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Hyperbolic 分佈在期貨避險之應用

在 Normal 分佈及鞅的假設下，各種最佳期貨避險比例和最小變異避險比例是等價的。然而在很多的實證市場裡，Normal 分佈的假設是被拒絕的。在這篇文章裡，我們提出用 hyperbolic 分佈來探討期貨避險的最佳避險比例問題，並且我們也實證探討 TAIEX, S & P 500, Nasdaq 100 期貨市場。

Generalized Hyperbolic Distributions in Futures Hedge

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Abstract

Under martingale and joint-normality assumptions, various optimal hedge ratios are identical to the minimum variance hedge ratio. As empirical studies usually reject the joint-normality assumption, we propose the generalized hyperbolic distribution as the joint log-return distribution of the spot and futures. Using the parameters in this distribution, we estimate several most widely-used optimal hedge ratios: minimum variance, maximum Sharpe measure and minimum generalized semivariance. Under mild assumptions on the parameters, we find that these hedge ratios are identical. Empirical studies show that our proposed models fit the TAIEX futures and S & P 500 futures very well. Numerical results for different optimal hedge ratios also verify our theoretical observations. Regarding the equivalence of these three optimal hedge ratios, our analysis suggests that the martingale property plays a much important role than the joint distribution assumption.

1 Introduction

Because of their low transaction cost, high liquidity, high leverage and ease of short position, stock index futures are among the most successful innovations in the financial markets. Besides the speculative trading, they are widely used to hedge against the market risk of the spot position. One of the most important issues for investors and portfolio managers is to calculate the optimal futures hedge ratio, the proportion of the position taken in futures to the size of the spot so that the risk exposure can be minimized.

The optimal hedge ratios typically depend on the objective functions under consideration. In literature on futures hedging, there are two different types of objective functions: the risk function to be minimized, and the utility function to be maximized. Johnson (1960) obtains the minimum variance hedge ratio by minimizing the variance of the change in the value of the hedged portfolios. On the other hand, as Adams and Montesi (1995) indicate, corporate managers are more concerned with the downside risk rather than the upside variation. A measure of the downside risk is the generalized semi-variance (GSV) where the risk is computed from the expectation of a power function of shortfalls from the target return (Bara (1975, 1978); Fishburn (1977)). De Jong et al. (1997) and Lien and Tse (1998, 2000, 2001) have calculated several GSV-minimizing hedge ratios. Regarding the utility function approach, we consider the Sharp measure (SM) criteria, i.e., the ratio of the portfolio's excess return to its volatility. Howard and D'Antonio (1984) formulate the optimal hedge ratio by maximizing the Sharp measure.

Normally, these optimal hedge ratios under different approaches are not the same. However, with the joint-normality and martingale assumptions, they are identical to the minimum variance hedge ratio. Unfortunately, many empirical studies indicate that major markets typically reject the joint-normality assumption (Chen et. al. (2001); Lien and Tse (1998)). In particular, the fat-tail property of the return distribution affects the hedging effectiveness substantially. It will be useful to find out the nature of the optimal hedge ratios under more realistic assumption. In this paper we introduce the bivariate generalized hyperbolic distributions as alternative joint distributions for returns in the spot and futures markets.

Barndorff-Nielsen (1977, 1978) develops the generalized hyperbolic (GH) distributions as a mixture of the normal distribution and the generalized inverse Gaussian (GIG) distribution first proposed in 1946 by Étienne Halphen. The class of the generalized hyperbolic distributions includes the hyperbolic distributions, the normal inverse Gaussian distributions and the variance-Gamma distributions, while the normal distribution is a limiting case of the generalized hyperbolic distributions. Uses of the generalized hyperbolic distributions have been increasing in finance literature. To model the log returns of some financial assets, Eberlein and Keller (1995) consider the hyperbolic distribution and Barndorff-Nielsen (1995) proposes the normal inverse Gaussian distribution. For more recent applications of the generalized hyperbolic distributions in finance, see Bibby and Sørensen (2003); Eberlein, Keller and Prause (1998); Rydberg (1997, 1999); Kuchler et. al. (1999); and Bingham and Kiesel (2001).

In terms of the parameters for the bivariate hyperbolic distributions, we have devel-

oped in this paper the minimum variance hedge ratio, GSV-minimizing hedge ratio and the SM-maximizing hedge ratio. Moreover, the relationships between these hedge ratios are explored. In particular, under the martingale assumption, we can still obtain the result that these hedge ratios are the same as the minimum variance hedge ratio. Based on the maximum likelihood estimation of the parameters and the numerical methods, we calculate and compare the different hedge ratios for TAIEX futures, S & P 500 futures.

The paper is divided into six sections. Section 2 first introduces the definitions and some basic properties for GIG and GH distributions. In Section 3, we study the optimal hedge ratios under different approaches and estimate these ratios in terms of the parameters for GH distributions. In Section 4, we employ the kernel density estimators and MLE method for our data. Based on these estimations of the parameters, the different hedge ratios are calculated in the fifth section. The last section provides the concluding remarks.

2 The Generalized Hyperbolic and Inverse Gaussian Distributions

2.1 The Generalized Inverse Gaussian Distributions

To introduce the generalized inverse Gaussian distribution, we first recall that the Bessel functions of the third kind with index λ can be written in integral form as

$$K_\lambda(x) = \frac{1}{2} \int_0^\infty u^{\lambda-1} e^{-\frac{1}{2}x(u^{-1}+u)} du, \quad x > 0. \quad (1)$$

From this we see that for any $\delta, \psi > 0$, the function

$$d_{GIG(\lambda, \delta, \psi)}(x) = \frac{(\psi/\delta)^\lambda}{2K_\lambda(\delta\psi)} x^{\lambda-1} e^{-\frac{1}{2}(\delta^2 x^{-1} + \psi^2 x)}, \quad x > 0. \quad (2)$$

is a probability density function on $(0, \infty)$. The distribution with the density function $d_{GIG(\lambda, \delta, \psi)}(x)$ on the positive half-line is called a generalized inverse Gaussian (**GIG**) distribution with parameters λ, δ, ψ , and denoted by $GIG(\lambda, \delta, \psi)$. For $\lambda = -1/2$, the $GIG(\lambda, \delta, \psi)$ reduces to the inverse Gaussian (IG) distribution $IG(\delta, \psi)$. By using the fact that $K_{-\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2}} x^{-\frac{1}{2}} e^{-x}$, we obtain

$$d_{IG(\delta, \psi)}(x) = \left(\frac{\delta^2}{2\pi}\right)^{\frac{1}{2}} x^{-\frac{3}{2}} e^{-\frac{\psi^2}{2x}(x - \frac{\delta}{\psi})^2}.$$

It worths to notice that $IG(\delta, \psi)$ is the law of the first time a standard Brownian motion with drift ψ reaches the level δ .

The moment generating function of the generalized inverse Gaussian distribution is given by

$$M_{GIG(\lambda, \delta, \psi)}(u) = \int_0^\infty e^{ux} d_{GIG(\lambda, \delta, \psi)}(x) dx = \left(\frac{\psi}{\sqrt{\psi^2 - 2u}}\right)^\lambda \frac{K_\lambda(\delta\sqrt{\psi^2 - 2u})}{K_\lambda(\delta\psi)} \quad (3)$$

with the restriction $2u < \psi^2$. From this and the fact that

$$K'_\lambda(u) = -\frac{\lambda}{u}K_\lambda(u) - K_{\lambda-1}(u),$$

we get

$$\mathbb{E}[GIG^r] = \left(\frac{\delta}{\psi}\right)^r \frac{K_{\lambda+r}(\delta\psi)}{K_\lambda(\delta\psi)}.$$

In particular we obtain

$$\begin{aligned}\mathbb{E}[GIG] &= \frac{\delta}{\psi} \frac{K_{\lambda+1}(\delta\psi)}{K_\lambda(\delta\psi)} \\ \text{Var}[GIG] &= \left(\frac{\delta}{\psi}\right)^2 \left[\frac{K_{\lambda+2}(\delta\psi)}{K_\lambda(\delta\psi)} - \frac{K_{\lambda+1}^2(\delta\psi)}{K_\lambda^2(\delta\psi)} \right].\end{aligned}$$

Moreover, by letting $\delta \rightarrow 0^+$ and using the fact that $K_\lambda(t) \sim \frac{1}{2}\Gamma(\lambda) \left(\frac{t}{2}\right)^{-\lambda} a s \quad t \rightarrow 0$, we get

$$\lim_{\delta \rightarrow 0^+} M_{GIG(\lambda, \delta, \psi)}(u) = \left(1 - \frac{2}{\psi^2}u\right)^{-\lambda}$$

which is the moment generating function of the Gamma distribution $\Gamma(\lambda, a)$, $a = \frac{\psi^2}{2}$, with the density function

$$d_{\Gamma(\lambda, a)}(x) = \frac{a^\lambda}{\Gamma(\lambda)} x^{\lambda-1} e^{-ax}, \quad x > 0 \quad (4)$$

where $\Gamma(\lambda)$ is the Gamma function. (Gamma distributions are often used in finance, in particular in the context of models for credit risk.) Similarly, for $\lambda < 0$ and $\delta > 0$, by letting $\psi \rightarrow 0^+$, we get the inverse Gamma distribution $I\Gamma(\lambda, \delta)$ with density

$$d_{I\Gamma(\lambda, \delta)}(x) = \left(\frac{2}{\delta^2}\right)^\lambda \frac{1}{\Gamma(-\lambda)} x^{\lambda-1} e^{-\frac{\delta^2}{2x}}, \quad x > 0.$$

