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中文摘要：

在這篇研究論文裡我們發展一套有系統的方法來研究高維度網格模型花樣生成問題。我們將定義 ordering matrices 藉由此矩陣來導出一個遞迴公式以便造成更大尺寸的 ordering matrices。給定一個可允許的局部花樣子集，相對應 ordering matrices 可定義出 transition matrices。我們的目的希望計算一個系統的複雜度，也就是計算它的熵，而藉由計算高尺寸的 transition matrices 的最大的特徵值，可讓我們得到此結果。這篇研究報告的成果可應用到網格動態系統和類神經網路的穩定解問題上。

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PATTERNS GENERATION AND TRANSITION MATRICES IN MULTI-DIMENSIONAL LATTICE MODELS

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Abstract. In this paper we develop a general approach for investigating pattern generation problems in multi-dimensional lattice models. Let $\mathcal{S}$ be a set of p symbols or colors, $\mathbf{Z}_N$ a fixed finite rectangular sublattice of $\mathbf{Z}^d$, $d \geq 1$ and N a d-tuple of positive integers. Functions $U : \mathbf{Z}^d \to \mathcal{S}$ and $U_N : \mathbf{Z}_N \to \mathcal{S}$ are called a global pattern and a local pattern on $\mathbf{Z}_N$, respectively. We introduce an ordering matrix $\mathbf{X}_N$ for Σ_N, the set of all local patterns on $\mathbf{Z}_N$. For a larger finite lattice $\mathbf{Z}_{\tilde{N}}$, $\tilde{N} \geq N$, we derive a recursion formula to obtain the ordering matrix $\mathbf{X}_{\tilde{N}}$ of $\Sigma_{\tilde{N}}$ from $\mathbf{X}_N$. For a given basic admissible local patterns set $\mathcal{B} \subset \Sigma_N$, the transition matrix $\mathbf{T}_N(\mathcal{B})$ is defined. For each $\tilde{N} \geq N$ denoted by $\Sigma_{\tilde{N}}(\mathcal{B})$ the set of all local patterns which can be generated from $\mathcal{B}$, the cardinal number of $\Sigma_{\tilde{N}}(\mathcal{B})$ is the sum of entries of the transition matrix $\mathbf{T}_{\tilde{N}}(\mathcal{B})$ which can be obtained from $\mathbf{T}_N(\mathcal{B})$ recursively. The spatial entropy $h(\mathcal{B})$ can be obtained by computing the maximum eigenvalues of a sequence of transition matrices $\mathbf{T}_n(\mathcal{B})$. The results can be applied to study the set of global stationary solutions in various Lattice Dynamical Systems and Cellular Neural Networks.

1. **Introduction.**

Many systems have been studied as models for spatial pattern formation in biology, chemistry, engineering and physics. Lattices play important roles in modeling underlying spatial structures. Notable examples include models arising from biology [7, 8, 21, 22, 23, 33, 34, 35], chemical reaction and phase transitions [4, 5, 11, 12, 13, 14, 24, 41, 43], image processing and pattern recognition [11, 12, 15, 16, 17, 18, 19, 25, 40], as well as materials science[9, 20, 26]. Stationary patterns play a critical role in investigating of the long time behavior of related dynamical systems. In general, multiple stationary patterns may induce complicated phenomena of such systems.

In Lattice Dynamical Systems(LDS), especially Cellular Neural Networks (CNN), the set of global stationary solutions (global patterns) has received considerable attention in recent years (e.g.[1, 2, 6, 10, 27, 28, 29, 30, 31, 32, 36, 37]). When the

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mutual interaction between states of a system is local, the state at each lattice point is influenced only by its finitely many neighborhood states. The admissible (or allowable) local patterns are introduced and defined on a certain finite lattice. The admissible global patterns on the entire lattice space are then glued together from those admissible local patterns. More precisely, let $\mathcal{S}$ be a finite set of p elements (symbols, colors or letters of an alphabet) where $\mathbf{Z}^d$ denotes the integer lattice on $\mathbf{R}^d$, and $d \geq 1$ is a positive integer representing the lattice dimension. Then, function $U : \mathbf{Z}^d \to \mathcal{S}$ is called a global pattern. For each $\alpha \in \mathbf{Z}^d$, we write $U(\alpha)$ as u_α. The set of all patterns $U : \mathbf{Z}^d \to \mathcal{S}$ is denoted by

$$\Sigma_p^d \equiv \mathcal{S}^{\mathbf{Z}^d},$$

i.e., Σ_p^d is the set of all patterns with p different colors in d-dimensional lattice. As for local patterns, i.e., functions defined on (finite) sublattices, for a given d-tuple $N = (N_1, N_2, \cdots, N_d)$ of positive integers, let

$$\mathbf{Z}_N = \{(\alpha_1, \alpha_2, \cdots, \alpha_d) : 1 \leq \alpha_k \leq N_k, 1 \leq k \leq d\}$$

be an $N_1 \times N_2 \times \cdots N_d$ finite rectangular lattice. Denoted by $\widetilde{N} \geq N$ if $\widetilde{N_k} \geq N_k$ for all $1 \leq k \leq d$. The set of all local patterns defined on $\mathbf{Z}_N$ is denoted by

$$\Sigma_N \equiv \Sigma_{N,p} \equiv \{U|_{\mathbf{Z}_N} : U \in \Sigma_p^d\}.$$

Under many circumstances, only a (proper) subset $\mathcal{B}$ of Σ_N is admissible (allowable or feasible). In this case, local patterns in $\mathcal{B}$ are called basic patterns and $\mathcal{B}$ is called the basic set. In a one dimensional case, $\mathcal{S}$ consists of letters of an alphabet, and $\mathcal{B}$ is also called a set of allowable words of length N.

Consider a fixed finite lattice $\mathbf{Z}_N$ and a given basic set $\mathcal{B} \subset \Sigma_N$. For larger finite lattice $\mathbf{Z}_{\widetilde{N}} \supset \mathbf{Z}_N$, the set of all local patterns on $\mathbf{Z}_{\widetilde{N}}$ which can be generated by $\mathcal{B}$ is denoted as $\Sigma_{\widetilde{N}}(\mathcal{B})$. Indeed, $\Sigma_{\widetilde{N}}(\mathcal{B})$ can be characterized by

$$\begin{aligned} \Sigma_{\widetilde{N}}(\mathcal{B}) \quad = \{ \quad & U \in \Sigma_{\widetilde{N}} : U_{\alpha+N} = V_N \ for\ any\ \alpha \in \mathbf{Z}^d\ with\ \mathbf{Z}_{\alpha+N} \subset \mathbf{Z}_{\widetilde{N}} \\ & and\ some\ V_N \in \mathcal{B}\}, \end{aligned}$$

where

$$\alpha + N = \{(\alpha_1 + \beta_1, \cdots, \alpha_d + \beta_d) : (\beta_1, \cdots, \beta_d) \in N\},$$

and

$$U_{\alpha+N} = V_N\ means\ u_{\alpha+\beta} = v_\beta\ for\ each\ \beta \in \mathbf{Z}_N.$$

Similarly, the set of all global patterns which can be generated by $\mathcal{B}$ is denoted by

$$\Sigma(\mathcal{B}) = \{U \in \Sigma_p^d : U_{\alpha+N} = V_N\ for\ any\ \alpha \in \mathbf{Z}^d\ with\ some\ V_N \in \mathcal{B}\}.$$

The following questions arise :

(1) Can we find a systematic means of constructing $\Sigma_{\widetilde{N}}(\mathcal{B})$ from $\mathcal{B}$ for $\mathbf{Z}_{\widetilde{N}} \supset \mathbf{Z}_N$?
(2) What is the complexity (or spatial entropy) of $\{\sum_{\widetilde{N}}(\mathcal{B})\}_{\widetilde{N} \geq N}$?

The spatial entropy $h(\mathcal{B})$ of $\Sigma(\mathcal{B})$ is defined as follows :
Let

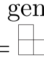
$$\Gamma_{\widetilde{N}}(\mathcal{B}) = card(\Sigma_{\widetilde{N}}(\mathcal{B})), \tag{1.1}$$

the number of distinct patterns in $\Sigma_{\widetilde{N}}(\mathcal{B})$. The spatial entropy $h(\mathcal{B})$ is defined as

$$h(\mathcal{B}) = \lim_{\widetilde{N} \to \infty} \frac{1}{\widetilde{N_1} \cdots \widetilde{N_d}} \log \Gamma_{\widetilde{N}}(\mathcal{B}), \tag{1.2}$$

where $\widetilde{N} = (\widetilde{N_1}, \widetilde{N_2}, ..., \widetilde{N_d})$ be a d-tuple positive integers, which is well-defined and exists (e.g. [13]). The spatial entropy, which is an analogue to topological entropy in dynamical system, has been used to measure a kind of complexity in LDS (e.g. [13], [42]).

In a one dimensional case, the above two questions can be answered by using transition matrix. Indeed, for a given basic set $\mathcal{B}$, we can associate the transition matrix $\mathbf{T}(\mathcal{B})$ to $\mathcal{B}$. Then the spatial entropy $h(\mathcal{B}) = \log \lambda$, where λ is the largest eigenvalue of $\mathbf{T}(\mathcal{B})$ (e.g. [29, 41]). On the other hand, for higher dimensional cases, constructing $\Sigma_{\widetilde{N}}(\mathcal{B})$ systematically and computing $\Gamma_{\widetilde{N}}(\mathcal{B})$ effectively for a large $\widetilde{N}$ are extremely difficult.

In the two dimensional case, Chow et al. [13] estimated lower bounds of the spatial entropy for some problems in LDS. Later, using a "building block" technique, Juang and Lin [29] studied the patterns generation and obtained lower bounds of the spatial entropy for CNN with square-cross or diagonal-cross templates. For CNN with general templates, Hsu et al [27] investigated the generation of admissible local patterns and obtained the basic set for any parameter, i.e., the first step in studying the patterns generation problem. Meanwhile, given a set of symbols $\mathcal{S}$ and a pair consisting of a horizontal transition matrix H and a vertical transition matrix V, Juang et al [30] defined m-th order transition matrices $T^{(m)}_{H,V}$ and $\bar{T}^{(m)}_{H,V}$ for each $m \geq 1$ and, in doing so, obtained the recursion formulas for both $T^{(m)}_{H,V}$ and $\bar{T}^{(m)}_{H,V}$. Furthermore, they proved that $T^{(m)}_{H,V}$ and $\bar{T}^{(m)}_{H,V}$ have the same maximum eigenvalue λ_m and spatial entropy $h(H,V) = \lim_{m\to\infty} \frac{\log \lambda_m}{m}$. For a certain class of H,V, the recursion formulas for $T^{(m)}_{H,V}$ and $\bar{T}^{(m)}_{H,V}$ yield recursion formulas for λ_m explicitly and the exact entropy. On the other hand, for the patterns generation problem Lin and Yang [37] worked on the 3-cell L-shaped lattice, i.e., N= . They developed an algorithm to investigate how patterns are generated on larger lattices from smaller one. Their algorithm treated all patterns in $\Sigma_{\widetilde{N}}(\mathcal{B})$ as entries and arranged them in a "counting matrix" $M_{\widetilde{N}}(\mathcal{B})$. A good arrangement of $M_{\widetilde{N}}(\mathcal{B})$ implies an easier extension to $M_{\widetilde{\widetilde{N}}}(\mathcal{B})$ for a larger lattice $\widetilde{\widetilde{N}} \supset \widetilde{N}$ and effective counting of the number of elements in $\Sigma_{\widetilde{N}}(\mathcal{B})$. Upper and lower bounds of spatial entropy were also obtained. Next, there are some relations with matrix shift [13], that details will appear in section 3.4.

