行政院國家科學委員會專題研究計畫　期中進度報告

推廣型類神經網路：函數和動態(1/3)

計畫類別：個別型計畫
計畫編號：NSC92-2115-M-009-011-
執行期間：92 年 08 月 01 日至 93 年 07 月 31 日
執行單位：國立交通大學應用數學系

計畫主持人：莊重

報告類型：精簡報告

處理方式：本計畫可公開查詢

中　華　民　國 93 年 5 月 25 日

BOUNDARY INFLUENCE ON THE ENTROPY OF A PROBLEM IN CELLULAR NEURAL NETWORKS

YU-CHUAN CHANG AND JONG JUANG

ABSTRACT. The purpose of this paper is to shed some light on the open problem raised by Afraimovich and Hsu [2003]. Specifically, under some mild conditions, we show that for any ℓ_1 and $n \in \mathbb{N}$, except possibly a few piece of $T^n\ell_1$, $T^n\ell_1$ is contained in an N-shaped tunnel for which its boundary point is an ω-limit point of ℓ_1 for T. Moreover, we show under a stronger condition, see (3.3), that the entropy $h_{\ell_1,\ell_2}(T)$, see Definition 1.1, of T with respect to ℓ_1 and ℓ_2 is independent of the choice of ℓ_1. It is also shown that $h(T) = h_D(T) = h_N(T) = \ln 3$, where $h_D(T)$ and $h_N(T)$ are the entropy of T with respect to Dirichlet and Neuman boundary conditions, respectively, see Remark 1.1-(2), and that $h_{\ell_1,\ell_2}(T)(= h_{\ell 2}(T))$ takes on two distinct values $\ln 3$ and 0. The necessary and sufficient conditions on ℓ_2 for which $h_{\ell_2}(T) = \ln 3$ are also obtained.

Key words: Boundary influence, dynamics of intersection, entropy, cellular neural networks.

1. **Introduction**

We consider one-dimensional Cellular Neural Networks (CNNs) of the form(e.g., [Ban *et al.*, 2002, 2001; Hsu 2000]).

$$\frac{dx_i}{dt} = -x_i + z + \alpha f(x_{i-1}) + af(x_i) + \beta f(x_{i+1}),\ i \in \mathbb{Z}, \tag{1.1a}$$

where f(x) is a piecewise-linear output function defined by

$$f(x) = \begin{cases} rx + 1 - r, & \text{if } x \geq 1, \\ x, & \text{if } |x| \leqslant 1, \\ lx + l - 1, & \text{if } x \leq -1. \end{cases} \tag{1.1b}$$

where r and l are positive constants. The quantity z is called threshold or bias term. The constants α, a and β are the interaction weights between neighboring cells. Such triple pair $[\alpha, a, \beta]$ of the interaction weights is called the template of the system (1). The complexity of the set of bounded stable (mosaic) stationary solutions of (1.1) has been intensively studied by many authors ([Ban *et al.*, 2002, 2001;

Chua, 1998; Chua and Yang 1998a; Hsu 2000; Juang and Lin 2000; Thiran 1997; Thiran *et al.*, 1995]). Those steady-state solutions $\{x_i\}_{i\in\mathbb{Z}}$, satisfy the equation

$$f(x_{i+1}) = \frac{1}{\beta}(x_i - z - \alpha f(x_{i-1}) - af(x_i)). \tag{1.2}$$

Set $u_i = f(x_i)$. Then (1.2) becomes

$$u_{i+1} = \frac{1}{\beta}(-\alpha u_{i-1} - z + f^{-1}(u_i) - au_i), \tag{1.3a}$$

or, equivalently,

$$T(u_{i-1}, u_i) = (u_i, u_{i+1}) = (u_i, \frac{1}{\beta}(-\alpha u_{i-1} - z + f^{-1}(u_i) - au_i)). \tag{1.3b}$$

Clearly, (1.3b) such induced is a Lozi-type map T.

$$(x_{i+1}, y_{i+1}) = T(x_i, y_i) = (y_i, F(y_i) - bx_i). \tag{1.4a}$$

Here

$$b = \frac{\alpha}{\beta}, \tag{1.4b}$$

and

$$F(y) = \begin{cases} a_1 y + a_0 - a_1 + \bar{a}_0 := a_1 y + \bar{a}_1, & \text{if } y \geq 1, \\ a_0 y + \bar{a}_0, & \text{if } |y| \leqslant 1, \\ a_{-1} y + a_{-1} - a_0 + \bar{a}_0 := a_{-1} y + \bar{a}_{-1}, & \text{if } y \leq -1. \end{cases} \tag{1.4c}$$

where

$$\begin{aligned} a_1 &= \tfrac{1}{\beta}(\tfrac{1}{r} - a) > 0, & a_0 &= \tfrac{1}{\beta}(1 - a) < 0, \\ a_{-1} &= \tfrac{1}{\beta}(\tfrac{1}{l} - a) > 0, & \bar{a}_0 &= \tfrac{-z}{\beta}. \end{aligned} \tag{1.4d}$$