2.2 The Generalized Hyperbolic Distributions

Barndorff-Nielsen (1977) introduced the class of generalized hyperbolic (GH) distributions as mean-variance mixtures of normal distributions. More precisely, one says that a random variable Z has the generalized hyperbolic distribution $GH(\lambda, \alpha, \beta, \delta, \mu)$ if

$$Z|Y = y \sim N(\mu + \beta y, y),$$

where Y is a random variable with distribution $GIG(\lambda, \delta, \sqrt{\alpha^2 - \beta^2})$ and $N(\mu + \beta y, y)$ denotes the normal distribution with mean $\mu + \beta y$ and variance y . From this, one can

easily verify that the density function for $GH(\lambda, \alpha, \beta, \delta, \mu)$ is given by the formula

$$\begin{aligned} d_{GH(\lambda, \alpha, \beta, \delta, \mu)}(x) &= \int_0^\infty d_{N(\mu + \beta y, y)}(x) d_{GIG(\lambda, \delta, \sqrt{\alpha^2 - \beta^2})}(y) dy \\ &= \left(\frac{\psi}{\delta}\right)^\lambda \frac{e^{(x-\mu)\beta}}{\sqrt{2\pi} K_\lambda(\delta\psi)} \left[\frac{\delta^2 + (x-\mu)^2}{\alpha^2} \right]^{\frac{\lambda-1}{2}} K_{\lambda-\frac{1}{2}}(\alpha\sqrt{\delta^2 + (x-\mu)^2}) \end{aligned}$$

where $\psi = \sqrt{\alpha^2 - \beta^2}$. The moment generating function for $GH(\lambda, \alpha, \beta, \delta, \mu)$ is easily computed as

$$\begin{aligned} M_{GH}(u) &= e^{u\mu} M_{GIG(\lambda, \delta, \sqrt{\alpha^2 - \beta^2})} \left(\frac{u^2}{2} + u\beta \right) \\ &= e^{u\mu} \left(\frac{\alpha^2 - \beta^2}{\alpha^2 - (\beta + u)^2} \right)^{\frac{\lambda}{2}} \frac{K_\lambda(\delta\sqrt{\alpha^2 - (\beta + u)^2})}{K_\lambda(\delta\sqrt{\alpha^2 - \beta^2})} \end{aligned} \quad (5)$$

whenever $|\beta + u| < \alpha$. As above, this implies that

$$\begin{aligned} \mathbb{E}[GH(\lambda, \alpha, \beta, \delta, \mu)] &= \mu + \beta \mathbb{E}[GIG(\lambda, \delta, \psi)], \\ Var[GH(\lambda, \alpha, \beta, \delta, \mu)] &= \mathbb{E}[GIG(\lambda, \delta, \psi)] + \beta^2 Var[GIG(\lambda, \delta, \psi)]. \end{aligned}$$

The class of hyperbolic distributions is the subclass of GH distributions obtained when λ is equal to 1. We write $H(\alpha, \beta, \delta, \mu)$ instead of $GH(1, \alpha, \beta, \delta, \mu)$. Using the fact that $K_{1/2}(z) = (\pi/2z)^{1/2} e^{-z}$, one obtains the density for $H(\alpha, \beta, \delta, \mu)$ is

$$d_{H(\alpha, \beta, \delta, \mu)}(x) = \frac{\sqrt{\alpha^2 - \beta^2}}{2\alpha\delta K_1(\delta\sqrt{\alpha^2 - \beta^2})} e^{-\alpha\sqrt{\delta^2 + (x-\mu)^2} + \beta(x-\mu)}. \quad (6)$$

The normal inverse Gaussian (NIG) distributions were introduced to finance in Barndorff-Nielsen (1995). It is a subclass of the generalized hyperbolic distributions obtained for λ equal to $-1/2$. The density of the **NIG** distribution is given by

$$d_{NIG(\alpha, \beta, \delta, \mu)}(x) = \frac{\delta}{\pi} \left[\frac{\alpha^2}{\delta^2 + (x-\mu)^2} \right]^{\frac{1}{2}} e^{\delta\psi + (x-\mu)\beta} K_1(\alpha\sqrt{\delta^2 + (x-\mu)^2}).$$

2.3 Multivariate modelling

In finance one does not look at a single asset, but at a bunch of assets. Since the assets in the market are typically highly correlated, it is natural to use multivariate distributions. A straightforward way for introducing multivariate generalized hyperbolic (MGH) distributions is via the mixtures of multivariate Normal distributions with the generalized inverse Gaussian distributions. In fact the multivariate generalized hyperbolic distributions were introduced and investigated in Barndorff-Nielsen (1978).

Let Δ be a symmetric positive-definite $d \times d$ - matrix with determinant $|\Delta| = 1$. We say that a d -dimensional random vector Z has the multivariate generalized hyperbolic distribution $MGH(\lambda, \alpha, \beta, \delta, \mu, \Delta)$ with parameters $(\lambda, \alpha, \beta, \delta, \mu, \Delta)$ if:

$$\mathbf{Z}|Y = y \sim N_d(\mu + y\Delta\beta, y\Delta),$$

where $N_d(A, B)$ denotes the d -dimensional Normal distribution with mean vector A and covariance matrix B , and Y distributed as $GIG(\lambda, \delta, \sqrt{\alpha^2 - \beta'\Delta\beta})$. Here we notice that $\lambda \in \mathbb{R}$, $\beta, \mu \in \mathbb{R}^d$, $\delta > 0$, $\alpha^2 > \beta'\Delta\beta$, and generalized hyperbolic distributions are symmetric if and only if $\beta = (0, \dots, 0)'$. For $\lambda = (d + 1)/2$ we obtain the multivariate hyperbolic distributions. For $\lambda = -1/2$ we obtain the multivariate normal inverse Gaussian distribution.

The density function of the distribution $MGH(\lambda, \alpha, \beta, \delta, \mu, \Delta)$ is given by the formula

$$d_{MGH}(x) = c_d \frac{K_{\lambda-d/2} \left(\alpha \sqrt{\delta^2 + (x - \mu)'\Delta^{-1}(x - \mu)} \right)}{\left(\alpha^{-1} \sqrt{\delta^2 + (x - \mu)'\Delta^{-1}(x - \mu)} \right)^{d/2-\lambda}} e^{(\beta'(x-\mu))} \quad (7)$$

where $c_d = \frac{(\sqrt{\alpha^2 - \beta'\Delta\beta}/\delta)^\lambda}{(2\pi)^{d/2} K_\lambda(\delta\sqrt{\alpha^2 - \beta'\Delta\beta})}$.

The moment generating function of $MGH(\lambda, \alpha, \beta, \delta, \mu, \Delta)$ distribution is given by

$$M_{MGH}(z) = e^{\mu'z} \left(\frac{\alpha^2 - \beta'\Delta\beta}{\alpha^2 - (\beta + \mu)'\Delta(\beta + z)} \right)^{\lambda/2} \frac{K_\lambda(\delta\sqrt{\alpha^2 - (\beta + z)'\Delta(\beta + z)})}{K_\lambda(\delta\sqrt{\alpha^2 - \beta'\Delta\beta})}, z \in \mathbb{R}^d$$

whenever $\alpha^2 > (\beta + u)'\Delta(\beta + u)$. The mean and covariance of MGH are given by

$$\mathbb{E}[MGH(\lambda, \alpha, \beta, \delta, \mu, \Delta)] = \mu + \Delta\beta\mathbb{E}[GIG(\lambda, \delta, \psi)], \quad (8)$$

$$Var[MGH(\lambda, \alpha, \beta, \delta, \mu, \Delta)] = \Delta\mathbb{E}[GIG(\lambda, \delta, \psi)] + \Delta\beta\beta'\Delta Var[GIG(\lambda, \delta, \psi)] \quad (9)$$

where $\psi = \sqrt{\alpha^2 - \beta'\Delta\beta}$. (For details, see, e.g., Bæsild (1981).)

3 Futures hedge ratios

We consider a decision maker. At the decision date ($t = 0$), the agent engages in the production of Q ($Q > 0$) commodity units for sale at the terminal date ($t = 1$) at the random cash price P_1 . In addition, at the decision date the agent can sell X commodity units in the futures market at the price F_0 , but must repurchase them back at the terminal date at the random futures price F_1 . Let the initial wealth be $V_0 = P_0Q$ and the end-of-period wealth be $V_1 = P_1Q + (F_0 - F_1)X$. Then we consider the wealth return that is

$$\begin{aligned} \tilde{r}_\theta &= \frac{V_1 - V_0}{V_0} = \frac{P_1Q + F_0X - F_1X - P_0Q}{P_0Q} \\ &= \frac{P_1 - P_0}{P_0} - \frac{F_1 - F_0}{F_0} \left(\frac{F_0 X}{P_0 Q} \right) = \tilde{r}_p - \theta \tilde{r}_f \end{aligned} \quad (10)$$

where $\tilde{r}_p = \frac{P_1 - P_0}{P_0}$ and $\tilde{r}_f = \frac{F_1 - F_0}{F_0}$ are one-period returns on the spot and futures positions, respectively, $h = \frac{X}{Q}$ is the hedge ratio and $\theta = h \frac{F_0}{P_0}$. (Note that θ is so-called the adjusted hedge ratio.) The main objective of hedging is to choose the optimal hedge ratio θ . However the optimal hedge ratio will depend on a particular objective function to be optimized. We recall some most widely used theoretical approaches to the optimal futures hedge ratios and compute explicitly these optimal ratios in terms of the parameters for MGH distributions. For a comprehensive review of futures hedge ratios, see Chen et al. (2002).