Motivated by the counting matrix $M_N(\mathcal{B})$ of [37] and the recursion formulas for transition matrices in [30], this work introduces the "ordering matrix" $\mathbf{X}_2$ for $\Sigma_{2\ell\times 2\ell}$ to study the patterns generation and obtain recursion formulas for $\mathbf{X}_n$ for $\Sigma_{2\ell\times n\ell}$ where $\ell \geq 1$ is a fixed positive integer and $n \geq 2$. The recursion formulas for $\mathbf{X}_n$ imply the recursion formula for the associated transition matrices $\mathbf{T}_n(\mathcal{B})$ of $\Sigma_{2\ell\times n\ell}(\mathcal{B})$, i.e., a generalization of the recursion formulas in [30]. Notably, a different ordering matrix $\widetilde{\mathbf{X}}_2$ for $\Sigma_{2\ell\times 2\ell}$ induces different recursion formulas of $\widetilde{\mathbf{X}}_n$ for $\Sigma_{2\ell\times n\ell}$ and $\widetilde{\mathbf{T}}_n(\mathcal{B})$. Among them, $\mathbf{X}_2$ defined in (2.9) yields a simple recursion formula (3.16) and rewriting rule (3.14), which enabling us to compute the maximum eigenvalue of $\mathbf{T}_n$ effectively. The computations or estimates of λ_n are interesting problems in linear algebra and numerical linear algebra. Owing to the similarity property of (3.16) or (3.14) of transition matrices $\{\mathbf{T}_n\}_{n=2}^{\infty}$, we show that for a certain class of $\mathcal{B}$, λ_n satisfies certain recursion relations and $h(\mathcal{B})$ can be computed explicitly.

In $d \geq 3$, the structure of ordering matrix and transition matrices has been explored, and it can be found in [3].

The rest of this paper is organized as follows. Section 2 describes a two dimensional case by thoroughly investigating $\Sigma_{2\times 2}$ and introducing the ordering matrix $\mathbf{X}_2$ of patterns in $\Sigma_{2\times 2}$. The ordering matrix $\mathbf{X}_n$ on $\Sigma_{2\times n}$ is then constructed from $\mathbf{X}_2$ recursively. Finally, section 3 derives higher order transition matrices $\mathbf{T}_n$ from $\mathbf{T}_2$ and computes λ_n explicitly for a certain type of $\mathbf{T}_2$.

2. **Two Dimensional Patterns.** This section describes two dimensional patterns generation. For clarity, we begin by the studying two symbols, i.e., $\mathcal{S} = \{0, 1\}$. On a fixed finite lattice $\mathbf{Z}_{m_1\times m_2}$, we first give a ordering $\chi = \chi_{m_1\times m_2}$ on $\mathbf{Z}_{m_1\times m_2}$ by

$$\chi((\alpha_1, \alpha_2)) = m_2(\alpha_1 - 1) + \alpha_2 \ , \tag{2.1}$$

i.e.,

m_2	$2m_2$		m_1m_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	m_2+1		$(m_1-1)m_2+1$

. (2.2)

The ordering χ of (2.1) on $\mathbf{Z}_{m_1\times m_2}$ can now be passed to $\Sigma_{m_1\times m_2}$. Indeed, for each $U = (u_{\alpha_1,\alpha_2}) \in \Sigma_{m_1\times m_2}$, define

$$\begin{aligned} \chi(U) \ &\equiv \ \chi_{m_1\times m_2}(U) \\ &= \ 1 + \sum_{\alpha_1=1}^{m_1} \sum_{\alpha_2=1}^{m_2} u_{\alpha_1\alpha_2} 2^{m_2(m_1-\alpha_1)+(m_2-\alpha_2)}. \end{aligned} \tag{2.3}$$

Obviously, there is an one-to-one correspondence between local patterns in $\Sigma_{m_1\times m_2}$ and positive integers in the set $\mathbf{N}_{2^{m_1m_2}} = \{k \in \mathbf{N} : 1 \leq k \leq 2^{m_1,m_2}\}$, where $\mathbf{N}$ is the set of positive integers. Therefore, U is referred to herein as the $\chi(U)$-th element in $\Sigma_{m_1\times m_2}$. By identifying the pictorial patterns by numbers $\chi(U)$, it becomes highly effective in proving theorems since computations can now be performed on $\chi(U)$. In a two dimensional case, we will keep the ordering (2.1)$\sim$ (2.3) χ on $\mathbf{Z}_{m_1\times m_2}$ and $\Sigma_{m_1\times m_2}$, respectively.

2.1. **Ordering Matrices.** For $1 \times n$ pattern $U = (u_k), 1 \leq k \leq n$ in $\Sigma_{1\times n}$, as in (2.3), U is assigned the number

$$i = \chi(U) = 1 + \sum_{k=1}^{n} u_k 2^{(n-k)}. \tag{2.4}$$

As denoted by the $1 \times n$ column pattern $x_{n;i}$,

$$x_{n;i} = \begin{bmatrix} u_n \\ \vdots \\ u_1 \end{bmatrix} \quad or \quad \boxed{\begin{array}{c} u_n \\ \hline \vdots \\ \hline u_1 \end{array}} \ . \tag{2.5}$$

In particular, when $n = 2$, as denoted by $x_i = x_{2;i}$,

$$i = 1 + 2u_1 + u_2$$

and

$$x_i = \begin{bmatrix} u_2 \\ u_1 \end{bmatrix} \quad or \quad \boxed{\begin{array}{c} u_2 \\ \hline u_1 \end{array}} \ . \tag{2.6}$$

A 2×2 pattern $U = (u_{\alpha_1 \alpha_2})$ can now be obtained by a horizontal direct sum of two 1×2 patterns, i.e.,

$$\begin{aligned} x_{i_1 i_2} &\equiv x_{i_1} \oplus x_{i_2} \\ &\equiv \begin{bmatrix} u_{12} & u_{22} \\ u_{11} & u_{21} \end{bmatrix} \quad or \quad \begin{array}{|c|c|}\hline u_{12} & u_{22} \\ \hline u_{11} & u_{21} \\ \hline \end{array}\ , \end{aligned} \tag{2.7}$$

where

$$i_k = 1 + 2u_{k1} + u_{k2}, \quad 1 \le k \le 2. \tag{2.8}$$

Therefore, the complete set of all $16 (= 2^{2\times 2})$ 2×2 patterns in $\Sigma_{2\times 2}$ can be listed by a 4×4 matrix $\mathbf{X}_2 = [x_{i_1 i_2}]$ with 2×2 pattern $x_{i_1 i_2}$ as its entries in

$$\begin{array}{c} \\ \begin{smallmatrix}0\\0\end{smallmatrix} \\ \\ \begin{smallmatrix}1\\0\end{smallmatrix} \\ \\ \begin{smallmatrix}0\\1\end{smallmatrix} \\ \\ \begin{smallmatrix}1\\1\end{smallmatrix} \end{array}
\begin{array}{c} \begin{array}{cccc} \begin{smallmatrix}0\\0\end{smallmatrix} & \begin{smallmatrix}1\\0\end{smallmatrix} & \begin{smallmatrix}0\\1\end{smallmatrix} & \begin{smallmatrix}1\\1\end{smallmatrix} \end{array} \\
\begin{pmatrix} \begin{smallmatrix}0&0\\0&0\end{smallmatrix} & \begin{smallmatrix}0&1\\0&0\end{smallmatrix} & \begin{smallmatrix}0&0\\0&1\end{smallmatrix} & \begin{smallmatrix}0&1\\0&1\end{smallmatrix} \\ \\
\begin{smallmatrix}1&0\\0&0\end{smallmatrix} & \begin{smallmatrix}1&1\\0&0\end{smallmatrix} & \begin{smallmatrix}1&0\\0&1\end{smallmatrix} & \begin{smallmatrix}1&1\\0&1\end{smallmatrix} \\ \\
\begin{smallmatrix}0&0\\1&0\end{smallmatrix} & \begin{smallmatrix}0&1\\1&0\end{smallmatrix} & \begin{smallmatrix}0&0\\1&1\end{smallmatrix} & \begin{smallmatrix}0&1\\1&1\end{smallmatrix} \\ \\
\begin{smallmatrix}1&0\\1&0\end{smallmatrix} & \begin{smallmatrix}1&1\\1&0\end{smallmatrix} & \begin{smallmatrix}1&0\\1&1\end{smallmatrix} & \begin{smallmatrix}1&1\\1&1\end{smallmatrix} \end{pmatrix} \end{array} \tag{2.9}$$