Any bounded trajectory $(x_{j+1}, y_{j+1}) = T(x_j, y_j)$ corresponds to a bounded steady-state solution of system (1.1).

Inspired by the open problems raised in [Arnold 1993], and [Afraimovich and Hsu 2003], respectively, we are led to consider the following problems. Define the line $\ell_{m,k}$ as

$$\ell_{m,k} = \{(x, y) : y = mx + k\}. \tag{1.5a}$$

Here

$$\ell_{\infty,k} \text{ is interpreted as } \{(x, y) : x = k\}. \tag{1.5b}$$

Denote by $\mathcal{N}(n, \ell_{m_1,k_1}, \ell_{m_2,k_2}, T)$ the number of points on the intersection of $T^n \ell_{m_1,k_1} \cap \ell_{m_2,k_2}$. Should no ambiguity arise, we will write ℓ_{m_i,k_i} as ℓ_i.

Definition 1.1. *The entropy $h_{\ell_1,\ell_2}(T)$ of T with respect to lines ℓ_1 and ℓ_2 is defined as the limit*

$$h_{\ell_1,\ell_2}(T) = \overline{\lim_{n\to\infty}} \frac{\ln \mathcal{N}(n, \ell_1, \ell_2, T)}{n}. \tag{1.6}$$

In case that the growth rate of $\mathcal{N}(n, \ell_1, \ell_2, T)$ is super exponential, $h_{\ell_1,\ell_2}(T)$ is defined to be ∞. For a local holomorphic mapping, preserving the origin, and two lines ℓ_1 and ℓ_2 passing the origin. Suppose all the images $T^n\ell_1$ are smooth [15] or that everything is algebraic (see [11], [16]). Then $h_{\ell_1,\ell_2}(T)$ exists and is finite. In our case, $\mathcal{N}(n, \ell_1, \ell_2, T) \leq 3^n$. We next recall the definition of the spatial entropy of system (1.1).

Now, set $\Gamma_{n,k}(T)$ to be the number of elements in the solution set $S_{n,k}$, $S_{n,k} = \{\{u_i\}_{i=k}^{n+k-1} : \{u_i\}_{i=-\infty}^{\infty}$ is a bounded steady-state solution of (1.1)$\}$. Here $k \in \mathbb{Z}$. Since the template of system (1.1) is space invariant, the steady-state solutions of (1.1) are also space invariant. That is to say if $\{u_i\}_{i=-\infty}^{\infty}$ is a steady state solution of (1), so is $\{u_{i+k}\}_{i=-\infty}^{\infty}$ for any $k \in \mathbb{Z}$. Hence, $\Gamma_{n,k}(T)$ is independent of the choice of k. Thus, we set $\Gamma_{n,k}(T) = \Gamma_n(T)$.

Definition 1.2. *The spatial entropy $h(T)$ of the system (1.1) is defined as the limit*

$$h(T) = \overline{\lim_{n\to\infty}} \frac{\ln \Gamma_n(T)}{n}.$$

We next consider how the behavior of solutions of a large but finite lattice is related to the behavior of steady-state solutions of (1.1). Let $\{u_i\}_{i=-\infty}^{\infty}$ be an orbit sequence generated by T as given in (1.3b). The number of distinct orbit sequences $\{u_i\}_{i=-\infty}^{\infty}$ of T satisfying

$$u_2 = m_1 u_1 + k_1 \text{ (or equivalently } y_1 = m_1 x_1 + k_1), \tag{1.7a}$$

and

$$u_{n+1} = m_2 u_n + k_2 \text{ (or equivalently } y_n = m_2 x_n + k_2), \tag{1.7b}$$

be denoted by $\Gamma_n(n, m_1, k_1, m_2, k_2, T)$.

Remark 1.1.

(1) It is easy to see that $\Gamma_n(n, m_1, k_1, m_2, k_2, T) = \mathcal{N}(n-1, \ell_1, \ell_2, T)$, where $\ell_1 = \ell_{m_1,k_1}$ and $\ell_2 = \ell_{m_2,k_2}$.