3.1 Minimum variance hedge ratio

The most widely-used hedge ratio is minimum variance hedge ratio which is known as the MV hedge ratio. The objective function to be minimized is the variance of \tilde{r}_θ . Clearly we have

$$Var[\tilde{r}_\theta] = \sigma_{r_p}^2 + \theta^2 \sigma_{r_f}^2 - 2\theta \rho \sigma_{r_p} \sigma_{r_f},$$

where σ_{r_p} and σ_{r_f} are standard deviations of \tilde{r}_p and \tilde{r}_f , respectively and ρ is the correlation coefficient between \tilde{r}_p and \tilde{r}_f . The MV hedge ratio is obtained by minimizing $Var[\tilde{r}_\theta]$. Simple calculation shows that the MV hedge ratio is given by

$$\theta_{MV}^* = \rho \frac{\sigma_{r_p}}{\sigma_{r_f}}. \quad (11)$$

Theorem 3.1. Assume $(\tilde{r}_f, \tilde{r}_p)'$ is distributed as $MGH(\lambda, \alpha, \beta, \delta, \mu, \Delta)$, where $\beta = (\beta_1, \beta_2)'$, $\mu = (\mu_1, \mu_2)'$, and $\Delta = \begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{pmatrix}$ is symmetry. Then we have

$$\theta_{MV}^* = \frac{\Delta_{12} \mathbb{E}[GIG] + \delta_{fp}}{\Delta_{11} \mathbb{E}[GIG] + \delta_{ff}} \quad (12)$$

where $GIG = GIG(\lambda, \delta, \sqrt{\alpha^2 - \beta' \Delta \beta})$ and

$$\begin{aligned} \delta_{ff} &= [\beta_1^2 \Delta_{11}^2 + 2\beta_1 \beta_2 \Delta_{11} \Delta_{12} + \beta_2^2 \Delta_{12}^2] Var[GIG] \\ \delta_{fp} &= [\beta_1^2 \Delta_{11} \Delta_{12} + \beta_1 \beta_2 (\Delta_{11} \Delta_{22} + \Delta_{12}^2) + \beta_2^2 \Delta_{12} \Delta_{22}] Var[GIG]. \end{aligned}$$

In particular, if $\beta = (0, \dots, 0)'$, then $\theta_{MV}^* = \frac{\Delta_{12}}{\Delta_{11}}$.

Proof. The second statement follows from the first one. We prove the first statement. By the equation (9), we obtain

$$Cov(\tilde{r}_f, \tilde{r}_p) = \Delta_{12} \mathbb{E}[GIG] + \delta_{fp}$$

and

$$\sigma_{r_f}^2 = \Delta_{11} \mathbb{E}[GIG] + \delta_{ff}.$$

Then our result follows by plugging these into the formula $\theta_{MV}^* = \frac{Cov(\tilde{r}_f, \tilde{r}_p)}{\sigma_{r_f}^2}$. \square

3.2 Sharpe hedge ratio

We consider the optimal hedge ratio that incorporates both risk and expected return. Howard and D'Antonio(1984) considered the optimal level of futures contracts by maximizing the ratio of the portfolio's excess return to its volatility, that is

$$\max_{\theta} \frac{\mu_{r_p} - \theta\mu_{r_f} - r_L}{\sigma_{\theta}}, \quad (13)$$

where σ_{θ} is the stanard deviation of \tilde{r}_{θ} , μ_{r_p}, μ_{r_f} are expected values for \tilde{r}_p and \tilde{r}_f , respectively, and r_L is the risk-free interest rate.

Consider the function

$$r(\theta) = \frac{\mu_{r_p} - \theta\mu_{r_f} - r_L}{\sigma_{\theta}}.$$

Then we have

$$r'(\theta) = \frac{\theta \left[-\sigma_{r_f}^2 (\mu_{r_p} - r_L) + \mu_{r_f} \sigma_{r_f r_p} \right] + (\mu_{r_p} - r_L) \sigma_{r_p r_f} - \sigma_{r_p}^2 \mu_{r_f}}{\sigma_{\theta}^3} \quad (14)$$

where $\sigma_{r_f r_p} = Cov(\tilde{r}_p, \tilde{r}_f)$ and, hence, the critical point for $r(\theta)$ is given by

$$\theta_s^* = \frac{\left(\frac{\sigma_{r_p}}{\sigma_{r_f}}\right)^2 \mu_{r_f} - \rho \frac{\sigma_{r_p}}{\sigma_{r_f}} (\mu_{r_p} - r_L)}{\rho \frac{\sigma_{r_p}}{\sigma_{r_f}} \mu_{r_f} - (\mu_{r_p} - r_L)}. \quad (15)$$

It follows from the equation (14) that if $\mu_{r_p} - r_L > \rho \frac{\sigma_{r_p}}{\sigma_{r_f}} \mu_{r_f}$, then $r'(\theta) > 0$ for $\theta < \theta_s^*$ and $r'(\theta) < 0$ for $\theta > \theta_s^*$. Hence the optimal hedge ratio(Sharpe hedge ratio) for the equation (13) is given by

$$\theta_s^* = \frac{\left(\frac{\sigma_{r_p}}{\sigma_{r_f}}\right)^2 \mu_{r_f} - \rho \frac{\sigma_{r_p}}{\sigma_{r_f}} (\mu_{r_p} - r_L)}{\rho \frac{\sigma_{r_p}}{\sigma_{r_f}} \mu_{r_f} - (\mu_{r_p} - r_L)}. \quad (16)$$

Similarly, if $\mu_{r_p} - r_L < \rho \frac{\sigma_{r_p}}{\sigma_{r_f}} \mu_{r_f}$, then $r(\theta)$ has a minimum at θ_s^* . (Note that if $\mu_{r_p} - r_L = \rho \frac{\sigma_{r_p}}{\sigma_{r_f}} \mu_{r_f}$, then $r(\theta)$ is strictly monotonic in θ .)

The measure of hedging effectiveness (abbreviated HE) is given in Howard and D'Antonio(1984) by

$$HE = r_s(\theta_s^*) / \left(\frac{\mu_{r_p} - r_L}{\sigma_{r_p}} \right). \quad (17)$$

Write

$$\zeta = \frac{\mu_{r_f} / \sigma_{r_f}}{(\mu_{r_p} - r_L) / \sigma_{r_p}} \quad (18)$$

(ζ is also-called the risk-return relative.) Then we have

$$\theta_s^* = \frac{\sigma_{r_p}}{\sigma_{r_f}} \left(\frac{\rho - \zeta}{1 - \zeta\rho} \right)$$

and

$$\text{HE} = \sqrt{\frac{(\rho - \zeta)^2}{1 - \rho^2} + 1}.$$

Clearly the last equality implies that

$$\text{HE} \begin{cases} > 1 & \text{when } \rho \neq \zeta \\ = 1 & \text{when } \rho = \zeta. \end{cases}$$

Moreover we have the following relationship between θ_s^* and θ_{MV}^* .

Proposition 3.2. *Assume $\mu_{r_p} > r_L$ and $1 > \zeta\rho$. Then we have*

$$\begin{cases} \theta_s^* > \theta_{MV}^* & \text{when } \mu_f < 0 \\ \theta_s^* = \theta_{MV}^* & \text{when } \mu_f = 0 \\ \theta_s^* < \theta_{MV}^* & \text{when } 0 < \mu_f. \end{cases}$$

Proof Assume that $\mu_{r_p} > r_L$. Then $1 > \zeta\rho$ if and only if $\mu_{r_p} - r_L > \rho \frac{\sigma_{r_p}}{\sigma_{r_f}} \mu_{r_f}$. Also μ_f has the same sign as that of ζ . From this, we observe

$$\begin{cases} \frac{\rho - \zeta}{1 - \rho\zeta} > \rho & \text{when } \mu_f < 0 \\ \frac{\rho - \zeta}{1 - \rho\zeta} = \rho & \text{when } \mu_f = 0 \\ \frac{\rho - \zeta}{1 - \rho\zeta} < \rho & \text{when } 0 < \mu_f. \end{cases}$$

Therefore if $\mu_f > 0$, then $\theta_s^* = \frac{\sigma_{r_p}}{\sigma_{r_f}} \frac{\rho - \zeta}{1 - \zeta\rho} < \frac{\sigma_{r_p}}{\sigma_{r_f}} \rho = \theta_{MV}^*$. Other cases follow similarly. \square

Therefore, if the expected return on the futures contract is zero and $\mu_{r_p} > r_L$, then the Sharpe hedge ratio reduces to the minimum variance hedge ratio.