It is easy to verify that

$$\chi(x_{i_1 i_2}) = 4(i_1 - 1) + i_2, \tag{2.10}$$

i.e, we are counting local patterns in $\Sigma_{2\times 2}$ by going through each row successively in Table (2.9). Correspondingly, $\mathbf{X}_2$ can be referred to as an ordering matrix for $\Sigma_{2\times 2}$. Similarly, a 2×2 pattern can also be viewed as a vertical direct sum of two 2×1 patterns, i.e,

$$y_{j_1 j_2} = y_{j_1} \oplus y_{j_2}, \tag{2.11}$$

where

$$y_{j_l} = \begin{bmatrix} u_{1l} & u_{2l} \end{bmatrix} \quad or \quad \begin{array}{|c|c|}\hline u_{1l} & u_{2l} \\ \hline \end{array}\ ,$$

and

$$j_l = 1 + 2u_{1l} + u_{2l}, \tag{2.12}$$

$1 \le l \le 2$. A 4×4 matrix $\mathbf{Y}_2 = [y_{j_1 j_2}]$ can also be obtained for $\Sigma_{2\times 2}$. i.e., we have

$$\begin{array}{c} \\ \begin{smallmatrix}0&0\end{smallmatrix} \\ \\ \begin{smallmatrix}0&1\end{smallmatrix} \\ \\ \begin{smallmatrix}1&0\end{smallmatrix} \\ \\ \begin{smallmatrix}1&1\end{smallmatrix} \end{array}
\begin{array}{c} \begin{array}{cccc} \begin{smallmatrix}0&0\end{smallmatrix} & \begin{smallmatrix}0&1\end{smallmatrix} & \begin{smallmatrix}1&0\end{smallmatrix} & \begin{smallmatrix}1&1\end{smallmatrix} \end{array} \\
\begin{pmatrix} \begin{smallmatrix}0&0\\0&0\end{smallmatrix} & \begin{smallmatrix}0&1\\0&0\end{smallmatrix} & \begin{smallmatrix}1&0\\0&0\end{smallmatrix} & \begin{smallmatrix}1&1\\0&0\end{smallmatrix} \\ \\
\begin{smallmatrix}0&0\\0&1\end{smallmatrix} & \begin{smallmatrix}0&1\\0&1\end{smallmatrix} & \begin{smallmatrix}1&0\\0&1\end{smallmatrix} & \begin{smallmatrix}1&1\\0&1\end{smallmatrix} \\ \\
\begin{smallmatrix}0&0\\1&0\end{smallmatrix} & \begin{smallmatrix}0&1\\1&0\end{smallmatrix} & \begin{smallmatrix}1&0\\1&0\end{smallmatrix} & \begin{smallmatrix}1&1\\1&0\end{smallmatrix} \\ \\
\begin{smallmatrix}0&0\\1&1\end{smallmatrix} & \begin{smallmatrix}0&1\\1&1\end{smallmatrix} & \begin{smallmatrix}1&0\\1&1\end{smallmatrix} & \begin{smallmatrix}1&1\\1&1\end{smallmatrix} \end{pmatrix} \end{array} \tag{2.13}$$

The relation between $\mathbf{X}_2$ and $\mathbf{Y}_2$ must be explored. Indeed, from (2.12), u_{kl} can be solved in terms of j_l, i.e., we have

$$u_{1l} = [\frac{j_l - 1}{2}] \tag{2.14}$$

and

$$u_{2l} = j_l - 1 - 2[\frac{j_l - 1}{2}], \tag{2.15}$$

where [ ] is the Gauss symbol, i.e., $[r]$ is the largest integer which is equal to or less than r. From (2.8), (2.12), (2.14) and (2.15), we have the following relations between indices i_1, i_2 and j_1, j_2.

$$j_1 = 1 + \sum_{k=1}^{2} [\frac{i_k - 1}{2}]\ 2^{2-k}, \tag{2.16}$$

$$j_2 = 1 + \sum_{k=1}^{2} \{\ i_k - 1 - 2\ [\frac{i_k - 1}{2}]\ \}\ 2^{2-k}, \tag{2.17}$$

and

$$i_1 = 1 + \sum_{l=1}^{2} [\frac{j_l - 1}{2}]\ 2^{2-l}, \tag{2.18}$$

$$i_2 = 1 + \sum_{l=1}^{2} \{\ j_l - 1 - 2\ [\frac{j_l - 1}{2}]\ \}\ 2^{2-l}. \tag{2.19}$$

From (2.16) and (2.17), (2.9) or $\mathbf{X}_2$ can also be represented by $y_{j_1 j_2}$ as

$$\mathbf{X}_2 = \begin{bmatrix} y_{11} & y_{12} & y_{21} & y_{22} \\ y_{13} & y_{14} & y_{23} & y_{24} \\ y_{31} & y_{32} & y_{41} & y_{42} \\ y_{33} & y_{34} & y_{43} & y_{44} \end{bmatrix}. \tag{2.20}$$

In (2.20), the indices $j_1 j_2$ are arranged by two Z-maps successively, as

$$\begin{bmatrix} 1 & \longrightarrow & 2 \\ & \swarrow & \\ 3 & \longrightarrow & 4 \end{bmatrix} \tag{2.21}$$

i.e., the path from 1 to 4 in (2.21) is Z shaped and is then called a Z-map. More precisely, $\mathbf{X}_2$ can be decomposed by

$$\mathbf{X}_2 = \begin{bmatrix} Y_{2;1} & Y_{2;2} \\ Y_{2;3} & Y_{2;4} \end{bmatrix} \tag{2.22}$$

and

$$Y_{2;k} = \begin{bmatrix} y_{k1} & y_{k2} \\ y_{k3} & y_{k4} \end{bmatrix}. \tag{2.23}$$

where $\mathbf{X}_2$ is arranged by a Z-map ($Y_{2;k}$) in (2.22) and each $Y_{2;k}$ is also arranged by a Z-map (y_{kl}) in (2.23). Therefore, the indices of y in (2.20) consist of two Z-maps.

The expression (2.20) of all local patterns in $\Sigma_{2\times 2}$ by y can be extended to all patterns in $\Sigma_{2\times n}$ for any $n \geq 3$. Indeed, a local pattern U in $\Sigma_{2\times n}$ can be viewed as the horizontal direct sum of two $1 \times n$ local patterns, i.e. U_1 and U_2, and also the vertical direct sums of n many 2×1 local patterns. As in (2.9), all patterns in $\Sigma_{2\times n}$ can be arranged by the ordering matrix

$$\mathbf{X}_n = \left[\ x_{n;i_1i_2}\ \right], \tag{2.24}$$

a $2^n\times 2^n$ matrix with entry $x_{n;i_1i_2} = x_{n;i_1}\oplus x_{n;i_2}$, where $\chi(U_1) = i_1$ and $\chi(U_2) = i_2$ as in (2.4) and (2.5), $1\leq i_1, i_2\leq 2^n$. On the other hand, for two 2×2 patterns $y_{j_1j_2}$ and $y_{j_2j_3}$, we can attach them together to become a 2×3 pattern $y_{j_1j_2j_3}$, since the second row in $y_{j_1j_2}$ and the first row of $y_{j_2j_3}$ are identical, i.e.,

$$\begin{aligned} y_{j_1j_2j_3} &\equiv y_{j_1j_2}\hat{\oplus}\, y_{j_2j_3} \\ &\equiv y_{j_1}\oplus y_{j_2}\oplus y_{j_3}. \end{aligned} \tag{2.25}$$

Herein, a wedge direct sum $\hat{\oplus}$ is used for 2×2 patterns whenever they can be attached together. In this way, a $2\times n$ pattern $y_{j_1\cdots j_n}$ is obtained from $n-1$ many 2×2 patterns $y_{j_1j_2}, y_{j_2j_3}, \cdots, y_{j_{n-1}j_n}$ by

$$\begin{aligned} y_{j_1\cdots j_n} &\equiv y_{j_1j_2}\hat{\oplus}\, y_{j_2j_3}\hat{\oplus}\cdots\hat{\oplus}\, y_{j_{n-1}j_n} \\ &\equiv y_{j_1}\oplus y_{j_2}\oplus\cdots\oplus y_{j_n}, \end{aligned} \tag{2.26}$$

where $1\leq j_k\leq 4$, and $1\leq k\leq n$. Now, $\mathbf{X}_n$ in y expression can be obtained as follows.

Theorem 2.1. *For any $n\geq 2$, $\Sigma_{2\times n} = \{y_{j_1\cdots j_n}\}$, where $y_{j_1\cdots j_n}$ is given in (2.26). Furthermore, the ordering matrix $\mathbf{X}_n$ can be decomposed by n Z-maps successively as*

$$\mathbf{X}_n = \begin{bmatrix} Y_{n;1} & Y_{n;2} \\ Y_{n;3} & Y_{n;4} \end{bmatrix}, \tag{2.27}$$

$$Y_{n;j_1\cdots j_k} = \begin{bmatrix} Y_{n;j_1\cdots j_k1} & Y_{n;j_1\cdots j_k2} \\ Y_{n;j_1\cdots j_k3} & Y_{n;j_1\cdots j_k4} \end{bmatrix}, \tag{2.28}$$

for $1\leq k\leq n-2$, and

$$Y_{n;j_1\cdots j_{n-1}} = \begin{bmatrix} y_{j_1\cdots j_{n-1}1} & y_{j_1\cdots j_{n-1}2} \\ y_{j_1\cdots j_{n-1}3} & y_{j_1\cdots j_{n-1}4} \end{bmatrix}. \tag{2.29}$$

Proof. From (2.12), (2.14) and (2.15), we have following table.

j_l	1	2	3	4
u_{1l}	0	0	1	1
u_{2l}	0	1	0	1

Table 2.1

For any $n\geq 2$, by (2.12),(2.14) and (2.15), it is easy to generalize (2.18) and (2.19) to

$$i_{n;1} = 1 + \sum_{l=1}^{n}[\frac{j_l-1}{2}]2^{n-l}, \tag{2.30}$$

and

$$i_{n;2} = 1 + \sum_{l=1}^{n} \{j_l - 1 - 2[\frac{j_l - 1}{2}]\}2^{n-l}. \tag{2.31}$$

From (2.30) and (2.31), we have

$$i_{n+1;1} = 2i_{n;1} - 1 + [\frac{j_{n+1} - 1}{2}], \tag{2.32}$$

and

$$i_{n+1;2} = 2i_{n;2} - 1 + \{j_{n+1} - 1 - 2[\frac{j_{n+1} - 1}{2}]\}. \tag{2.33}$$

Now, by induction on n the theorem follows from the last two formulas and the table 2.1. The proof is complete. □

Remark 2.2. *The ordering matrix on $\Sigma_{m\times n}$ can also be introduced accordingly. However, when spatial entropy $h(\mathcal{B})$ of $\Sigma(\mathcal{B})$ is computed, only λ_n, the largest eigenvalue of $\mathbf{T}_n(\mathcal{B})$ must be known. Section 3 provides further details.*

2.2. **More Symbols on Larger Lattices.** The idea introduced in the last section can be generalized to more symbols on $\mathbf{Z}_{m\times m}$, where $m \geq 3$. We first treat a case when m is even. Indeed, assume that $m = 2\ell$, $\ell \geq 2$ and $\mathcal{S}$ contains p elements. Now, we introduce the ordering matrices $\mathbf{X}_2 = [x_{i_1 i_2}]$ and $\mathbf{Y}_2 = [y_{j_1 j_2}]$ to $\Sigma_{2\ell\times 2\ell}$ as follows. Let $q = p^{\ell^2}$, $\mathbf{X}_2$ can be expressed by $y_{j_1 j_2}$, i.e.,

$$\mathbf{X}_2 = \begin{bmatrix} Y_1 & Y_2 & \cdots & Y_q \\ Y_{q+1} & Y_{q+2} & \cdots & Y_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{(q-1)q+1} & Y_{(q-1)q+2} & \cdots & Y_{q^2} \end{bmatrix}_{q\times q}, \tag{2.34}$$

with

$$Y_{j_1} = \begin{bmatrix} y_{j_1,1} & \cdots & y_{j_1,q} \\ y_{j_1,q+1} & \cdots & y_{j_1,2q} \\ \vdots & \ddots & \vdots \\ y_{j_1,(q-1)q+1} & \cdots & y_{j_1,q^2} \end{bmatrix}_{q\times q}. \tag{2.35}$$

Now, we can state recursion formulas for higher ordering matrix $\mathbf{X}_n = [x_{n;i_1 i_2}]_{q^n\times q^n}$ as follows and omit the proof for brevity.