(2) When $(m_1, k_1) = (\infty, 0)$ and $(m_2, k_2) = (0, 0)$ (resp., $(m_1, k_1) = (m_2, k_2) = (1, 0)$), $h_{\ell_1,\ell_2}(T)$ is the so-called the spatial entropy of system (1.1) with Dirichlet (resp., Neumann) boundary conditions. We write such entropy as $h_D(T)$ (resp., $h_N(T)$).

(3) For other choices of ℓ_1 and ℓ_2, $h_{\ell_1,\ell_2}(T)$ is called the spatial entropy of system (1.1) with Robbin's boundary conditions.

In [Afraimovich and Hsu, 2003], the following open problems were raised.

(P1): Is it true that, in general, $h(T) = h_D(T) = h_N(T) = h_{\ell_1,\ell_2}(T)$?

(P2): If it is not true, then which parameters m_i and k_i, $i = 1, 2$, are responsible for the values of $h(T)$. What kind of bifurcations occurs if the lines $\ell_{m,b}$ move?

The purpose of this paper is to shed some light on those two problems. Specifically, under some mild conditions, we show that for any ℓ_1 and $n \in \mathbb{N}$, except possibly a few piece of $T^n\ell_1$, $T^n\ell_1$ is contained in an N-shaped tunnel for which its boundary point is an ω-limit point of ℓ_1 for T. Moreover, we show under a stronger condition, see (3.3), that the entropy $h_{\ell_1,\ell_2}(T)$ of T with respect to ℓ_1 and ℓ_2 is independent of the choice of ℓ_1. It is also shown that $h(T) = h_D(T) = h_N(T) = \ln 3$, and that $h_{\ell_1,\ell_2}(T)(= h_{\ell_2}(T))$ takes on two distinct values $\ln 3$ and 0. The necessary and sufficient conditions on ℓ_2 for which $h_{\ell_2}(T) = \ln 3$ are also obtained. Those main results are recorded in Section 3. In Section 2, we study the dynamics of a certain two-dimensional map induced from $T^n\ell_1$. We conclude this introductory section by mentioning some related work. Shih [2000] studied the influence of periodic, Neumann and Dirichlet boundary conditions on a problem also arising in two dimensional CNNs. Since their output function f, as given in (1.1b), is flat at infinity, i.e. , $r = l = 0$, the formulation of the problem is much different from those in [Afraimovich and Hsu 2003]. Consequently, the techniques used in both situations are also quite different.

We also remark that the problem of the asymptotic behavior of the number of points on the intersection $f^k L_1 \cap L_2$, where L_1, L_2 are submanifolds of a smooth manifold, and f is a smooth map, is said to be a problem of dynamics of the intersection. These problems arise in various branches of analysis. There are some general results (see, e.g., p.261 of [Arnold 1993]) obtained for such problems. However, no approaches are available to solve specific problems.

2. Dynamics of Certain Maps Induced From $T^n\ell_{m,k}$

We begin with the calculation of $T^n\ell_{m,k}$. Now, for $m \neq 0$,

$$T(x', mx' + k) = (mx' + k, F(mx' + k) - bx').$$

Set $x = mx' + k$, $y = F(mx' + k) - bx'$, we see immediately that

$$y = F(x) - \frac{b(x-k)}{m} = \begin{cases} (a_1 - \frac{b}{m})x + (\bar{a}_1 + \frac{bk}{m}), & \text{if } x \geq 1, \\ (a_0 - \frac{b}{m})x + (\bar{a}_0 + \frac{bk}{m}), & \text{if } |x| \leqslant 1, \\ (a_{-1} - \frac{b}{m})x + (\bar{a}_{-1} + \frac{bk}{m}), & \text{if } x \leq -1. \end{cases} \tag{2.1}$$

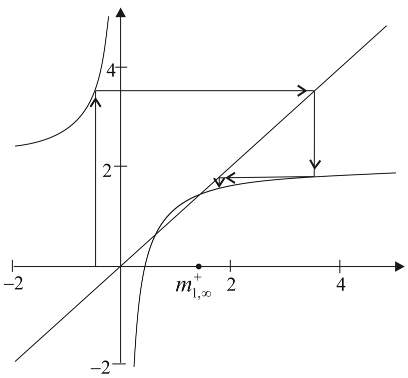

FIGURE 2.1

Using (2.1), we define two dimensional maps $G_i(x, y)$, $i = 1, 0, -1$, of the form

$$G_i(x, y) = (a_i - \frac{b}{x}, \bar{a}_i + \frac{b}{x}y) =: (g_{i,1}(x), g_{i,2}(x, y)). \tag{2.2}$$