Recall that $\sigma_{r_f r_p} = \text{Cov}(\tilde{r}_p, \tilde{r}_f)$. Then we have

$$\theta_s^* = \frac{\sigma_{r_p}^2 \mu_{r_f} - \sigma_{r_f r_p} (\mu_{r_p} - r_L)}{\sigma_{r_f r_p} \mu_{r_f} - \sigma_{r_f}^2 (\mu_{r_p} - r_L)}. \quad (19)$$

From this and by equations (8) and (9), we obtain

Theorem 3.3. *Assume $(\tilde{r}_f, \tilde{r}_p)'$ is distributed as in Theorem 3.1. Assume that*

$$\zeta_{fp} [(\mu_1 + \beta_1 \Delta_{11} + \beta_2 \Delta_{12}) \mathbb{E}[GIG]] < \zeta_{ff} [(\mu_2 + \beta_1 \Delta_{21} + \beta_2 \Delta_{22}) \mathbb{E}[GIG] - r_L].$$

Then we have

$$\theta_s^* = \frac{\zeta_{pp} [(\mu_1 + \beta_1 \Delta_{11} + \beta_2 \Delta_{12}) \mathbb{E}[GIG]] - \zeta_{fp} [(\mu_2 + \beta_1 \Delta_{21} + \beta_2 \Delta_{22}) \mathbb{E}[GIG] - r_L]}{\zeta_{fp} [(\mu_1 + \beta_1 \Delta_{11} + \beta_2 \Delta_{12}) \mathbb{E}[GIG]] - \zeta_{ff} [(\mu_2 + \beta_1 \Delta_{21} + \beta_2 \Delta_{22}) \mathbb{E}[GIG] - r_L]} \quad (20)$$

where $\delta_{ff}, \delta_{fp}, GIG$ are the same as in Theorem 3.1 and

$$\begin{aligned}\delta_{pp} &= [\beta_1^2 \Delta_{21}^2 + 2\beta_1\beta_2 \Delta_{21}\Delta_{22} + \beta_2^2 \Delta_{22}^2] \text{Var}[GIG] \\ \zeta_{ff} &= \Delta_{11}\mathbb{E}[GIG] + \delta_{ff} \\ \zeta_{fp} &= \Delta_{12}\mathbb{E}[GIG] + \delta_{fp} \\ \zeta_{pp} &= \Delta_{22}\mathbb{E}[GIG] + \delta_{pp}.\end{aligned}$$

3.3 Minimum generalized semivariance hedge ratio

In this case the optimal hedge ratio is obtained by minimizing the generalized semi-variance(GSV) given below:

$$L_n(c, X) = \int_{-\infty}^c (c-x)^n dF(x), \quad n > 0, \quad (21)$$

where $F(\cdot)$ is the probability distribution function of the return X . The GSV is specified by two parameters: the target return c and the power of the shortfall n . (Note that if the density function of X is symmetric at c , then we obtain $L_2(c, X) = \frac{\text{Var}(X)}{2}$. Hence in this case, the GSV approach is the same as that of the minimum variance.) The GSV, due to its emphasis on the returns below the target return, is consistent with the risk perceived by managers(see Lien and Tse (2001)).

For futures hedge, we write

$$L_n(c, \theta) = L_n(c, \tilde{r}_f - \theta \tilde{r}_p). \quad (22)$$

Assume that $n > 1$ and $h(r_f, r_p)$ is a joint density function of \tilde{r}_f and \tilde{r}_p . Then we have

$$L_n(c, \theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{c+\theta r_f} (c - r_p + \theta r_f)^n h(r_f, r_p) dr_p dr_f.$$

Simple calculation gives

$$\frac{\partial}{\partial \theta} L_n(c, \theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{c+\theta r_f} n r_f (c - r_p + \theta r_f)^{n-1} h(r_f, r_p) dr_p dr_f$$

and

$$\frac{\partial^2}{\partial \theta^2} L_n(c, \theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{c+\theta r_f} n(n-1) r_f^2 (c - r_p + \theta r_f)^{n-2} h(r_f, r_p) dr_p dr_f.$$

Since $\frac{\partial^2}{\partial \theta^2} L_n(c, \theta) > 0$, the minimum for $L_n(c, \theta)$ occurs at the unique critical point(if it exists.) If the futures and spot returns are jointly normally distributed and if the future price is unbiased (i.e., expected futures price change is zero), Lien and Tse (1998) showed that the minimum GSV hedge ratio is the same as the minimum-variance hedge ratio. For later application, we summarize their arguments here.

Suppose that $(\tilde{r}_f, \tilde{r}_p)$ is bivariate normal distributed. The joint density $h(r_f, r_p)$ is characterized by means $\mu_{r_i} = E(\tilde{r}_i)$, $i = p, f$, and by the covariance $\sigma_{r_i r_j} = \text{cov}(\tilde{r}_i, \tilde{r}_j)$, $i, j = p, f$. Write $x = c + \theta r_f - r_p$ and $y = r_f$. Then we have

$$\frac{\partial}{\partial \theta} L_n(c, \theta) = \int_0^\infty \int_{-\infty}^\infty n x^{n-1} y h(y, c - x + \theta y) dy dx. \quad (23)$$

The function $h(y, c - x + \theta y)$ can be decomposed as $\frac{1}{2\pi\Lambda} e^{-\frac{1}{2}AB}$, where $\Lambda^2 = \sigma_{r_p}^2 \sigma_{r_f}^2 - (\sigma_{r_p r_f})^2$, $\sigma_{r_\theta}^2 = \theta^2 \sigma_{r_f}^2 - 2\theta \sigma_{r_p r_f} + \sigma_{r_p}^2$, and

$$\begin{aligned} A &= \left[y + (c - x - \mu_{r_p})(\theta \sigma_{r_f}^2 - \sigma_{r_p r_f}) \sigma_{r_\theta}^{-2} - \mu_{r_f}(\sigma_{r_p}^2 - \theta \sigma_{r_p r_f}) \sigma_{r_\theta}^{-2} \right]^2 \Lambda^{-2} \sigma_{r_\theta}^2 \\ B &= (c - x - \mu_{r_p} + \theta \mu_{r_f})^2 \sigma_{r_\theta}^{-2}. \end{aligned}$$

Plugging this into equation (23) and then integrating with respect to y give

$$\frac{\partial}{\partial \theta} L_n(c, \theta) = \int_0^\infty \frac{1}{\sqrt{2\pi}} n x^{n-1} \sigma_{r_\theta}^{-3} k_c(x, \theta) e^{-\frac{1}{2} \left(\frac{c-x-\mu_{r_p}+\theta\mu_{r_f}}{\sigma_{r_\theta}} \right)^2} dx \quad (24)$$

where $k_c(x, \theta) = -(c - x - \mu_{r_p})(\theta \sigma_{r_f}^2 - \sigma_{r_p r_f}) + \mu_{r_f}(\sigma_{r_p}^2 - \theta \sigma_{r_p r_f})$. In the case of an unbiased futures market (i.e., $\mu_{r_f} = 0$), then the above equation implies that $\frac{\sigma_{r_p r_f}}{\sigma_{r_f}^2}$ is a critical point for $L_n(c, \theta)$. Hence, by remark above, the minimum generalized semivariance hedge ratio is established at $\theta_{GSV}^* = \frac{\sigma_{r_p r_f}}{\sigma_{r_f}^2} = \theta_{MV}^*$, $\forall n > 1$.

Next, we consider the case that \tilde{r}_f and \tilde{r}_p are distributed as symmetric bivariate generalized hyperbolic distribution with parameters $(\lambda, \alpha, 0, \delta, \mu, \Delta)$. Recall that given $Y = y$, \tilde{r}_f and \tilde{r}_p are distributed jointly as bivariate normal distribution with mean vector $(\mu_1, \mu_2)'$ and covariance matrix $y\Delta$. Write $h_{(\mu, y\Delta)}(r_f, r_p)$ for the joint density of this distribution. Then the generalized semivariance is given by the formula

$$L_n(c, \theta) = \int_{-\infty}^\infty \int_{-\infty}^{c+\theta r_f} \int_0^\infty (c - r_p + \theta r_f)^n h_{(\mu, y\Delta)}(r_f, r_p) d_{GIG}(y) dy dr_p dr_f$$

where $d_{GIG}(y)$ is the density function for the $GIG(\lambda, \delta, \alpha)$ distribution.

Similar arguments as above and using the formula (24) give

$$\frac{\partial}{\partial \theta} L_n(c, \theta) = \int_0^\infty \int_0^\infty \frac{1}{\sqrt{2\pi}} n x^{n-1} \sigma_\theta^{-3} y k_c(x, \theta) e^{-\frac{1}{2} \left(\frac{c-x-\mu_2+\theta\mu_1}{\sigma_\theta} \right)^2} d_{GIG}(y) dy dx$$

where $\sigma_\theta^2 = y(\Delta_{22} + \theta^2 \Delta_{11} - 2\theta \Delta_{12})$ and $k_c(x, \theta) = -(c - x - \mu_2)(\theta \Delta_{11} - \Delta_{12}) + \mu_1(\Delta_{22} - \theta \Delta_{12})$. Clearly, if $\mu_1 = 0$ and $\theta = \frac{\Delta_{12}}{\Delta_{11}}$, then $k_c(x, \theta) = 0$. Hence $L_n(c, \theta)$ has a critical point at $\frac{\Delta_{12}}{\Delta_{11}}$. From this, we obtain the following.