Theorem 2.3. *Suppose we have p symbols, $p \geq 2$ and let $q = p^{\ell^2}$, $\ell \geq 2$. For any $n \geq 2$, $\Sigma_{2\ell\times n\ell} = \{y_{j_1 j_2\cdots j_n}\}$, where $y_{j_1 j_2\cdots j_n} \equiv y_{j_1 j_2}\hat{\oplus} y_{j_2 j_3}\hat{\oplus}\cdots\hat{\oplus} y_{j_{n-1} j_n}$, $1 \leq j_k \leq q^2$ and $1 \leq k \leq n$. Furthermore, the ordering matrix $\mathbf{X}_n$ can be decomposed by n Z-maps successively as*

$$\mathbf{X}_n = \begin{bmatrix} Y_{n;1} & Y_{n;2} & \cdots & Y_{n;q} \\ Y_{n;q+1} & Y_{n;q+2} & \cdots & Y_{n;2q} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{n;(q-1)q+1} & Y_{n;(q-1)q+2} & \cdots & Y_{n;q^2} \end{bmatrix} \tag{2.36}$$

$$Y_{n;j_1\cdots j_k} =$$

$$\begin{bmatrix} Y_{n;j_1,\cdots,j_k,1} & Y_{n;j_1,\cdots,j_k,2} & \cdots & Y_{n;j_1,\cdots,j_k,q} \\ Y_{n;j_1,\cdots,j_k,q+1} & Y_{n;j_1,\cdots,j_k,q+2} & \cdots & Y_{n;j_1,\cdots,j_k,2q} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{n;j_1,\cdots,j_k,(q-1)q+1} & Y_{n;j_1,\cdots,j_k,(q-1)q+2} & \cdots & Y_{n;j_1,\cdots,j_k,q^2} \end{bmatrix} \quad (2.37)$$

for $1 \le k \le n-2$,

$$Y_{n;j_1\cdots j_{n-1}} =$$

$$\begin{bmatrix} y_{j_1,\cdots,j_{n-1},1} & y_{j_1,\cdots,j_{n-1},2} & \cdots & y_{j_1,\cdots,j_{n-1},q} \\ y_{j_1,\cdots,j_{n-1},q+1} & y_{j_1,\cdots,j_{n-1},q+2} & \cdots & y_{j_1,\cdots,j_{n-1},2q} \\ \vdots & \vdots & \ddots & \vdots \\ y_{j_1,\cdots,j_{n-1},(q-1)q+1} & y_{j_1,\cdots,j_{n-1},(q-1)q+2} & \cdots & y_{j_1,\cdots,j_{n-1},q^2} \end{bmatrix}. \quad (2.38)$$

3. **Transition matrices.** This section derives the transition matrices $\mathbf{T}_n$ for a given basic set $\mathcal{B}$. For simplicity, the study of two symbols $\mathcal{S} = \{0, 1\}$ on 2×2 lattice $\mathbf{Z}_{2\times 2}$ in two dimensional lattice space $\mathbf{Z}^2$ is of particular focus. The results can be extended to general cases.

3.1. 2×2 **systems.** Given a basic set $\mathcal{B} \subset \Sigma_{2\times 2}$, horizontal and vertical transition matrices H_2 and V_2 can be defined by

$$H_2 = [h_{i_1 i_2}] \text{ and } V_2 = [v_{j_1 j_2}], \quad (3.1)$$

two 4×4 matrices with entries either 0 or 1 according to following rules:

$$\begin{cases} h_{i_1 i_2} & = 1 \quad if \quad x_{i_1 i_2} \in \mathcal{B}, \\ & = 0 \quad if \quad x_{i_1 i_2} \in \Sigma_{2\times 2} - \mathcal{B}, \end{cases} \quad (3.2)$$

and

$$\begin{cases} v_{j_1 j_2} & = 1 \quad if \quad y_{j_1 j_2} \in \mathcal{B}, \\ & = 0 \quad if \quad y_{j_1 j_2} \in \Sigma_{2\times 2} - \mathcal{B}. \end{cases} \quad (3.3)$$

Obviously, $h_{i_1 i_2} = v_{j_1 j_2}$, where (i_1, i_2) and (j_1, j_2) are related according to (2.16)$\sim$(2.19). Now, the transition matrix $\mathbf{T}_2$ for $\mathcal{B}$ can be defined by

$$\mathbf{T}_2 \equiv \mathbf{T}_2(\mathcal{B})$$

$$= \begin{bmatrix} v_{11} & v_{12} & v_{21} & v_{22} \\ v_{13} & v_{14} & v_{23} & v_{24} \\ v_{31} & v_{32} & v_{41} & v_{42} \\ v_{33} & v_{34} & v_{43} & v_{44} \end{bmatrix}. \quad (3.4)$$

Define

$$v_{j_1 j_2 \cdots j_n} = v_{j_1 j_2} \cdot v_{j_2 j_3} \cdots v_{j_{n-1} j_n}, \quad (3.5)$$

and

$$\mathbf{T}_n = [v_{j_1 j_2 \cdots j_n}],$$

then the transition matrix $\mathbf{T}_n$ for $\mathcal{B}$ defined on $\mathbf{Z}_{2\times n}$ is a $2^n \times 2^n$ matrix with entries $v_{j_1\cdots j_n}$, which are either 1 or 0, by substituting $y_{j_1\cdots j_n}$ by $v_{j_1\cdots j_n}$ in $\mathbf{X}_n$, see (2.27)$\sim$(2.29).

In the following, we give some interpretations for $\mathbf{T}_n$, one from an algebraic perspective and the other from Lindenmayer system (for details see Remark 3.2). For clarity, $\mathbf{T}_3$ can be written in a complete form as

$$\begin{bmatrix} v_{11}v_{11} & v_{11}v_{12} & v_{12}v_{21} & v_{12}v_{22} & v_{21}v_{11} & v_{21}v_{12} & v_{22}v_{21} & v_{22}v_{22} \\ v_{11}v_{13} & v_{11}v_{14} & v_{12}v_{23} & v_{12}v_{24} & v_{21}v_{13} & v_{21}v_{14} & v_{22}v_{23} & v_{22}v_{24} \\ v_{13}v_{31} & v_{13}v_{32} & v_{14}v_{41} & v_{14}v_{42} & v_{23}v_{31} & v_{23}v_{32} & v_{24}v_{41} & v_{24}v_{42} \\ v_{13}v_{33} & v_{13}v_{34} & v_{14}v_{43} & v_{14}v_{44} & v_{23}v_{33} & v_{23}v_{34} & v_{24}v_{43} & v_{24}v_{44} \\ v_{31}v_{11} & v_{31}v_{12} & v_{32}v_{21} & v_{32}v_{22} & v_{41}v_{11} & v_{41}v_{12} & v_{42}v_{21} & v_{42}v_{22} \\ v_{31}v_{13} & v_{31}v_{14} & v_{32}v_{23} & v_{32}v_{24} & v_{41}v_{13} & v_{41}v_{14} & v_{42}v_{23} & v_{42}v_{24} \\ v_{33}v_{31} & v_{33}v_{32} & v_{34}v_{41} & v_{34}v_{42} & v_{43}v_{31} & v_{43}v_{32} & v_{44}v_{41} & v_{44}v_{42} \\ v_{33}v_{33} & v_{33}v_{34} & v_{34}v_{43} & v_{34}v_{44} & v_{43}v_{33} & v_{43}v_{34} & v_{44}v_{43} & v_{44}v_{44} \end{bmatrix} \quad (3.6)$$

From an algebraic perspective, $\mathbf{T}_3$ can be defined through the classical Kronecker product (or tensor product) $\otimes$ and Hadamard product $\odot$. Indeed, for any two matrices $A = (a_{ij})$ and $B = (b_{kl})$, the Kronecker product of $A \otimes B$ is defined by

$$A \otimes B = (a_{ij}B). \quad (3.7)$$

On the other hand, for any two $n \times n$ matrices

$$C = (c_{ij}) \textit{ and } D = (d_{ij}),$$

where c_{ij} and d_{ij} are numbers or matrices. Then, Hadamard product of $C \odot D$ is defined by

$$C \odot D = (c_{ij} \cdot d_{ij}), \quad (3.8)$$

where the product $c_{ij} \cdot d_{ij}$ of c_{ij} and d_{ij} may be multiplication of numbers, numbers and matrices or matrices whenever it is well-defined. For instance, c_{ij} is number and d_{ij} is matrix.
Denoted by

$$\mathbf{T}_2 = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}, \quad (3.9)$$

where T_k is a 2×2 matrix with

$$T_k = \begin{bmatrix} v_{k1} & v_{k2} \\ v_{k3} & v_{k4} \end{bmatrix}. \quad (3.10)$$

Then, using Hadamard product, (3.6) can be written as

$$\mathbf{T}_3 = \begin{bmatrix} v_{11} & v_{12} & v_{21} & v_{22} \\ v_{13} & v_{14} & v_{23} & v_{24} \\ v_{31} & v_{32} & v_{41} & v_{42} \\ v_{33} & v_{34} & v_{43} & v_{44} \end{bmatrix} \odot \begin{bmatrix} T_1 & T_2 & T_1 & T_2 \\ T_3 & T_4 & T_3 & T_4 \\ T_1 & T_2 & T_1 & T_2 \\ T_3 & T_4 & T_3 & T_4 \end{bmatrix}, \quad (3.11)$$

and can also be written by Kronecker product with Hadamard product as

$$\mathbf{T}_3 = \left(\mathbf{T}_2 \right)_{4\times 4} \odot \left[\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \otimes \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \right], \quad (3.12)$$

where $(\mathbf{T}_2)_{4\times4}$ is interpreted as a 4×4 matrix given as in (3.4). Hereinafter, $(M)_{k\times k}$ is used as the $k \times k$ matrix; its entries may also be matrices.