We call $g_{i,1}(x)$, $i = 1, 0, -1$, the slope maps of T. For $g_{i,1}(x)$, $i = 1, 0, -1$, denote, respectively, the slopes of $T\ell_{x,y}$ in the regions.

$$R_1 = \{(x, y) : x \geq 1\},\ R_0 = \{(x, y) : |x| \leq 1\} \text{ and } R_{-1} = \{(x, y) : x \leq -1\}. \tag{2.3}$$

Moreover, $g_{i,2}(x, y)$ are to be termed the intercept maps. We next consider the dynamics of the slope and intercept maps $g_{i,1}$ and $g_{i,2}$.

Proposition 2.1. *Let $b > 0$, $a_i > 2\sqrt{b}$, $i = 1, -1$ and $-a_0 > 2\sqrt{b}$. Then* (i) $m^{\pm}_{i,\infty} := \frac{a_i \pm \sqrt{a_i^2 - 4b}}{2}$ *are two fixed points of the slope maps $g_{i,1}$.* (ii) *Moreover, the attracting interval of $m^+_{i,\infty}$, $i = 1, -1$, is $R - \{m^-_{i,\infty}\}$. That is to say if $x \in R - \{m^-_{i,\infty}\}$, then, for $i = 1, -1$, $\lim_{n\to\infty} g^n_{i,1}(x) = m^+_{i,\infty}$.* (iii) *The attracting interval of $m^-_{0,\infty}$ is $R - \{m^+_{0,\infty}\}$.* (iv) *Suppose $a_i = 2\sqrt{b}$. Then $m^+_{i,\infty} = m^-_{i,\infty}$ is the globally attracting fixed point of $g_{i,1}$, $i = 1, 0, -1$.*

Proof. We illustrate only $i = 1$. Clearly, two fixed points of $g_{1,1}$ are $m^{\pm}_{1,\infty}$. The attracting interval of $g_{1,1}$ can be easily concluded by using graphical analysis on Figure 2.1. □

Proposition 2.2. *Suppose*

$$b > 0,\ a_i > 1 + b,\ i = 1, -1 \text{ and } -a_0 > 1 + b. \tag{2.4}$$

(i) *For fixed* $x = m^+_{i,\infty}$, $i = 1, -1$, *then* $k_{i,\infty} := \frac{m^+_{i,\infty}\bar{a}_i}{m^+_{i,\infty}-b}$ *is a globally attracting fixed point of the intercept maps* $g_{i,2}(m^+_{i,\infty}, y)$. *Moreover,* (ii) *for fixed* $x = m^-_{0,\infty}$, $k_{0,\infty} = \frac{m^-_{0,\infty}\bar{a}_0}{m^-_{0,\infty}-b}$ *is also a globally attracting fixed point of* $g_{0,2}(m^-_{0,\infty}, y)$.

Proof. It suffices to show that $0 < \frac{b}{m^+_{i,\infty}} < 1$, $i = 1, -1$, and $-1 < \frac{b}{m^-_{0,\infty}} < 0$.

We illustrate only $i = 1$. Now,

$$0 < \frac{b}{m^+_{1,\infty}} = \frac{2b}{a_1 + \sqrt{a_1^2 - 4b}} = \frac{a_1 - \sqrt{a_1^2 - 4b}}{2} < 1. \tag{2.5}$$

The last inequality is justified by the fact that $a_1 > 1 + b \geq 2\sqrt{b} > 0$. □

Theorem 2.1. *Suppose (2.4) holds.* (i) *The two dimensional map* G_i, *as defined in (2.2),* $i = 1, 0, -1$, *have two fixed points* $(m^\pm_{i,\infty}, \frac{m^\pm_{i,\infty}\bar{a}_i}{m^\pm_{i,\infty}-b}) =: A^\pm_i$. (ii) *Moreover, the attracting regions of* $m^+_{i,\infty}$, $i = 1, -1$, *and* $m^-_{0,\infty}$, *are, respectively,* $\mathbb{R}^2 - \{A^-_i\}$, $i = 1, -1$, *and* $\mathbb{R}^2 - \{A^+_0\}$. *That is to say, for any* $(m, k) \in \mathbb{R}^2 - \{A^-_i\}$, $i = 1, -1$.*(resp.,*$(m, k) \in \mathbb{R}^2 - \{A^+_0\}$*),* $\lim_{n\to\infty} G^n_i(m, k) = A^+_i$, $i = 1, -1$,*(resp.,* $\lim_{n\to\infty} G^n_0(m, k) = A^+_0$*).*