Theorem 3.4. *Assume $(\tilde{r}_f, \tilde{r}_p)$ is the same as in Theorem 3.1. If $\beta = 0$ and $\mu_1 = \mu_{r_f} = 0$, then the minimum GSV hedge ratio is the same as the minimum variance hedge ratio (i.e., $\theta_{GSV}^* = \theta_{MV}^* = \frac{\Delta_{12}}{\Delta_{11}}$).*

In empirical studies, the true distribution is unknown or complicated. Then θ_{GSV}^* can be estimated from the sample by using the so-called empirical distribution method adapted in, e.g., Price et al. (1982) and Harlow (1991). Suppose we have m observations of $(\tilde{r}_f, \tilde{r}_p)$, say, $(r_f(i), r_p(i)), i = 1, 2, \dots, m$. From this, the GSV can be estimated by the formula:

$$L_n^{obs}(c, \theta) = \frac{1}{m} \sum_{i=1}^m (c - r_{i,\theta})^n I_{r_{i,\theta} \leq c}, \quad (25)$$

where $r_{i,\theta} = r_p(i) - \theta r_f(i)$. Given c and n , numerical methods can be used to search the hedge ratio that minimizing the sample GSV, $L_n^{obs}(c, \theta)$. (We will write θ_{GSV}^{obs} for this numerical value.)

4 Estimation and Simulation

4.1 Kernel Density Estimators

Assumed that we have n independent observations x_1, \dots, x_n from the random variable X with the unknown density function f . The kernel density estimator for the estimation of f is given by

$$\hat{f}_h(x) = \frac{1}{nh} \sum_{i=1}^n K\left(\frac{x - x_i}{h}\right), \quad x \in R \quad (26)$$

where K is a so-called kernel function and h is the bandwidth. In this paper we work with the Gaussian kernel: $K(x) = 1/\sqrt{2\pi} \exp\{-x^2/2\}$ and $h = (\frac{4}{3})^{1/5} \sigma n^{-1/5}$. (For more details, see Scott(1979).) Meanwhile it is worth noting that Lien and Tse (2000) proposed the kernel density estimation method to estimate the probability distribution of the portfolio return for every θ , and then grid search methods was adapted to find the optimum GSV hedge ratio.

Table 1: Mean, standard deviation, skewness and kurtosis of daily log returns of major indices and futures
Insert Table 1 here.

In Table 1 we summarize the empirical mean and standard deviation for the daily log returns of major indices and futures over the period from January 2000 until December of 2004. Figure 1 shows the Gaussian kernel density estimators together with the fitted normal distributions, with parameters from Table 1. We see that the Gaussian kernel density estimators have sharp peaked distributions and heavy tail behavior than that of normal distributions.

4.2 Maximum-Likelihood Estimation

We focus on how to estimate the parameters of a density function $f(x; \Theta)$, where Θ is the set of parameters to be estimated. Suppose that we have n independent observations x_1, \dots, x_n of a random variable X with the density function $f(x; \Theta)$. The maximum likelihood estimator $\hat{\theta}_{MLE}$ is the parameter set that maximizes the likelihood function

$$L(\Theta) = \prod_{i=1}^n f(x_i; \Theta).$$

Clearly this is equivalent to maximizing the logarithm of the likelihood function:

$$\log L(\Theta) = \sum_{i=1}^n \log f(x_i; \Theta).$$

The log-likelihood function for hyperbolic distribution $H(\alpha, \beta, \delta, \mu)$ is given by

$$\begin{aligned} \ell_{H(\alpha, \beta, \delta, \mu)}(\Theta) &= n \left(\log \sqrt{\alpha^2 - \beta^2} - \log 2 - \log \alpha - \log \delta - \log K_1(\delta \sqrt{\alpha^2 - \beta^2}) \right) \\ &\quad + \sum_{i=1}^n \left[-\alpha \sqrt{\delta^2 + (x_i - \mu)^2} + \beta(x_i - \mu) \right]. \end{aligned}$$

The first derivatives are:

$$\begin{aligned} \frac{\partial \ell_H}{\partial \alpha} &= n \left[\frac{2\alpha}{\alpha^2 - \beta^2} - \frac{1}{\alpha} + \frac{\alpha\delta}{\sqrt{\alpha^2 - \beta^2}} \frac{K_0(\delta \sqrt{\alpha^2 - \beta^2})}{K_1(\delta \sqrt{\alpha^2 - \beta^2})} \right] - \sum_{i=1}^n \sqrt{\delta^2 + (x_i - \mu)^2} \\ \frac{\partial \ell_H}{\partial \beta} &= n \left[\frac{-2\beta}{\alpha^2 - \beta^2} - \frac{\delta\beta}{\sqrt{\alpha^2 - \beta^2}} \frac{K_0(\delta \sqrt{\alpha^2 - \beta^2})}{K_1(\delta \sqrt{\alpha^2 - \beta^2})} - \mu \right] + \sum_{i=1}^n x_i \\ \frac{\partial \ell_H}{\partial \delta} &= n \left[\frac{\sqrt{\alpha^2 - \beta^2} K_0(\delta \sqrt{\alpha^2 - \beta^2})}{K_1(\delta \sqrt{\alpha^2 - \beta^2})} \right] - \alpha\delta \sum_{i=1}^n \frac{1}{\sqrt{\delta^2 + (x_i - \mu)^2}} \\ \frac{\partial \ell_H}{\partial \mu} &= \alpha \sum_{i=1}^n \frac{(x_i - \mu)}{\sqrt{\delta^2 + (x_i - \mu)^2}} - n\beta. \end{aligned}$$

Table 2: MLE parameters for hyperbolic distribution (d=1) Insert Table 2 here.

Figure 2 shows the Gaussian kernel density estimators based on the daily log returns of major indices and futures over the period from 2000 until the end of 2004, together with the fitted hyperbolic distributions, with parameters from Table 2. Compared with Figure 1, in which the normal counterpart were plotted, we see a significant improvement.

The symmetric MGH density function is given by the formula

$$\frac{(\alpha/\delta)^\lambda}{(2\pi)^{d/2} K_\lambda(\alpha\delta)} \frac{K_{\lambda - \frac{d}{2}}(\alpha \sqrt{\delta^2 + (x - \mu)' \Delta^{-1} (x - \mu)})}{(\alpha^{-1} \sqrt{\delta^2 + (x - \mu)' \Delta^{-1} (x - \mu)})^{\frac{d}{2} - \lambda}}.$$

Note that $K_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2}}z^{-\frac{1}{2}}e^{-z}$. Hence the two-dimensional symmetric hyperbolic distributions (i.e., $\beta = 0$ and $\lambda = \frac{3}{2}$) has the density

$$\mathbf{H}_2 = \frac{(\alpha/\delta)^{3/2}}{2^{3/2}\sqrt{\pi}\alpha K_{\frac{3}{2}}(\alpha\delta)} e^{-\alpha\sqrt{\delta^2+(x-\mu)'\Delta^{-1}(x-\mu)}}.$$

From this, we obtain the log-likelihood function for two-dimensional symmetric hyperbolic distributions :

$$\begin{aligned} \ell_{H_2} = & n \left[\frac{3}{2} \log \frac{\alpha}{\delta} - \frac{3}{2} \log 2 - \frac{1}{2} \log \pi - \log \alpha - \log K_{\frac{3}{2}}(\alpha\delta) \right] \\ & - \alpha \sum_{i=1}^n \sqrt{\delta^2 + (x_i - \mu)'\Delta^{-1}(x_i - \mu)}. \end{aligned}$$

Table 3: MLE parameters for symmetric hyperbolic distribution (d=2) Insert Table 3 here.

Figure 4 shows the fitted symmetric hyperbolic distributions with parameters from Table 3.

4.3 Comparison of the Estimates

Various distances between two distributions have been proposed in literature. The Kolmogorov-Smirnov distance is defined as the supremum over the absolute difference between the two cumulative density functions, i.e.,

$$D_K = \max_{x \in \mathbb{R}} |F_{emp}(x) - F_{est}(x)|$$

where F_{emp} and F_{est} are the empirical and the estimated CDFs. The Anderson and Darling statistic is given by

$$D_{AD} = \max_{x \in \mathbb{R}} \frac{|F_{emp}(x) - F_{est}(x)|}{\sqrt{F_{est}(x)(1 - F_{est}(x))}}.$$

In Table 4, for both Kolmogorov-Smirnov and Anderson-Darling distances, we get better results for the GH distributions than for the normal distributions.

Table 4: Distance between the estimated and empirical cumulative density functions Insert Table 4 here.

4.4 Simulation of Generalized Hyperbolic Random Variables

From the representation of GH distribution as a conditional normal distribution mixed with the generalized inverse Gaussian, a schematic representation of the algorithm reads as follows.

1. Sample Y from $\mathbf{GIG}(\lambda, \delta, \psi)$ distribution;
2. Sample ε from $N(0, 1)$;
3. Return $X = \mu + \beta Y + \sqrt{Y}\varepsilon$.