Furthermore, by (3.9) and (3.12), $\mathbf{T}_3$ can also be written as

$$\mathbf{T}_3 = \begin{bmatrix} T_1 \odot \mathbf{T}_2 & T_2 \odot \mathbf{T}_2 \\ T_3 \odot \mathbf{T}_2 & T_4 \odot \mathbf{T}_2 \end{bmatrix}. \tag{3.13}$$

Now, from the perspective of Lindenmayer system, (3.13) can be interpreted as a rewriting rule as follows:

To construct $\mathbf{T}_3$ from $\mathbf{T}_2$, simply replace T_k in (3.9) by $T_k \odot \mathbf{T}_2$, i.e,

$$T_k \longmapsto T_k \odot \mathbf{T}_2 = \begin{bmatrix} v_{k1}T_1 & v_{k2}T_2 \\ v_{k3}T_3 & v_{k4}T_4 \end{bmatrix}. \tag{3.14}$$

Now, $\mathbf{T}_3$ can be written as

$$\mathbf{T}_3 = \begin{bmatrix} v_{11}T_1 & v_{12}T_2 & v_{21}T_1 & v_{22}T_2 \\ v_{13}T_3 & v_{14}T_4 & v_{23}T_3 & v_{24}T_4 \\ v_{31}T_1 & v_{32}T_2 & v_{41}T_1 & v_{42}T_2 \\ v_{33}T_3 & v_{34}T_4 & v_{43}T_3 & v_{44}T_4 \end{bmatrix}. \tag{3.15}$$

Since v_{kj} is either 0 or 1. The entries of $\mathbf{T}_3$ in (3.15) are T_k, i.e, T_k can be taken as the "basic element" in constructing $\mathbf{T}_n$, $n \geq 3$. As demonstrated later that(3.14) is an efficient means of constructing $\mathbf{T}_{n+1}$ from $\mathbf{T}_n$ for any $n \geq 2$.

Now, by induction on n, the following properties of transition matrix $\mathbf{T}_n$ on $\mathbf{Z}_{2\times n}$ can be easily proven.

Theorem 3.1. *Let $\mathbf{T}_2$ be a transition matrix given by (3.4). Then, for higher order transition matrices $\mathbf{T}_n$, $n \geq 3$, we have the following three equivalent expressions*
(I) $\mathbf{T}_n$ can be decomposed into n successive 2×2matrices (or n-successive Z-maps) as follows:

$$\mathbf{T}_n = \begin{bmatrix} T_{n;1} & T_{n;2} \\ T_{n;3} & T_{n;4} \end{bmatrix},$$

$$T_{n;j_1\cdots j_k} = \begin{bmatrix} T_{n;j_1\cdots j_k 1} & T_{n;j_1\cdots j_k 2} \\ T_{n;j_1\cdots j_k 3} & T_{n;j_1\cdots j_k 4} \end{bmatrix},$$

for $1 \leq k \leq n-2$ and

$$T_{n;j_1\cdots j_{n-1}} = \begin{bmatrix} v_{j_1\cdots j_{n-1}1} & v_{j_1\cdots j_{n-1}2} \\ v_{j_1\cdots j_{n-1}3} & v_{j_1\cdots j_{n-1}4} \end{bmatrix}.$$

Furthermore,

$$T_{n;k} = \begin{bmatrix} v_{k1}T_{n-1;1} & v_{k2}T_{n-1;2} \\ v_{k3}T_{n-1;3} & v_{k4}T_{n-1;4} \end{bmatrix}. \tag{3.16}$$

(II) Starting from

$$\mathbf{T}_2 = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix},$$

with

$$T_k = \begin{pmatrix} v_{k1} & v_{k2} \\ v_{k3} & v_{k4} \end{pmatrix},$$

$\mathbf{T}_n$ can be obtained from $\mathbf{T}_{n-1}$ by replacing T_k by $T_k \odot \mathbf{T}_2$ according to (3.14).

(III)
$$\mathbf{T}_n = (\mathbf{T}_{n-1})_{2^{n-1}\times 2^{n-1}} \odot \left(E_{2^{n-2}} \otimes \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} \right),$$

where E_{2^k} is the $2^k \times 2^k$ matrix with 1 as its entries.

Proof.

(I)The proof is to simply replace $Y_{n;j_1\cdots j_k}$ and $y_{j_1\cdots j_n}$ by $T_{n;j_1\cdots j_k}$ and $v_{j_1\cdots j_n}$ in Theorem 2.1, respectively.

(II) follows from (I) directly.

(III) follows from (I), we have

$$\mathbf{T}_n = \begin{bmatrix} T_{n;1} & T_{n;2} \\ T_{n;3} & T_{n;4} \end{bmatrix}.$$

And by (3.16), we get following formula.

$$\begin{aligned} \mathbf{T}_n &= \begin{bmatrix} v_{11}T_{n;1} & v_{12}T_{n;2} & v_{21}T_{n;1} & v_{22}T_{n;2} \\ v_{13}T_{n;3} & v_{14}T_{n;4} & v_{23}T_{n;3} & v_{24}T_{n;4} \\ v_{31}T_{n;1} & v_{32}T_{n;2} & v_{41}T_{n;1} & v_{42}T_{n;2} \\ v_{33}T_{n;3} & v_{34}T_{n;4} & v_{43}T_{n;3} & v_{44}T_{n;4} \end{bmatrix} \\ &= (\mathbf{T}_{n-1})_{2^{n-1}\times 2^{n-1}} \odot \left(E_{2^{n-2}} \otimes \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} \right). \end{aligned}$$

The proof is complete. □

Remark 3.2. *While studying the growth processes of plants, Lindenmayer, e.g.[39], derived a developmental algorithm, i.e., a set of rules which describes plant development in time. Thereafter, a system with a set of rewriting rules was called Lindenmayer system or L-system. From Theorem 3.1(III), the family of transition matrices $\{\mathbf{T}_n\}_{n\geq 2}$ is a two-dimensional L-system with a rewriting rule(3.16). Similar to many L-systems, our system $\mathbf{T}_n$ also enjoys the simplicity of recursion formulas and self-similarity.*

As for spatial entropy $h(\mathcal{B})$, we have the following theorem.

Theorem 3.3. *Given a basic set $\mathcal{B} \subset \Sigma_{2\times 2}$, let λ_n be the largest eigenvalue of the associated transition matrix $\mathbf{T}_n$ which is defined in Theorem 3.1. Then,*

$$h(\mathcal{B}) = \lim_{n\to\infty} \frac{\log \lambda_n}{n}. \tag{3.17}$$

Proof. By the same arguments as in [13], the limit (1.2) is well-defined and exists. From the construction of $\mathbf{T}_n$, we observe that for $m \geq 2$,

$$\Gamma_{m\times n}(\mathcal{B}) = \sum_{1\leq i,j\leq 2^n} (\mathbf{T}_n^{m-1})_{i,j} \equiv \#(\mathbf{T}_n^{m-1}). \tag{3.18}$$

As in a one dimensional case, we have

$$\lim_{m\to\infty} \frac{\log \#(\mathbf{T}_n^{m-1})}{m} = \log \lambda_n,$$

e.g. [42]. Therefore,

$$\begin{aligned} h(\mathcal{B}) &= \lim_{m,n\to\infty} \frac{\log \Gamma_{m\times n}(\mathcal{B})}{mn} \\ &= \lim_{n\to\infty} \frac{1}{n}\Big(\lim_{m\to\infty} \frac{\log \Gamma_{m\times n}(\mathcal{B})}{m}\Big) \\ &= \lim_{n\to\infty} \frac{\log \lambda_n}{n}. \end{aligned}$$

The proof is complete. □

3.2. **Computation of Maximum Eigenvalues and Spatial Entropy.** Given a transition matrix $\mathbf{T}_2$, for any $n \geq 2$, the characteristic polynomials $|\mathbf{T}_n - \lambda|$ are of degree 2^n. In general, computing or estimating the largest eigenvalue $\lambda_n = \lambda_n(\mathbf{T}_2)$ of $|\mathbf{T}_n - \lambda|$ for a large n is relatively difficult. However, in this section, we present a class of $\mathbf{T}_2$ in which $\lambda_n(\mathbf{T}_2)$ can be computed explicitly. Indeed, assume that $\mathbf{T}_2$ has the form of $\begin{bmatrix} A & B \\ B & A \end{bmatrix}$ in (3.9), i.e.,

$$T_1 = T_4 = A = \begin{bmatrix} a & a_2 \\ a_3 & a \end{bmatrix}, \tag{3.19}$$

and

$$T_2 = T_3 = B = \begin{bmatrix} b & b_2 \\ b_3 & b \end{bmatrix}, \tag{3.20}$$

where a, a_2, a_3, b, b_2 and b_3 are either 0 or 1.

We need the following lemma.

Lemma 3.4. *Let A and B be non-negative and non-zero $m \times m$ matrices, respectively, and α and β are positive numbers. The maximum eigenvalue of $\begin{bmatrix} A & \alpha B \\ \beta B & A \end{bmatrix}$ is then the maximum eigenvalue of*

$$A + \sqrt{\alpha\beta}B.$$

Proof. Consider

$$\begin{vmatrix} A - \lambda & \alpha B \\ \beta B & A - \lambda \end{vmatrix} = 0.$$

For $|A - \lambda| \neq 0$, the last equation is equivalent to

$$\begin{vmatrix} A - \lambda & B \\ 0 & (A - \lambda) - \alpha\beta B(A - \lambda)^{-1}B \end{vmatrix} = 0,$$

or

$$|I - \alpha\beta((A - \lambda)^{-1}B)^2| = 0.$$

Then, we have

$$|A + \sqrt{\alpha\beta}B - \lambda| = 0 \quad \text{or} \quad |A - \sqrt{\alpha\beta}B - \lambda| = 0.$$

Since A and B are non-negative and α and β are positive, verifying that the maximum eigenvalue λ of $\begin{bmatrix} A & \alpha B \\ \beta B & A \end{bmatrix}$ and $A + \sqrt{\alpha\beta}B$ are equal is relatively easy. The proof is complete. □

Now, we can state our computation results for $\lambda_n(\mathbf{T}_2)$ when $\mathbf{T}_2$ satisfies (3.19) and (3.20).

