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測度值隨機過程與財務應用(2/3)

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中文摘要：

Affine 隨機過程是取值在 $R_+^m \times R^n$ 上的一種馬可夫過程。這種隨機過程具有多樣的隨機性質(如平均迴歸、跳躍、隨機波動)及良好的可分析性，所以在財務領域裡有非常廣泛的應用。最近 Duffie, Filipovic 和 Schachermayer[2003]建構所有平滑的 affine 隨機過程。同時，他們也建立平滑 affine 隨機過程和超過程的密切關聯。應用這個關係，我們建構更一般的 affine 隨機過程及研究他們的路徑性質及其在財務上的應用。

關鍵詞：affine 過程、超過程、有限基空間、利率結構、債券選擇權定價、違約風險

SOME REMARKS ON AFFINE PROCESSES

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ABSTRACT. Affine processes is a class of Markov processes taking values in $R_+^m \times R^n$. The rich variety of alternative types of random behavior (e.g., mean reversion, stochastic volatility, and jumps) and analytically tractable for affine processes make them ideal models for financial applications. Duffie, Filipovic and Schachermayer[DFS03] characterized all regular affine processes. Connections between regular affine processes and superprocesses with a finite base space were also established. Based on this observation, we construct more general affine processes and investigate sample path properties and financial applications of these processes.

1. AFFINE PROCESSES

A Markov transition function in a measurable space (E, \mathcal{B}) is a function $p(r, x; t, B)$, $r < t \in \mathcal{R}$, $x \in E$, $B \in \mathcal{B}$ which is \mathcal{B} -measurable in x and which is a measure in \mathcal{B} subject to the conditions:

- (A) $\int_E p(r, x; t, dy)p(t, y; u, B) = p(r, x; u, B)$ for all $r < t < u$, $x \in E$ and all $B \in \mathcal{B}$.
- (B) $p(r, x; t, E) \leq 1$ for all r, x, t .

To every Markov transition function p there corresponds a family of linear operator T_t^r acting on functions by the formula

$$T_t^r f(x) = \int_E p(r, x; t, dy) f(y).$$

It follows from (B) that $T_s^r T_t^s = T_t^r$ for all $r < s < t \in \mathcal{R}$. We call T the Markov semigroup corresponding to the transition function p .

We assume that (E, \mathcal{B}) is a measurable Luzin space. To every Markov transition function p there corresponds a Markov process $\xi = (\xi_t, \mathcal{F}(I), \Pi_{r,x})$ such that

$$\Pi_{r,x} \{\xi_t \in B\} = p(r, x; t, B),$$

$$\Pi_{r,x} \{\xi_{t_1} \in B_1, \dots, \xi_{t_n} \in B_n\} = \int_{B_1 \times \dots \times B_n} p(r, x; t_1, dy_1) p(t_1, y_1; t_2, dy_2) \dots p(t_{n-1}, y_{n-1}; t_n, dy_n)$$

for $n \geq 2$, $t_1 < t_2 < \dots < t_n$.

If the transition function $p(r, x; t, dy)$ satisfies a condition

$$p(r, x; t, B) = p(r + s, x; t, B)$$

for all r, s, t, B , then $\xi = (\xi_t, \mathcal{F}(I), \Pi_{r,x})$ is time homogeneous. In this case we consider only process $\xi = (\xi_t, \mathcal{F}_t, \Pi_x)$ where $\Pi_x = \Pi_{0,x}$, $\mathcal{F}_t = \mathcal{F}[0, t]$, and ξ_t is defined for all $t \geq 0$.

Key words and phrases. affine process, superprocess, finite base space, term structure of interest rates, bond option pricing, default risk.

From this point on we consider $E = \mathcal{R}_+^m \times \mathcal{R}^n$ and write $d = m + n$. We say that $\xi = (\xi_t, \mathcal{F}(I), \Pi_{r,x})$ is affine if for every $r < t \in \mathcal{R}$ and $\lambda \in \mathcal{C}^d$, there exists $v(r, t, \lambda) \in \mathcal{C}$ and $u(r, t, \lambda) \in \mathcal{C}^d$ such that

$$\Pi_{r,x}[\exp\{i \langle \lambda, \xi_t \rangle\}] = \exp\{v(r, t, \lambda) + u(r, t, \lambda) \cdot x\}$$

for all $x \in E$. Clearly if ξ is time homogeneous, then we have $u(r, t, \lambda) = u(t - r, \lambda)$ and $v(r, t, \lambda) = v(t - r, \lambda)$.