Similarly, for simulating a MGH distributed random vector, we have :

1. Set $\Delta = L^T L$ via Cholesky decomposition;
2. Sample Y from $\mathbf{GIG}(\lambda, \delta, \psi)$ distribution;
3. Sample Z from $N(0, I)$, where I is $d \times d$ -identity matrix;
4. Return $X = \mu + Y\Delta\beta + \sqrt{Y}L^T Z$.

The efficiency of the above algorithms depends on the method of sampling the generalized inverse Gaussian distributions. Atkinson (1982) applied the method of rejection algorithm to sampling GIG. We summarize his results below.

Consider a generalized inverse Gaussian distribution $GIG(\lambda, \delta, \psi)$. We write the density function as $d_{GIG(\lambda, \delta, \psi)} = ce(\lambda, a, b)$ and set

$$t = m(\lambda, a, b) = \begin{cases} \frac{\lambda-1+\sqrt{(1-\lambda)^2+ab}}{b}, & b > 0 \\ \frac{\gamma}{2(1-\lambda)}, & b = 0 \end{cases}$$

where $e_{(\lambda, a, b)}(x) = x^{\lambda-1}e^{-0.5(ax^{-1}+bx)}$, $a = \delta^2$, and $b = \psi^2$. For any two constants s and p (to be chosen later), set

$$\begin{aligned} S_1 &= e_{(\lambda, a, b+2s)}(x_L) \quad \text{where} \quad x_L = m(\lambda, a, b+2s) \\ S_2 &= e_{(\lambda, a, b-2p)}(x_R) \quad \text{where} \quad x_R = m(\lambda, a, b-2p) \end{aligned}$$

and

$$k = \frac{(e^{st} - 1)/s}{S_2} + \frac{e^{-pt}/p}{S_1}.$$

Write $k_1 = 1/kS_2$, $k_2 = 1/kS_1$ and $r = k_1 \frac{(e^{st}-1)}{s}$.

ALGORITHM GIG

1. Generate independent U and U^* , where U and U^* are uniformly distributed on $(0, 1)$. If $U > r$, go to 2.
 $x = \frac{1}{s} \log \left(1 + \frac{sU}{k_1} \right)$.
 If $\log U^* > \log \{e_{(\lambda, a, b+2s)}(x)/S_1\}$ go to 1. Otherwise return x .

2. $x = \frac{-1}{p} \log \left\{ \frac{p}{k_2} (1 - U) \right\}$.
 If $\log U^* > \log \{e_{(\lambda, a, b-2p)}(x)/S_2\}$ go to 1. Otherwise return x .

The algorithm with highest efficiency is found by choosing the values of s and p to minimize the function $S_1 \frac{(e^{st}-1)}{s} + S_2 \frac{e^{-pt}}{p}$.

5 Empirical Analysis

This section empirically analyzes the futures hedging for the spot markets of TAIEX, S&P 500 and Nasdaq 100 using the optimal hedge ratio formulae developed in the earlier sections. We consider the spot and futures prices in our tests. For the futures series, the closing settlement prices of the contracts of the nearest month are used. For the spot market, we adopt the closing indexes as the prices series. All data consist of daily observations of these prices from January 2000 through December 2004.

Based on the MLE estimation procedures discussed in Section 4, parameters of the distributions are obtained. We use different statistical tests to examine the goodness of fit of the estimated distributions. Table 5 depicts the result of the Kolmogorov-Smirnov test of normal distribution in various markets. It shows that the normality hypotheses are rejected significantly for all the spot and the futures markets, with P-values ranging from 0.000037 to 0.0133. On the other hand, under the hyperbolic distributions, the data fit very well at a 5% significance level as shown in Table 6. Moreover the bivariate symmetric hyperbolic distribution hypothesis is not rejected at the level of 5%, for TAIEX and S & P 500. However the same hypothesis is rejected for Nasdaq 100. (See Table 7.)

Table 5: Kolmogorov-Smirnov test of normal distribution
 Insert Table 5 here.

Table 6: Kolmogorov-Smirnov test of hyperbolic distribution
 Insert Table 6 here.

Table 7: χ^2 -test of symmetric bivariate hyperbolic distribution
 Insert Table 7 here.

Table 8 provides the minimum variance hedge ratios for TAIEX and S&P 500 using the estimated parameters of the symmetric bivariate hyperbolic distribution. As for the Sharpe hedge ratio, we first estimate the value of $\mu_{r_p} - \rho \frac{\sigma_{r_p}}{\sigma_{r_f}} \mu_{r_f}$. (See Table 9.) For existence of maximum, we need the condition that $r_L < \mu_{r_p} - \rho \frac{\sigma_{r_p}}{\sigma_{r_f}} \mu_{r_f}$ (see section 3.2). In our cases, it does not exist for any reasonable value of r_L . The dependence of the Sharp measure on the hedge ratio (for $r_L = 10^{-4}$) are shown in Figure 4. From the figure, it is seen that we obtain the minimum instead of the maximum.

Table 8: Estimated minimum variance hedge ratios under H_2 distribution
 Insert Table 8 here.

Table 9: Estimated values of $\mu_{r_p} - \rho(\sigma_{r_p}/\sigma_{r_f})\mu_{r_f}$ under H_2 distribution Insert Table 9 here.

To find the GSV-minimizing hedge ratios, we consider several target returns (TR=0.005, 0, and 0.005) with power of shortfall (n) equal to 1.5 (see Table 10) and 2 (see Table 11). For comparison purpose, the optimal hedge ratios are calculated by Monte Carlo method from parameters of the bivariate hyperbolic distribution and of the normal distribution. The hedge ratios from the empirical distribution obtained from the samples are also provided. As symmetry property is not rejected for TAIEX or S&P 500 markets (see Table 7), it is noted that GSV-minimizing hedge ratios (see Table 10 and 11) are very close to the minimum variance hedge ratios (see Table 8), which are consistent with Theorem 3.4 as discussed in the earlier section.

Table 10: GSV-minimizing hedge ratios ($n=1.5$) Insert Table 10 here.

Table 11: GSV-minimizing hedge ratios ($n=2$) Insert Table 11 here.

Overall, our empirical data from TAIEX spot and futures and from S&P 500 spot and futures support the proposed model very well. Under the bivariate symmetric hyperbolic distribution, the optimal hedge ratios calculated from different approaches are found consistent with the theoretical implication.

6 Concluding Remarks

Although there are many different theoretical approaches to the optimal futures hedge ratios, under the martingale and joint-normality assumptions, various optimal hedge ratios are identical to the minimum variance hedge ratio. However empirical studies show that major market data reject the joint-normality assumption. In this paper we propose the generalized hyperbolic distribution as the joint log-return distribution of the spot and futures. Using the parameters for generalized hyperbolic distributions, we estimate several most widely-used optimal hedge ratios: minimum variance, maximum Sharpe measure and minimum generalized semivariance. In particular, under mild assumptions on the parameters, we obtain that these theoretical approaches are equivalent. Empirical studies show that our proposed models fit the TAIEX futures and S & P 500 futures very well. Numerical results for different optimal hedge ratios also verify our theoretical observations. Moreover, regarding the equivalence of these three optimal hedge ratios, our empirical studies and simulation results suggest that the martingale property plays a much important role than the joint distribution assumption.

Although our empirical studies show that the value of complicated and sophisticated estimation methods for GSV hedge ratio is negligible, which is consistent with the findings of Lence (1995), conditional heteroskedasticity and stochastic volatility are observed in

many spot and futures price series. This implies that the optimal hedge strategy should be time-dependent. To account for this dynamic property, parametric specifications of the joint distribution are required. Based on our work here, it is natural to extend the results to time-varying hedge ratios, which will be the task in the near future.

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Table 1: Mean, standard deviation, skewness and kurtosis of daily log returns of major indices and futures

	mean $\times 10^{-4}$	standard deviation	skewness	kurtosis
TAIEX Futures	-2.8313	0.0204	-0.0807	4.8848
TAIEX Index	-2.8154	0.0177	0.0103	4.1432
S&P 500 Futures	-1.5092	0.0128	0.0562	4.8801
S&P 500 Index	-1.4577	0.0127	0.1221	4.7822
Nasdaq 100 Futures	-6.8444	0.0265	0.1478	5.3631
Nasdaq 100 Index	-6.7680	0.0269	0.3078	5.5348

Table 2: MLE parameters for hyperbolic distribution(d=1)

	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\delta}$	$\hat{\mu}$
TAIEX Futures	69.995	1.5937	0.0027	-9.5635×10^{-4}
TAIEX Index	89.647	3.3658	0.0095	-0.0014
S&P 500 Futures	122.474	-5.8050	0.0059	8.0431×10^{-4}
S&P 500 Index	125.312	-3.4285	0.0069	4.1696×10^{-4}
Nasdaq 100 Futures	53.510	-3.8168	0.0028	0.0021
Nasdaq 100 Index	54.506	-2.2591	0.0071	9.9398×10^{-4}

Table 3: MLE parameters for symmetric hyperbolic distributions(d=2)

	$\hat{\alpha}$	$\hat{\delta}$	$\hat{\mu} \times 10^{-4}$	$\hat{\Delta}_{11}$	$\hat{\Delta}_{12}$	$\hat{\Delta}_{21}$
TAIEX			-2.8313			
Futures/Index	161.1448	0.0027	-2.8154	3.3632	2.7491	2.5445
S&P500			-1.5092			
Futures/Index	298.1549	0.0017	-1.4577	4.3914	4.2464	4.3339
Nasdaq100			-6.8444			
Futures/Index	157.1192	0.0000	-6.7680	4.5243	4.4855	4.6681