Theorem 3.5. *Assume that $\mathbf{T}_2 = \begin{bmatrix} A & B \\ B & A \end{bmatrix}$ and $A = \begin{bmatrix} a & a_2 \\ a_3 & a \end{bmatrix}$ and $B = \begin{bmatrix} b & b_2 \\ b_3 & b \end{bmatrix}$ where $a, b, a_2, a_3, b_2, b_3 \in \{0, 1\}$. For $n \geq 2$, let λ_n be the largest eigenvalue of*

$$|\mathbf{T}_n - \lambda| = 0.$$

Then

$$\lambda_n = \alpha_{n-1} + \beta_{n-1}, \tag{3.21}$$

where α_k *and* β_k *satisfy the following recursion relations:*

$$\alpha_{k+1} = a\alpha_k + b\beta_k, \tag{3.22}$$

$$\beta_{k+1} = \sqrt{(a_2\alpha_k + b_2\beta_k)(a_3\alpha_k + b_3\beta_k)}, \tag{3.23}$$

for $k \geq 0$*, and*

$$\alpha_0 = \beta_0 = 1. \tag{3.24}$$

Furthermore, the spatial entropy $h(\mathbf{T}_2)$ *is equal to* $\log \xi_*$*, where* ξ_* *is the maximum root of the following polynomials* $Q(\xi)$*:*
(I) if $a_2 = a_3 = 1$*,*

$$Q(\xi) \equiv 4\xi^2(\xi-a)^2 + (\gamma^2 - 4\delta)(\xi-a)^2 - \gamma^2\xi^2 - 2\gamma(2b - a\gamma)\xi - (2b-a\gamma)^2, \tag{3.25}$$

where

$$\gamma = b_2 + b_3 \text{ and } \delta = b_2 b_3. \tag{3.26}$$

(II) if $a_2a_3 = 0$ *and* $a_2b_3 + a_3b_2 = 1$*,*

$$Q(\xi) \equiv \xi^3 - a\xi^2 - \delta\xi + a\delta - b. \tag{3.27}$$

Moreover, if $a_2a_3 = 0$ *and* $a_2b_3 + a_3b_2 = 0$*, then* $h(\mathbf{T}_2) = 0$*.*

Proof. Owing to the special structure of $\mathbf{T}_2$, it is easy to verify that for any $k \geq 2$, we have

$$\mathbf{T}_k = \begin{bmatrix} A_k & B_k \\ B_k & A_k \end{bmatrix},$$

and

$$\mathbf{T}_{k+1} = \begin{bmatrix} A_{k+1} & B_{k+1} \\ B_{k+1} & A_{k+1} \end{bmatrix},$$

here

$$A_{k+1} = \mathbf{T}_k \odot A = \begin{bmatrix} aA_k & a_2B_k \\ a_3B_k & aA_k \end{bmatrix}, \tag{3.28}$$

and

$$B_{k+1} = \mathbf{T}_k \odot B = \begin{bmatrix} bA_k & b_2B_k \\ b_3B_k & bA_k \end{bmatrix}, \tag{3.29}$$

$A_2 = A$ and $B_2 = B$. Now by Lemma 3.4, $|\mathbf{T}_{n+1} - \lambda_{n+1}| = 0$, so

$$|A_{n+1} + B_{n+1} - \lambda_{n+1}| = 0. \tag{3.30}$$

Let

$$\alpha_0 = 1 \ \ and \ \ \beta_0 = 1.$$

By induction on k, $1 \leq k \leq n$, and using (3.28),(3.29),(3.30) and Lemma 3.4, it is straightforward to derive

$$|\alpha_k A_{n-k+1} + \beta_k B_{n-k+1} - \lambda_{n+1}| = 0, \tag{3.31}$$

with α_k and β_k satisfy (3.22) and (3.23). In particular,

$$\alpha_n = a\alpha_{n-1} + b\beta_{n-1}, \tag{3.32}$$

$$\beta_n = \{(a_2\alpha_{n-1} + b_2\beta_{n-1})(a_3\alpha_{n-1} + b_3\beta_{n-1})\}^{\frac{1}{2}}, \tag{3.33}$$

and

$$\lambda_{n+1} = \alpha_n + \beta_n.$$

This proves the first part of the theorem.

The remainder of the proof, demonstrates that $h(\mathbf{T}_2) = \log \lambda_*$ where λ_* is the maximum root of $Q(\lambda)$. From (3.33), we have

$$\begin{aligned} \beta_n^2 = a_2 a_3 \alpha_{n-1}^2 \quad &+ \quad (a_2 b_3 + a_3 b_2)\alpha_{n-1}\beta_{n-1} \\ &+ \quad b_2 b_3 \beta_{n-1}^2. \end{aligned} \tag{3.34}$$

Now, in (3.34), we first solve α_{n-1} in terms of β_{n-1} and β_n, then substitute α_{n-1} and α_n into (3.32) to obtain difference equations involving β_{n+1}, β_n and β_{n-1}. There are two cases:

Case I. If $a_2 = a_3 = 1$, then we have

$$\alpha_{n-1} = \frac{1}{2}\{-\gamma\beta_{n-1} + (4\beta_n^2 + (\gamma^2 - 4\delta)\beta_{n-1}^2)^{\frac{1}{2}}\}. \tag{3.35}$$

Substituting (3.35) into (3.32), yields

$$\begin{aligned} \{4\beta_{n+1}^2 + (\gamma^2 - 4\delta)\beta_n^2\}^{\frac{1}{2}} \quad &= \quad \gamma\beta_n + (2b - a\gamma)\beta_{n-1} \\ &+ \quad a\{4\beta_n^2 + (\gamma^2 - 4\delta)\beta_{n-1}^2\}^{\frac{1}{2}}. \end{aligned} \tag{3.36}$$

Now, let

$$\xi_n = \frac{\beta_n}{\beta_{n-1}}, \tag{3.37}$$

and after dividing (3.36) by β_{n-1}, we have

$$\xi_n\{4\xi_{n+1}^2 + (\gamma^2 - 4\delta)\}^{\frac{1}{2}} = \gamma\xi_n + (2b - a\gamma) + a\{4\xi_n^2 + (\gamma^2 - 4\delta)\}^{\frac{1}{2}}. \tag{3.38}$$

(3.38) can be written as the following iteration map:

$$\xi_{n+1} = G_1(\xi_n), \tag{3.39}$$

where

$$G_1(\xi) = \frac{1}{2}\{4\delta + 2\gamma g(\xi) + g^2(\xi)\}^{\frac{1}{2}}, \tag{3.40}$$

and

$$g(\xi) = (2b - a\gamma)\xi^{-1} + a\{4 + (\gamma^2 - 4\delta)\xi^{-2}\}^{\frac{1}{2}}. \tag{3.41}$$

We first observe the fixed point ξ_* of $G_1(\xi)$, i.e., $\xi_* = G(\xi_*)$, is a root of $Q(\xi)$. Indeed, by letting $\xi_n = \xi_{n+1} = \xi_*$ in (3.38), we have

$$(\xi_* - a)(4\xi_*^2 + (\gamma^2 - 4\delta))^{\frac{1}{2}} = \gamma\xi_* + (2b - a\gamma),$$

which gives us $Q(\xi_*) = 0$.

It can be proven that the maximum fixed point of $G_1(\xi)$ or the maximum root ξ_* of $Q(\xi) = 0$ satisfies $1 \leq \xi_* \leq 2$ and

$$\xi_n \to \xi_* \quad as \quad n \to \infty. \tag{3.42}$$

Details are omitted here for brevity. By (3.21), (3.35) and (3.37), we can also prove

$$\frac{\lambda_{n+1}}{\lambda_n} \to \xi_* \quad as \quad n \to \infty. \tag{3.43}$$

Hence, $h(\mathbf{T}_2) = \log \xi_*$.

	A	B	$Q(\lambda)$	λ_*
(1)	$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	$\lambda - 2$	2
(2)	$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ *or* $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$	$\lambda^3 - 2\lambda^2 + \lambda - 1$	(i)
(3)(α)	$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ *or* $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	$\lambda^2 - \lambda - 1$	g
(3)(β)	$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\lambda^2 - \lambda - 1$	g
(3)(γ)	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	$\lambda^2 - \lambda - 1$	g
(4)	$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$	$\lambda^3 - \lambda^2 - 1$	(ii)
	$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$		
(5)	$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ *or* $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	$\lambda^3 - \lambda - 1$	(iii)
(6)	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ *or* $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$	$\lambda^4 - \lambda - 1$	(iv)

(i) $\lambda_* \doteq 1.75488$, (ii) $\lambda_* \doteq 1.46557$, (iii) $\lambda_* \doteq 1.32472$, (iv) $\lambda_* \doteq 1.22074$
where, $g \doteq 1.61803$, is the golden mean, a root of $\lambda^2 - \lambda - 1 = 0$.

Table 3.1

Case II. If $a_2a_3 = 0$ and $a_2b_3 + a_3b_2 = 1$, then, from (3.33), we have

$$\alpha_{n-1} = \beta_n^2 \beta_{n-1}^{-1} - \delta\beta_{n-1}. \tag{3.44}$$

Again, substituting (3.44) into (3.32) and letting (3.37) lead to

$$\xi_{n+1}^2 \xi_n - a\xi_n^2 - \delta\xi_n + a\delta - b = 0, \tag{3.45}$$

i.e., $\xi_{n+1} = G_2(\xi_n)$, where

$$G_2(\xi) = \{a\xi + \delta + (b - a\delta)\xi^{-1}\}^{\frac{1}{2}}. \tag{3.46}$$

The maximum fixed point ξ_* of (3.46) is the maximum root of $Q(\xi) = 0$ in (3.27). It can also be proven that (3.42) and (3.43) holds in this case.

Finally, if $a_2a_3 = 0$ and $a_2b_3 + a_3b_2 = 0$, then β_n are all equal for $n \geq 1$. Hence, α_n is at most linear growth in n, implying that $h(\mathbf{T}_2) = 0$. The proof is thus complete. □

For completeness, we list all $\mathbf{T}_2$ which satisfy (3.19) and (3.20) and have positive entropy $h(\mathbf{T}_2)$. The table is arranged based on the magnitude of $h(T_2)$. The polynomial $Q(.)$ in either (3.25) or (3.27) has been simplified whenever possible.

The recursion formulas for λ_n are

(1)		$\lambda_n = 2^n,$
(2)		$\lambda_{n+1} = \lambda_n + (\lambda_n\lambda_{n-1})^{\frac{1}{2}},$
(3)	(α)	$\lambda_{n+1} = \lambda_n + (\lambda_n(\lambda_n - \lambda_{n-1}))^{\frac{1}{2}},$
	(β)	$\lambda_{n+1} = \lambda_n + \lambda_{n-1},$
	(γ)	$\lambda_{n+1} = \lambda_n + \lambda_{n-1},$
(4)		$\lambda_{n+1} = \lambda_n + (\lambda_{n-1}(\lambda_n - \lambda_{n-1}))^{\frac{1}{2}},$
(5)		$\lambda_{n+1} = (\lambda_n\beta_{n-1})^{\frac{1}{2}} + \beta_{n-1},$ where $\beta_{n-1} = \lambda_n - \lambda_{n-1} + \cdots + (-1)^n,$
(6)		$\lambda_{n+1} = \lambda_n + (\lambda_n\beta_{n-2})^{\frac{1}{2}} - \beta_{n-2}.$

Table 3.2

Remark 3.6.
(i) According to Table 3.2, for cases (1)∼(4), λ_{n+1} depends only on two preceding terms, λ_n and λ_{n-1}. However, in (5) and (6), λ_{n+1} depends on all of its preceding terms $\lambda_1, \cdots, \lambda_n$.
(ii) From Lemma 3.4 and Theorem 3.5, in addition to the maximum eigenvalue we can obtain a complete set of eigenvalues of $\mathbf{T}_n$ explicitly.
(iii) In Theorem 3.5, polynomial $Q(\xi)$ given in (3.25) or (3.27) is the limiting equation for $\lambda_n^{\frac{1}{n}}$. It is interesting to know if there is any limiting equation for general $\mathbf{T}_n$.

