\phi|}\right)^2 + \hat{w}\left(1 + \left(\frac{w_k}{|\nabla\phi|}\right)^2\right)$$
$$+\left(\frac{\hat{w}}{|\nabla\phi|}\right)^2\left(\hat{\phi}_{ij}\frac{\phi_i}{|\nabla\phi|}\frac{\phi_j}{|\nabla\phi|} + \frac{1}{2}\hat{w}\right) \text{ if } |\nabla\phi|(x_0, t_0) \ne 0;$$

$$\phi_t \ge -(\delta_{ij} - \xi_i\xi_j)\overline{w}_{ij} - \overline{\phi}_{ij}\eta_i\eta_j - \overline{\phi}_{ij}\xi_i\xi_j(\xi_k + \eta_k)^2$$
$$-\overline{w}(\xi_k + \eta_k)^2 + \hat{\phi}_{ij}(\xi_i + \eta_i)(\xi_j + \eta_j)$$
$$+\hat{\phi}_{ij}\xi_i\xi_j\eta_k^2 + \hat{w}(1 + \eta_k^2)$$
$$+c^2\left(\hat{\phi}_{ij}\xi_i\xi_j + \frac{1}{2}\hat{w}\right) \text{ for some } \eta \in R^3, c \in R.$$
$$\text{if } |\nabla\phi|(x_0, t_0) = 0.$$

We can define weak subsolution in the same way, and a weak solution is both weak supersolution and a weak subsolution. This formulation of weak solution is based on the idea of the mean curvature flow defined by Evans and Spruck [ES]. The main difficult for obtain the existence of weak solution will be that there are two distinct functions in this formulation..

3. Generalized Willmore Flow

There are explicit examples which show that for decreasing the Willmore energy, there are more natural way, called the generalized Willmore flow

$$X_t = -\rho^{-2}(\Delta H + \Phi H)e_3 + \text{tangential component}.$$

The first example is the torus

$$X = ((1 + r\cos\theta)\cos\phi, (1 + r\cos\theta)\sin\phi, 1 + r\sin\theta).$$

where $r = r(t)$, $0 < r_0 = r(0) < 1$. We find that X satisfied the generalized Willmore flow

$$X_t = -\rho^{-2}(\Delta H + \Phi H)e_3 + \text{tangential component},$$

where $\rho^2 = r^3(1 + r\cos\theta)^3$. In fact, $r(t)=r_0$ if $r_0 = \frac{1}{\sqrt{2}}$; $r(t) = \frac{1}{\sqrt{2}}\frac{(1+\sqrt{2}r_0)e^{2\sqrt{2}t} + \sqrt{2}r_0 - 1}{(1+\sqrt{2}r_0)e^{2\sqrt{2}t} - \sqrt{2}r_0 + 1}$ if $r_0 \ne \frac{1}{\sqrt{2}}$. We note that X converges to the Clifford torus as t tends to infinity.

The second example is the ellipsoid

$$X = (a\cos\theta\cos\phi, a\cos\theta\sin\phi, b\sin\theta),$$

where $a = a(t)$, $b = b(t)$. We find that X satisfied the generalized Willmore flow

$$X_t = -\rho^{-2}(\Delta H + \Phi H)e_3 + \text{tangential component},$$

where $\rho^2 = \frac{-A(c)\cos^4\theta - B(c)\cos^2\theta - C(c)}{a^4(1 + (c^2 - 1)\cos^2\theta)^4}$,

c=b/a, A, B and C are functions of c. In fact, $a(t) = b(t) = a(0) = b(0)$ if $a(0) = b(0)$;

$c(t)=\frac{pe^{2t}}{pe^{2t}\pm 1}$ if $c_0\neq 1$. We note that X converges to the round sphere as t tends to infinity.

We want to find a possible interpretation for these explicit examples. For a solution of the Willmore flow $x: M\rightarrow R^3$ and a family of conformal transformations depending on t, $T: R^3\times[0,\infty)\rightarrow R^3$, there are two distinct induced metrics on R^3. Let ds_0^2 be the usual metric of R^3 and $d\bar{s}^2=(T^{-1})^* ds_0^2 = e^{2u} ds_0^2$ for fixed t, where u is a smooth function defined on $R^3\times[0,\infty)$. Since the Willmore energy $W(\bar{X})$ is invariant under conformal transformations of R^3,

$$(\int_M \bar{\Phi}\}_t = (\int_M \Phi\}_t$$

$$= \int_M (\Delta H+\Phi H)X_t\cdot e_3\ \omega_1\Lambda\omega_2$$

$$= \int_M (\Delta H+\Phi H)X_t\cdot e_3\ e^{-2u}\ \bar{\omega}_1\Lambda\bar{\omega}_2,$$

where $\bar{X}=T\circ X$. In this sense, the equation of Willmore flow will be

$\bar{X}_t=-e^{-2u}(\Delta H+\Phi H)\,e_3+\text{tangential component}.$

It is expected that the tangential component comes from diffeomorphisms of M, and the scaling factor comes from the conformal transformation T. For finding the origin solution X of the Willmore flow, let $X=T^{-1}\circ\bar{X}$. Therefore we need to solve the solution in a natural way, that is, finding the solution of a generalized Willmore flow, finding the associated diffeomorphisms and conformal transformations. According to the classical Liouville's theorem, a conformal transformation of R^3 is the composition of a motion a homothety and an inverse, we need finding conformal transformations in the conformal group. On the other hand, isometric diffeomorphisms depends on M. Following from the procedure of section 2, we can obtain the corresponding equation for the generalized Willmore flow in the level set form.

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