Table 4: Distance between the estimated and empirical cumulative density functions

Kolmogorov-Smirnov		
	GH	Normal
TAIEX Futures	0.0153	0.0655
TAIEX Index	0.0152	0.0434
S&P 500 Futures	0.0148	0.0488
S&P 500 Index	0.0142	0.0445
Nasdaq 100 Futures	0.0217	0.0619
Nasdaq 100 Index	0.0234	0.0628
Anderson-Darling		
TAIEX Futures	0.0748	0.4326
TAIEX Index	0.0608	0.2466
S&P 500 Futures	0.0779	1.0848
S&P 500 Index	0.0653	0.6985
Nasdaq 100 Futures	0.0827	16.2749
Nasdaq 100 Index	0.1192	95.3173

Table 5: Kolmogorov-Smirnov test of normal distribution

Markets	P -value
TAIEX Futures	3.7457×10^{-5}
TAIEX Index	0.0168
S&P 500 Futures	0.0049
S&P 500 Index	0.0133
Nasdaq 100 Futures	1.2275×10^{-4}
Nasdaq 100 Index	9.3381×10^{-5}

Table 6: Kolmogorov-Smirnov test of hyperbolic distribution

Markets	P -value
TAIEX Futures	0.9277
TAIEX Index	0.9334
S&P 500 Futures	0.9450
S&P 500 Index	0.9628
Nasdaq 100 Futures	0.5959
Nasdaq 100 Index	0.4994

Table 7: χ^2 -test of symmetric bivariate hyperbolic distribution

Markets	$\hat{\chi}^2_1$	n_1	P_1 -value	$\hat{\chi}^2_2$	n_2	P_2 -value
TAIEX						
Futures/Index	28.45	26	0.0555	18.78	21	0.1302
S&P 500						
Futures/Index	30.24	32	0.1769	34.28	33	0.1021
Nasdaq 100						
Futures/Index	65.47	29	1.8364×10^{-6}	63.21	30	7.4001×10^{-6}

*(To avoid any problems arising from partition sensitivity, two different estimation procedures were considered. To do this, we partition the whole space into cells of equal size, and compute the expected number of each cell. Now the first procedure is to count cells of which the expected value of observations greater than five, and then integrate all other cells of which the expected value of observations less than five into a new cell. The second procedure is very much the same as the first, but now combine all the cells of which the expected value of observations less than five with the random chosen cell of which the expected number is greater than 5. n_i is the number of the modified cells in procedure i .)

Table 8: Estimated minimum variance hedge ratios under \mathbf{H}_2 distribution

	θ_{MV}^*
TAIEX	0.8175
S&P500	0.9670

Table 9: Estimated value of $\mu_{r_p} - \rho \frac{\sigma_{r_p}}{\sigma_{r_f}} \mu_{r_f}$ under \mathbf{H}_2 distribution

	$\mu_{r_s} - \rho \frac{\sigma_{r_p}}{\sigma_{r_f}} \mu_{r_f}$
TAIEX	-5.0106×10^{-5}
S&P500	1.6494×10^{-7}

Table 10: GSV-minimizing Hedge Ratios (n=1.5)

	T R	$\theta_{GSV}^{N_2}$	$\theta_{GSV}^{H_2}$	θ_{GSV}^{sample}
TAIEX	-0.005	0.8165 (2.54)	0.8182 (2.99)	0.8266
	0	0.8163 (1.42)	0.8183 (1.41)	0.8224
	0.005	0.8164 (1.89)	0.8185 (1.56)	0.8320
S&P 500	-0.005	0.9650 (3.80)	0.9696 (4.70)	0.9743
	0	0.9670 (0.80)	0.9701 (0.71)	0.9749
	0.005	0.9659 (2.43)	0.9672 (1.99)	0.9863

TR= Target Return. $\theta_{GSV}^{N_2}$ = GSV-based optimal hedge ratio from Monte Carlo Simulation using estimated normal distribution. $\theta_{GSV}^{H_2}$ = GSV-based optimal hedge ratio from Monte Carlo Simulation using estimated H_2 distribution, and θ_{GSV}^{sample} =GSV-based optimal hedge ratio computed from sampled market data. The rows in each box without brackets are the means and the rows with brackets are the variances measured in units of 10^{-4} .

Table 11: GSV-minimizing Hedge Ratios (n=2)

	T R	$\theta_{GSV}^{N_2}$	$\theta_{GSV}^{H_2}$	θ_{GSV}^{sample}
TAIEX	-0.005	0.8121 (1.65)	0.8123 (3.65)	0.7884
	0	0.8143 (1.24)	0.8176 (2.12)	0.8131
	0.005	0.8161 (1.51)	0.8208 (1.90)	0.8271
S&P 500	-0.005	0.9608 (1.85)	0.9648 (6.86)	0.8764
	0	0.9656 (0.74)	0.9688 (1.09)	0.9643
	0.005	0.9687 (1.70)	0.9750 (1.36)	0.9678

TR= Target Return. $\theta_{GSV}^{N_2}$ = GSV-based optimal hedge ratio from Monte Carlo Simulation using estimated normal distribution. $\theta_{GSV}^{H_2}$ = GSV-based optimal hedge ratio from Monte Carlo Simulation using estimated H_2 distribution, and θ_{GSV}^{sample} =GSV-based optimal hedge ratio computed from sampled market data. The rows in each box without brackets are the means and the rows with brackets are the variances measured in units of 10^{-4} .

Figure 1: Normal density and Gaussian kernel density estimators

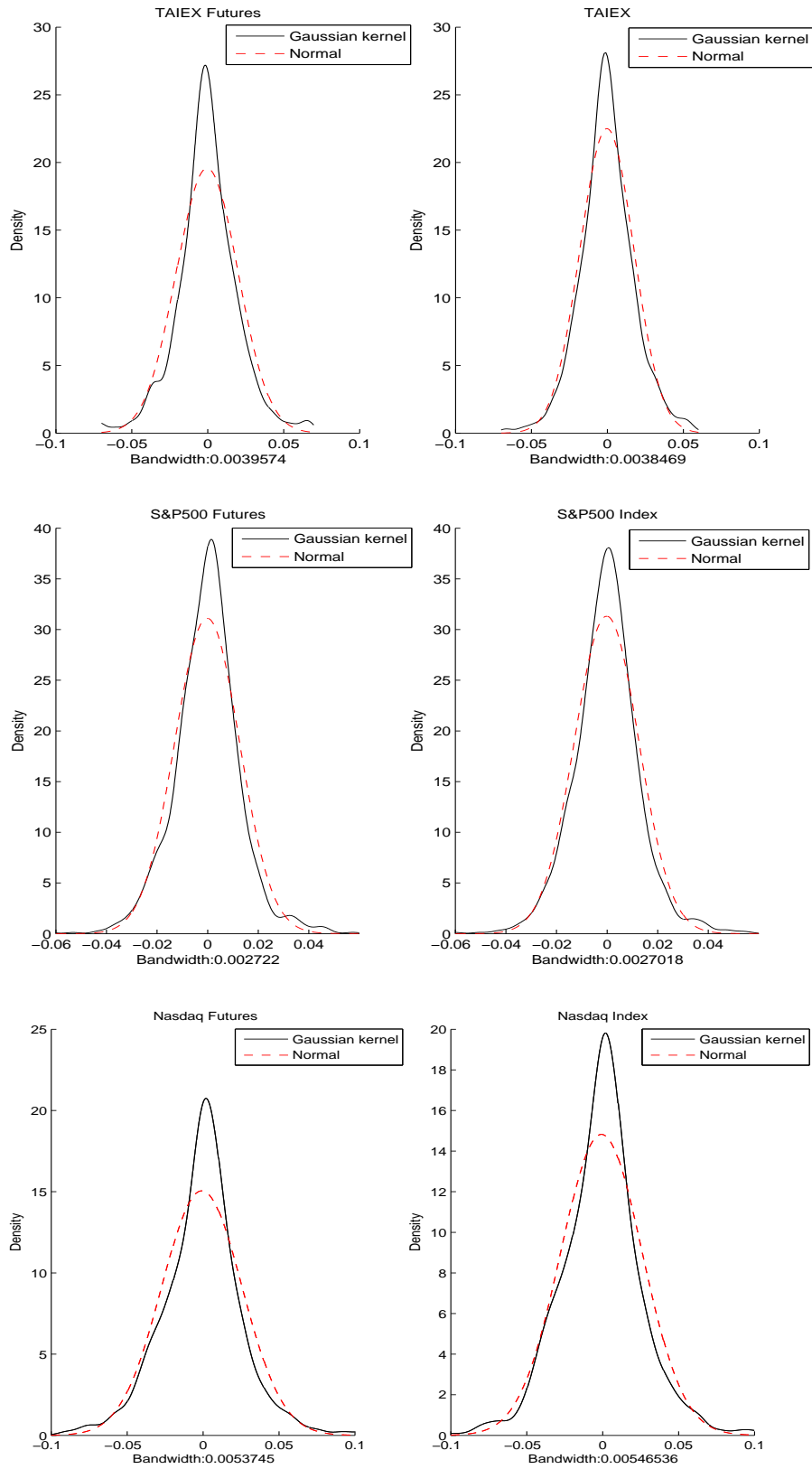


Figure 2: Log-densities of daily log returns of major indices and futures(2000-2004)