Remark 3.7. *Similar to the concept in Theorem 3.5, if $\mathbf{T}_2$ does not satisfy (3.19) and (3.20), another special structure can allow us to obtain explicit recursion formulas of λ_n and compute its spatial entropy $h(\mathbf{T}_2)$ explicitly.*

3.3. $2\ell \times 2\ell$ **Systems.** The results in last two subsections can be generalized to p-symbols on $\mathbf{Z}_{2\ell\times 2\ell}$. Given a basic set $\mathcal{B} \subset \Sigma_{2\ell\times 2\ell}$, horizontal and vertical transition matrices $H_2 = [h_{i_1i_2}]_{q^2\times q^2}$ and $V_2 = [v_{j_1j_2}]_{q^2\times q^2}$, where $q = p^{\ell^2}$, can be defined according the rules (3.2) and (3.3) by replacing $\Sigma_{2\times 2}$ with $\Sigma_{2\ell\times 2\ell}$, respectively.

Then the transition matrix $\mathbf{T}_2(\mathcal{B})$ for $\mathcal{B}$ can be defined by

$$\mathbf{T}_2 = \mathbf{T}_2(\mathcal{B}) = \begin{bmatrix} V_1 & V_2 & \cdots & V_q \\ V_{q+1} & V_{q+2} & \cdots & V_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ V_{(q-1)q+1} & V_{(q-1)q+2} & \cdots & V_{q^2} \end{bmatrix} \tag{3.47}$$

where

$$V_m = \begin{bmatrix} v_{m,1} & v_{m,2} & \cdots & v_{m,q} \\ v_{m,(q+1)} & v_{m,q+2} & \cdots & v_{m,2q} \\ \vdots & \vdots & \ddots & \vdots \\ v_{m,(q-1)q+1} & v_{m,(q-1)q+2} & \cdots & v_{m,q^2} \end{bmatrix}, \tag{3.48}$$

$1 \leq m \leq q^2$. The higher order transition matrix $\mathbf{T}_n = [v_{j_1 j_2 \cdots j_n}]$ for $\mathcal{B}$ defined on $\mathbf{Z}_{2\ell\times n\ell}$ is a $q^n \times q^n$ matrix, where $v_{j_1 j_2 \cdots j_n}$ is given by (3.5) which are either 1 or 0, by substituting $y_{j_1\cdots j_n}$by $v_{j_1\cdots j_n}$ in $\mathbf{X}_n$, see (2.36)∼(2.38). For completeness, we state the following theorem for $\mathbf{T}_n$ and omit the proof for brevity.

Theorem 3.8. *Let $\mathbf{T}_2$ be a transition matrix given by (3.47) and (3.48). Then for higher order transition matrices $\mathbf{T}_n$, $n \geq 3$, we have the following three equivalent expressions*

(I) *$\mathbf{T}_n$ can be decomposed into n successive $q \times q$ matrices as follows:*

$$\mathbf{T}_n = \begin{bmatrix} T_{n;1} & \cdots & T_{n;q} \\ T_{n;q+1} & \cdots & T_{n;2q} \\ \vdots & \ddots & \vdots \\ T_{n;(q-1)q+1} & \cdots & T_{n;q^2} \end{bmatrix}$$

$$T_{n;j_1\cdots j_k} = \begin{bmatrix} T_{n;j_1,\cdots,j_k,1} & \cdots & T_{n;j_1,\cdots,j_k,q} \\ T_{n;j_1,\cdots,j_k,q+1} & \cdots & T_{n;j_1,\cdots,j_k,2q} \\ \vdots & \ddots & \vdots \\ T_{n;j_1,\cdots,j_k,(q-1)q+1} & \cdots & T_{n;j_1,\cdots,j_k,q^2} \end{bmatrix}$$

for $1 \leq k \leq n-2$ and

$$T_{n;j_1\cdots j_{n-1}} = \begin{bmatrix} v_{j_1,\cdots,j_{n-1},1} & \cdots & v_{j_1,\cdots,j_{n-1},q} \\ v_{j_1,\cdots,j_{n-1},q+1} & \cdots & v_{j_1,\cdots,j_{n-1},2q} \\ \vdots & \ddots & \vdots \\ v_{j_1,\cdots,j_{n-1},(q-1)q+1} & \cdots & v_{j_1,\cdots,j_{n-1},q^2} \end{bmatrix}.$$

Furthermore,

$$T_{n;k} = \begin{bmatrix} v_{k,1}T_{n-1;1} & \cdots & v_{k,q}T_{n-1;q} \\ v_{k,q+1}T_{n-1;q+1} & \cdots & v_{k,2q}T_{n-1;2q} \\ \vdots & \ddots & \vdots \\ v_{k,(q-1)q+1}T_{n-1;(q-1)q+1} & \cdots & v_{k,q^2}T_{n-1;q^2} \end{bmatrix}$$

(II) *Starting from*

$$\mathbf{T}_2 = \begin{bmatrix} T_1 & \cdots & T_q \\ T_{q+1} & \cdots & T_{2q} \\ \vdots & \ddots & \vdots \\ T_{(q-1)q+1} & \cdots & T_{q^2} \end{bmatrix},$$

with

$$T_k = \begin{bmatrix} v_{k,1} & \cdots & v_{k,q} \\ v_{k,q+1} & \cdots & v_{k,2q} \\ \vdots & \ddots & \vdots \\ v_{k,(q-1)q+1} & \cdots & v_{k,q^2} \end{bmatrix},$$

$\mathbf{T}_n$ *can be obtained from* $\mathbf{T}_{n-1}$ *by replacing* T_k *by* $T_k \odot \mathbf{T}_2$ *according to*

$$T_k \mapsto T_k \odot \mathbf{T}_2 = \begin{bmatrix} v_{k,1}T_1 & \cdots & v_{k,q}T_q \\ v_{k,q+1}T_{q+1} & \cdots & v_{k,2q}T_{2q} \\ \vdots & \ddots & \vdots \\ v_{k,(q-1)q+1}T_{(q-1)q+1} & \cdots & v_{k,q^2}T_{q^2} \end{bmatrix}$$

(III)

$$\mathbf{T}_n = (\mathbf{T}_{n-1})_{q^{n-1}\times q^{n-1}} \odot (E_{q^{n-2}} \otimes \mathbf{T}_2).$$

For the spatial entropy $h(\mathcal{B})$, we have a similar result as in Theorem 3.3.

Theorem 3.9. *Given a basic set* $\mathcal{B} \subset \Sigma_{m_1\times m_2}$*, let* ℓ *be the smallest integer such that* $2\ell \geq m_1$ *and* $2\ell \geq m_2$*, and let* $\widetilde{\mathcal{B}} = \Sigma_{2\ell\times 2\ell}(\mathcal{B})$*. Suppose* $\lambda_{n;\ell}$ *be the largest eigenvalue of the associated transition matrix* $\mathbf{T}_n$*, which is defined in Theorem 3.8. Then*

$$h(\mathcal{B}) \quad = \tfrac{1}{\ell^2}\lim_{n\to\infty}\tfrac{log\lambda_{n;\ell}}{n}.$$

Proof. As in Theorem 3.3,

$$\begin{aligned} h(\mathcal{B}) &= \lim_{m,n\to\infty}\frac{log\Gamma_{m\ell\times n\ell}(\widetilde{\mathcal{B}})}{m\ell\times n\ell} \\ &= \frac{1}{\ell^2}\lim_{n\to\infty}\frac{1}{n}(\lim_{m\to\infty}\frac{log\#(T_n^{m-1}(\widetilde{\mathcal{B}}))}{m}) \\ &= \frac{1}{\ell^2}\lim_{n\to\infty}\frac{1}{n}(\lim_{m\to\infty}\frac{log\lambda_{n;\ell}^{m-1}}{m}) \\ &= \frac{1}{\ell^2}\lim_{n\to\infty}\frac{log\lambda_{n;\ell}}{n}. \end{aligned}$$

The proof is complete. □

3.4. **Relation with Matrix Shifts.** Under many circumstances, we are given a pair of horizontal transition matrix $H = (h_{ij})_{p\times p}$ and vertical transition matrix $V = (v_{ij})_{p\times p}$, where h_{ij} and $v_{ij} \in \{0,1\}$, e.g. [13, 29, 30, 32]. Now, the set of all admissible patterns which can be generated by H and V on $\mathbf{Z}_{m_1\times m_2}$ and $\mathbf{Z}^2$ are denoted by $\Sigma_{m_1\times m_2}(H;V)$ and $\Sigma(H;V)$, respectively. Furthermore, $\Sigma_{m_1\times m_2}(H;V)$ and $\Sigma(H;V)$ can be characterized by

$$\begin{aligned}\Sigma_{m_1\times m_2}(H;V) = \{&U\in\Sigma_{m_1\times m_2,p} : h_{u_\alpha u_{\alpha+e_1}} = 1 \ and\ v_{u_\beta u_{\beta+e_2}} = 1,\\ &where\ e_1=(1,0),\ e_2=(0,1),\ \alpha=(\alpha_1,\alpha_2),\ \beta=(\beta_1,\beta_2)\\ &with\ 1\le\alpha_1\le m_1-1\ ,\ 1\le\alpha_2\le m_2\ and\ 1\le\beta_1\le m_1\ 1\le\beta_2\le m_2-1\}\end{aligned} \tag{3.49}$$

and

$$\begin{aligned}\Sigma(H;V) = \{U\in\Sigma_p^2 : h_{u_\alpha u_{\alpha+e_1}} = 1\ and\ v_{u_\beta u_{\beta+e_2}} = 1\\ for\ all\ \alpha,\beta\in\mathbf{Z}^2\}.\end{aligned} \tag{3.50}$$

In literature, $\Sigma(H;V)$ is often called a Matrix shift, Markov shift or subshift of finite types, e.g. [13, 30, 32, 38]

As before, we are concerned about constructing $\Sigma_{m_1\times m_2}(H;V)$. We first show that the established theories can be applied to answer this question. Indeed, we introduce $\mathcal{S}=\{0,1,2,\cdots,p-1\}$. On $\mathbf{Z}_{2\times 2}$, consider local pattern $U=(u_{\alpha_1\alpha_2})$ with $u_{\alpha_1\alpha_2}\in\ \mathcal{S}$. Define the ordering matrices $\mathbf{X}_2=[x_{i_1i_2}]_{p^2\times p^2}$ and $\mathbf{Y}_2=[y_{j_1j_2}]_{p^2\times p^2}$ for $\Sigma_{2\times 2}$. Now, the basic set $\mathcal{B}(H;V)$ determined by H and V can be expressed as

$$\mathcal{B}(H;V)=\{U=(u_{\alpha_1\alpha_2})\in\Sigma_{2\times 2} : h_{u_{11}u_{21}}h_{u_{12}u_{22}}v_{u_{11}u_{12}}v_{u_{21}u_{22}}=1\}. \tag{3.51}$$

Therefore, the transition matrix $\mathbf{T}_2=\mathbf{T}_2(H;V)$ can be expressed as $\mathbf{T}_2=[t_{j_1j_2}]_{p^2\times p^2}$ with $t_{j_1j_2}=1$ if and only if $y_{j_1j_2}\in\ \mathcal{B}(H;V)$, i.e., $t_{j_1j_2}=1$ if and only if

$$h_{u_{11}u_{21}}h_{u_{12}u_{22}}v_{u_{11}u_{12}}v_{u_{21}u_{22}}=1, \tag{3.52}$$

where j_l is related to $u_{\alpha_1\alpha_2}$ according to (2.12) similarly.