Example (Ornstein-Uhlenbeck process)

Consider an Ornstein-Uhlenbeck process

$$d\xi_t = \alpha(l - \xi_t)dt + \sigma dw_t$$

where w is a standard Brownian and α, l and σ are positive constants. Through an application of Ito's formula, we get

$$\xi_t = e^{-\alpha t}[\xi_0 + l(e^{\alpha t} - 1) + \sigma \int_0^t e^{\alpha s} dw_s].$$

This implies that ξ_t is normally distributed with mean

$$\Pi_x \xi_t = e^{-\alpha t}[x + l(e^{\alpha t} - 1)]$$

and variance

$$Var \xi_t = \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t}).$$

This implies that

$$\Pi_x e^{i\lambda \xi_t} = \exp\{v(t, \lambda) + u(t, \lambda)x\}$$

with

$$v(t, \lambda) = i\lambda e^{-\alpha t}l(e^{\alpha t} - 1) - \frac{\lambda^2 \alpha^2}{4\alpha}(1 - e^{-2\alpha t})$$

and

$$u(t, \lambda) = i\lambda e^{-\alpha t}.$$

Example (Feller's diffusions)

Feller considered a class of processes that includes the square-root diffusions

$$d\xi_t = \alpha(l - \xi_t)dt + \sigma \sqrt{\xi_t} dw_t$$

where w is a standard Brownian. We consider the case that α, l and σ are positive constants. Based on results of Feller[Fe51], Cox, Ingersoll and Ross[CIR85] noted that the distribution of ξ_t given ξ_u for some $u < t$, is distributed as $\sigma^2(1 - e^{-\alpha(t-u)})/4\alpha$ times a noncentral chi-square distribution $\chi_\nu^2(\lambda)$ with degree of freedom

$$\nu = \frac{4l\alpha}{\sigma^2}$$

and noncentrality

$$\lambda = \frac{4\alpha e^{-\alpha(t-u)}}{\sigma^2(1 - e^{-\alpha(t-u)})} \xi_u.$$

Therefore the Laplace transform of ξ_t is given by

$$\Pi_x e^{-\lambda \xi_t} = \frac{1}{(2\lambda c + 1)^{2l\alpha/\sigma^2}} \exp\left\{-\frac{\lambda c f}{2\lambda c + 1}\right\}$$

with $c = \sigma^2/4\alpha(1 - e^{-\alpha t})$ and $f = 4x\alpha/(\sigma^2(e^{\alpha t} - 1))$.

2. CHARACTERIZATIONS AND APPLICATIONS OF AFFINE PROCESSES

When $m = 1$ and $n = 0$, the affine process ξ takes values in \mathcal{R}_+ and is also called a continuous- state process with immigration. It was first studied by Kawazu and Watanabe[KW71] as a continuous limit of Galton-Watson branching processes with immigration. Kawazu and Watanabe[KW71] showed that if ξ is a stochastically continuous affine process in \mathcal{R}_+ , then for every $\lambda > 0, t > 0$ and $x \in \mathcal{R}_+$, we have

$$\Pi_x e^{-\lambda \xi_t} = \exp\{-u_t x - \int_0^t \phi(u_s) ds\}$$

where $u_t = u(t, \lambda)$ satisfies

$$\frac{du}{dt} = -\varphi(u) \quad , u(0) = \lambda$$

with

$$\varphi(u) = \alpha u^2 - \beta u - \gamma + \int_{\mathcal{R}_+} [e^{-uy} - 1 + u(1 \wedge y)] \mu(dy)$$

and

$$\phi(u) = c + bu + \int_{\mathcal{R}_+} (1 - e^{-uy}) \nu(dy).$$

(Here we assume that

$$\alpha \geq 0, \gamma \geq 0, b \geq 0, c \geq 0, \beta \in \mathcal{R}$$

and μ, ν are two measures on $(0, \infty)$ satisfying

$$\int_0^\infty (1 \wedge y) \mu(dy) < \infty$$

and

$$\int_0^\infty (1 \wedge y) \nu(dy) < \infty.)$$

For general m and n , Duffie, Filipovic and Schachermayer[DFS03] characterized all regular affine processes. In particular they obtained that if ξ is regular, then

$$\Pi_x e^{\langle \lambda, x_t \rangle} = \exp\{\langle u(t, \lambda), x \rangle + \int_0^t \phi(u(s, \lambda)) ds\}$$

where u satisfies some generalized Riccati equations and ϕ is in a class of nonlinear functions. It is worth noting that if $n = 0$, then u solves the differential log-Laplace equation corresponding to some superprocess with a finite base space. Based on this observation, we can construct more general affine processes through general superprocesses. We also study sample path properties for general affine processes.

The rich variety of alternative types of random behavior(e.g., mean reversion, stochastic volatility, and jumps) and analytically tractable for affine processes make them ideal models for financial applications(see, e.g., Duffie, Filipovic and Schachermayer[DFS03] and references therein.)

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