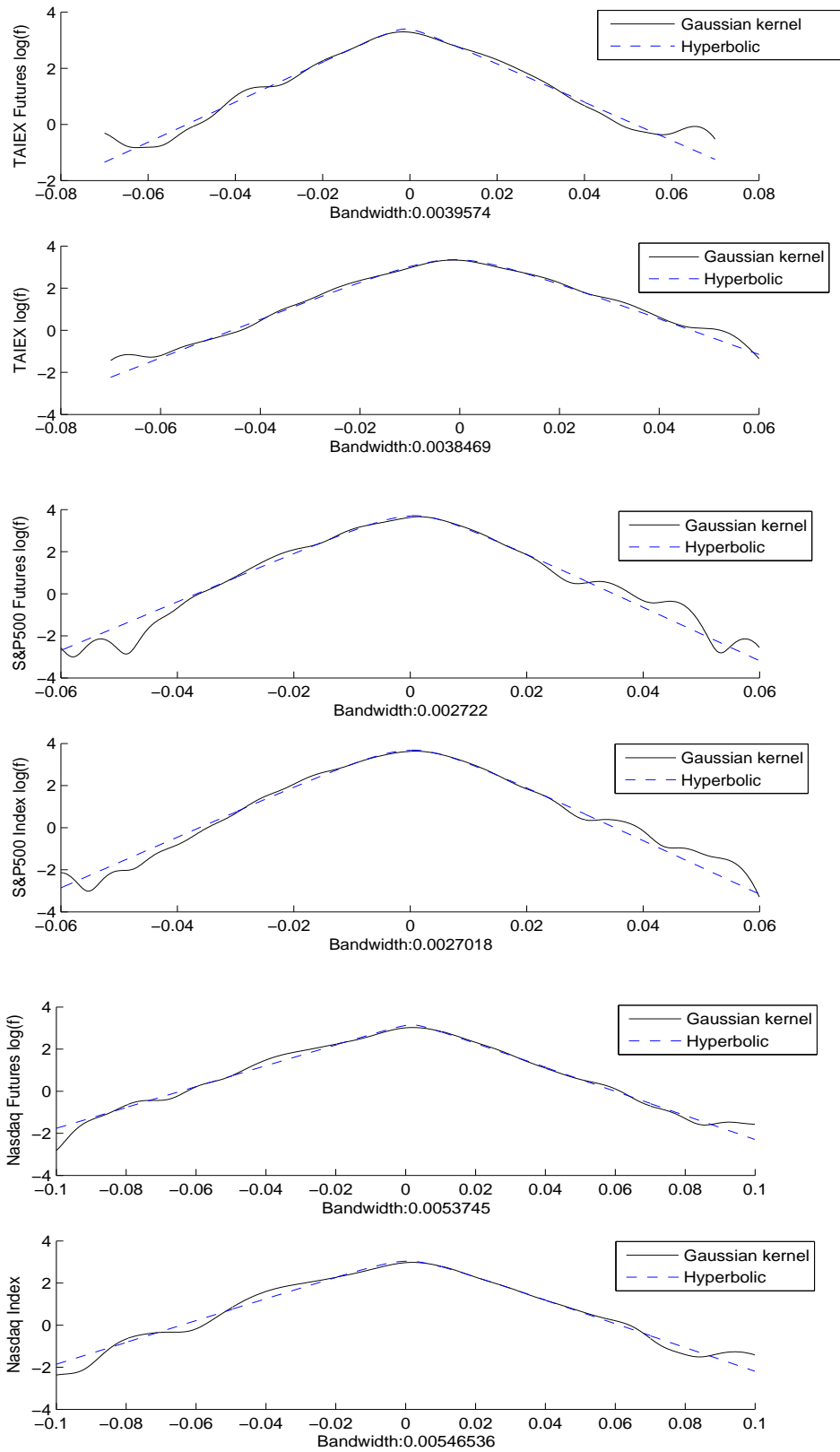


Figure 3: Estimated symmetric H_2 distributions

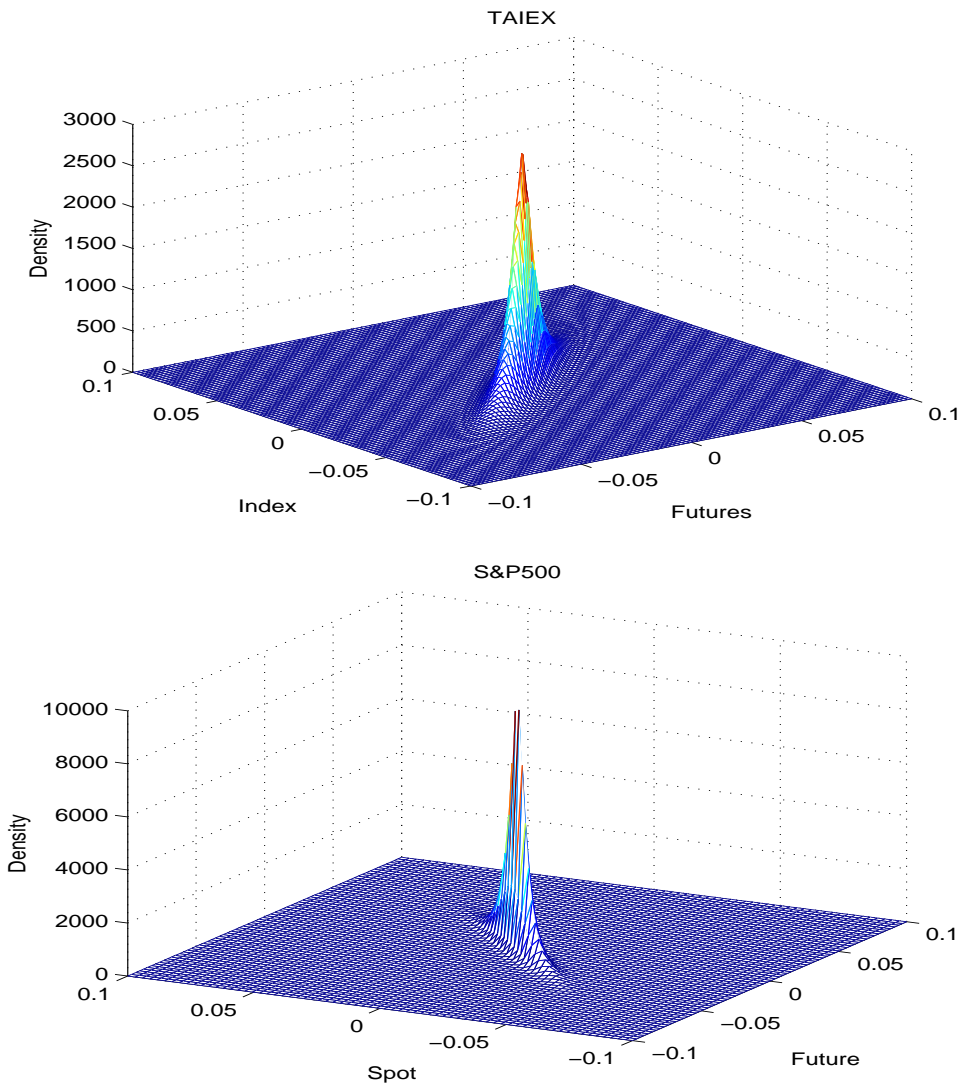
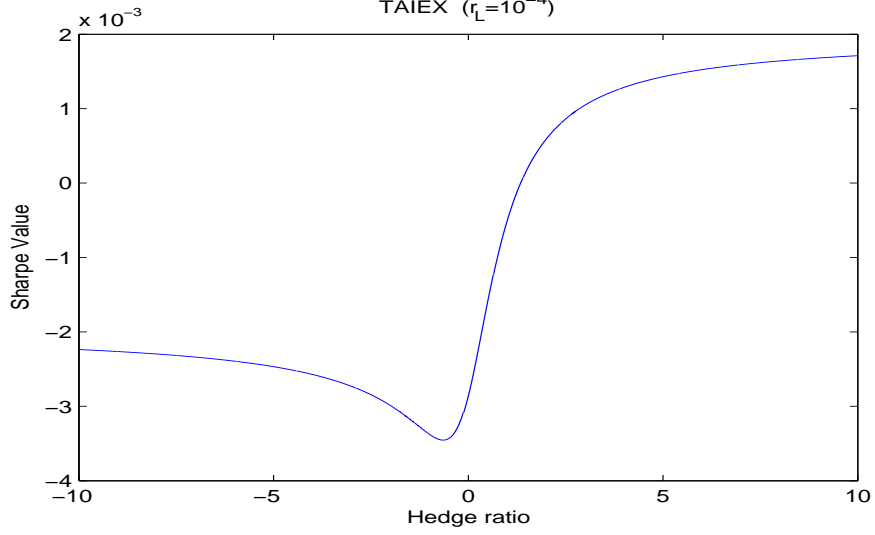


Figure 4:

TAIEX ($r_L=10^{-4}$)



S&P500 ($r_L=10^{-4}$)