Now, $\mathbf{T}_n=\mathbf{T}_n(H;V)$ can be constructed recursively from $\mathbf{T}_2(H;V)$ by Theorem 3.8. Then λ_n and spatial entropy $h(H;V)$ can be studied by Theorem 3.9. It is easy to verify $\mathbf{T}_n(H;V)=\overline{\mathbf{T}}^{(n)}_{H,V}$, the transition matrix obtained by Juang et al in [30]. Furthermore, $T^{(n)}_{H,V}$ in [30] can also be obtained by deleting the rows and columns formed by zeros in $\mathbf{T}_n(H;V)$.

On the other hand, given a basic set $\mathcal{B}\subset\Sigma_{2\times 2,p}$ (or $\Sigma_{2l\times 2l,p}$), in general there is no horizontal transition matrix $H=(h_{ij})_{p\times p}$ and vertical transition matrix $V=(v_{ij})_{p\times p}$ such that $\mathcal{B}=\mathcal{B}(H;V)$ given by (3.51). Indeed, the number of subsets of $\Sigma_{2\times 2,p}$ is 2^{p^4} and the number of $\mathcal{B}(H;V)$ is at most 2^{2p^2} and $2^{2p^2}<2^{p^4}$ for any $p\ge 2$. However, as mentioned in p.468[38], one can recode any shift of finite type to a matrix subshift.

Notably, the n-th order transition matrix $\mathbf{T}_n(\mathcal{B})$ is a $q^n\times q^n$ matrix with $q=p^{\ell^2}$ and the n-th order transition matrix $\mathbf{T}_n(H(\mathcal{B});V(\mathcal{B})))$ generated by $\mathbf{T}_2(H(\mathcal{B});V(\mathcal{B})))$ is a $m^n\times m^n$ matrix. Consequently, if $m=\#\mathcal{B}$ is relatively small compared with $q=p^{l^2}$, we may study the eigenvalue problems of $\mathbf{T}_n(H(\mathcal{B});V(\mathcal{B}))$. It is clear, small m generates less admissible patterns and then smaller entropy. For $\mathcal{B}$ with positive entropy $h(\mathcal{B})$ as in Table 3.1, $\#\mathcal{B}$ is much larger than $q=2$. Therefore, in general working on $\mathbf{T}_n(\mathcal{B})$ is better than on $\mathbf{T}_n(H(\mathcal{B});V(\mathcal{B})))$.

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REFERENCES

[1] J.C. Ban, K.P. Chien, S.S. Lin, Spatial disorder of CNN-with asymmetric output funtion, International J. of Bifurcation and Chaos, 11(2001), pp. 2085-2095.

[2] J.C. Ban, C.H. Hsu, S.S. Lin, Spatial disorder of Cellular Neural Network-with biased term, International J. of Bifurcation and Chaos, 12(2002), pp. 525-534.
[3] J.C. Ban, S.S. Lin, Y.H. Lin, Patterns Generation and Transition Matrices in Higher Dimensional Lattice Models, preprint (2004).
[4] P.W. Bates, A. Chmaj, A discrete convolution model for phase transitions, Arch. Rat. Mech. Anal, 150(1999), pp. 281-305
[5] P.W. Bates, K. Lu, B. Wang, Attractors for lattice dynamical systems, International J. of Bifurcation and Chaos, 11(2001),pp. 143-153.
[6] J.C. Ban, S.S. Lin, C.W. Shih, Exact number of mosaic patterns in cellular neural networks International J. of Bifurcation and Chaos, 11(2001), pp. 1645-1653.
[7] J. Bell, Some threshold results for modes of myelinated nerves, Math. Biosci., 54 (1981), pp. 181-190.
[8] J. Bell, C. Cosner, Threshold behavior and propagation for nonlinear differential-difference systems motivated by modeling myelinated axons, Quart. Appl. Math., 42(1984), pp. 1-14.
[9] J.W. Cahn, Theory of crystal growth and interface motion in crystalline materials, Acta Metallurgica, 8(1960), pp. 554-562.
[10] S.S Chen, C.W Shih, Asymptotic behaviors in a transiently chaotic neural network, Discrete Contin. Dynam. Systems, 10(2004), pp. 805-836.
[11] S.N. Chow, J. Mallet-paret, Pattern formation and spatial chaos in lattice dynamical systems II, IEEE Trans. Circuits Systems, 42(1995), pp. 752-756.
[12] S.N. Chow, J. Mallet-paret, E.S. Van vleck, Dynamics of lattice differential equations, International J. of Bifurcation and Chaos, 9(1996), pp. 1605-1621.
[13] S.N. Chow, J. Mallet-paret, E.S. Van vleck, Pattern formation and spatial chaos in spatially discrete evolution equations, Random Comput. Dynam., 4(1996), pp. 109-178.
[14] S.N. Chow, W. Shen, Dynamics in a discrete Nagumo equation: Spatial topological chaos, SIAM J. Appl. Math, 55(1995), pp. 1764-1781.
[15] L.O. Chua, CNN: A paradigm for complexity. World Scientific Series on Nonlinear Science, Series A,31. World Scietific, Singapore.(1998)
[16] L.O. Chua, K. R. Crounse, M. Hasler, P. Thiran, Pattern formation properties of autonomous cellular neural networks, IEEE Trans. Circuits Systems, 42(1995), pp. 757-774.
[17] L.O. Chua, T. Roska, The CNN paradigm, IEEE Trans. Circuits Systems, 40(1993), pp. 147-156.
[18] L.O. Chua, L. Yang, Cellular neural networks: Theory, IEEE Trans. Circuits Systems, 35(1988), pp. 1257-1272.
[19] L.O. Chua, L. Yang, Cellular neural networks: Applications, IEEE Trans. Circuits Systems, 35(1988), pp. 1273-1290.
[20] H.E. Cook, D.De Fontaine, J.E. Hilliard, A model for diffusion on cubic lattices and its application to the early stages of ordering, Acta Metallurgica, 17(1969), pp. 765-773.
[21] G.B. Ermentrout, Stable periodic solutions to discrete and continuum arrays of weakly coupled nonlinear oscillators, SIAM J. Appl. Math., 52 (1992), pp. 1665-1687.
[22] G.B. Ermentrout, N. Kopell, Inhibition-produced patterning in chains of coupled nonlinear oscillators, SIAM J. Appl. Math., 54 (1994), pp. 478-507.
[23] G.B. Ermentrout, N. Kopell, T.L. Williams, On chains of oscillators forced at one end, SIAM J. Appl. Math., 51(1991), pp.1397-1417.
[24] T. Eveneux, J.P. Laplante, Propagation failure in arrays of coupled bistable chemical reactors, J. Phys. Chem., 96(1992), pp. 4931-4934.
[25] W.J. Firth, Optical memory and spatial chaos, Phys. Rev. Lett., 61(1988), pp. 329-332.
[26] M. Hillert, A solid-solution model for inhomogeneous systems, Acta Metallurgica, 9(1961), pp.525-535.
[27] C.H. Hsu, J. Juang, S.S. Lin, W.W. Lin, Cellular neural networks: local patterns for general template, International J. of Bifurcation and Chaos, 10(2000), pp.1645-1659.
[28] C.H. Hsu, S.S. Lin, Traveling waves in lattice dynamical systems, J. Differential Equations, 164(2000),pp.431-450.
[29] J. Juang, S.S. Lin, Cellular Neural Networks: Mosaic pattern and spatial chaos, SIAM J. Appl. Math., 60(2000), pp.891-915, 2000.
[30] J. Juang, S.S. Lin, W.W. Lin, S.F. Shieh, Two dimensional spatial entropy, International J. of Bifurcation and Chaos, 10(2000), pp.2845-2852.
[31] J. Juang, S.S. Lin, Cellular Neural Networks: Defect pattern and spatial chaos, preprint.

[32] J. Juang, S.F. Shieh, On the spatial entropy of two-dimensional golden mean, International J. of Bifurcation and Chaos, 14(2004), pp.309-319.

[33] J.P. Keener, Propagation and its failure in coupled systems of discrete excitable cells, SIAM J. Appl. Math., 47(1987), pp.556-572.

[34] J.P. Keener, The effects of discrete gap junction coupling on propagation in myocardium, J. Theor. Biol., 148(1991), pp.49-82.

[35] A.L. Kimball, A. Varghese, R.L. Winslow, Simulating cardiac sinus and atrial network dynamics on the connection machine, Phys. D, 64(1993), pp.281-298.

[36] S.S. Lin, T.S. Yang, Spatial entropy of one dimensional celluar neural network, International J. of Bifurcation and Chaos, 10(2000), pp.2129-2140.

[37] S.S. Lin, T.S. Yang, On the spatial entropy and patterns of two-dimensional cellular neural network, International J. of Bifurcation and Chaos, 12(2002), pp.115-128.

[38] D. Lind, B. Marcus, An introduction to symbolic dynamics and coding, Cambrige University Press, New York, 1995.

[39] A. Lindenmayer, P. Prusinkiewicz, The algorithmic beauty of plants, Springer-Verlag, New York, 1990.

[40] R.S. Mackay, J.A. Sepulchre, Multistability in networks of weakly coupled bistable units, Phys. D, 82(1995), pp. 243-254.

[41] J. Mallet-paret, S.N. Chow, Pattern formation and spatial chaos in lattice dynamical systems I, IEEE Trans. Circuits Systems, 42(1995), pp. 746-751.

[42] C. Robinson, Dynamical Systems, CRC Press, London (1995).

[43] W. Shen, Lifted lattices, hyerbolic structures, and topological disorders in coupled map lattices, SIAM J. Appl. Math, 56(1996), 1379-1399.

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