Proof. We only illustrate $i = 1$. The cases for $i = 0, -1$ are similar. Define $g^n_{1,1}(m) = m_{1,n}$ and $G^n_1(m, k) = (m_{1,n}, k_{1,n})$. If $m \neq m^-_{1,\infty}$,then given $\varepsilon > 0$, there exists an $N_\varepsilon \in \mathbb{N}$ such that for every $n \geq N_\varepsilon$, we have

$$m^+_{1,\infty} - \varepsilon < m_{1,n} < m^+_{1,\infty} + \varepsilon. \tag{2.6}$$

It follows from (2.6) that for any $k \in \mathbb{R}$, and n sufficiently large,

$$min\{\bar{a}_1 + \frac{bk}{m^+_{1,\infty} - \varepsilon}, \bar{a}_1 + \frac{bk}{m^+_{1,\infty} + \varepsilon}\} < \bar{a}_1 + \frac{bk}{m_{1,n}} < max\{\bar{a}_1 + \frac{bk}{m^+_{1,\infty} - \varepsilon}, \bar{a}_1 + \frac{bk}{m^+_{1,\infty} + \varepsilon}\}. \tag{2.7}$$

It follows from (2.5) and Proposition 2.2 that for sufficiently small $\varepsilon > 0$, $\lim_{n\to\infty} g^n_{1,2}(m^+_{1,\infty} \pm \varepsilon, k)$ exist and that

$$\lim_{n\to\infty} g^n_{1,2}(m^+_{1,\infty} \pm \varepsilon, k) = \frac{\bar{a}_1(m^+_{1,\infty} \pm \varepsilon)}{m^+_{1,\infty} \pm \varepsilon - b} =: k_{1\pm\varepsilon}.$$

Using (2.7), we see inductively that

$$min\{g^n_{1,2}(m^+_{1,\infty}+\varepsilon, k), g^n_{1,2}(m^+_{1,\infty}-\varepsilon, k)\} < g^n_{1,2}(m_{1,n}, k) < max\{g^n_{1,2}(m^+_{1,\infty}+\varepsilon, k), g^n_{1,2}(m^+_{1,\infty}-\varepsilon, k)\}.$$

However, it is easy to see that the single limits $\lim_{n\to\infty} g^n_{1,2}(m^+_{1,\infty} \pm \varepsilon, k)$ and $\lim_{\varepsilon\to 0} g^n_{1,2}(m^+_{1,\infty} \pm \varepsilon, k)$ exist and the convergence of $\lim_{n\to\infty} g^n_{1,2}(m^+_{1,\infty} \pm \varepsilon, k)$ is uniform for all sufficiently small $\varepsilon > 0$. So the double limits of $g^n_{1,2}(m^+_{1,\infty} \pm \varepsilon, k)$ exist. Consequently, the double limits of $g^n_{1,2}(m^+_{1,\infty} \pm \varepsilon, k)$ exist and equal. Hence, the double limit of $g_{1,2}^n(m_{1,n},k)$ exists. Moreover, for each $\varepsilon > 0$ the limit $\lim_{n\to\infty} g_{1,2}^n(m_{1,n},k)$ exists. So the iterated limit $\lim_{\varepsilon\to 0}(\lim_{n\to\infty} g_{1,2}^n(m_{1,n},k))$ exists. And so $\lim_{n\to\infty} g_{1,2}^n(m_{1,n},k)$ exists and is equal to $\frac{m_{1,\infty}^+ \bar{a}_1}{m_{1,\infty}^+ - b}$. It is then easy to see that, for $(m,k) \in \mathbb{R}^2 - \{A_1^+\}$, $\lim_{n\to\infty} G_1^n(m,k) = (m_{1,\infty}^+, \dfrac{m_{1,\infty}^+ \bar{a}_1}{m_{1,\infty}^+ - b})$. We then complete the proof of theorem. $\square$

We are now in a position to study $T^n \ell_{m,k}$. To this end, we consider the lines ℓ_{i_∞}, $i = 1, 0, -1$, defined as follows. The (m,k)-pairs of ℓ_{i_∞} are, respectively, $(m_{i,\infty}^+, \frac{m_{i,\infty}^+ \bar{a}_i}{m_{i,\infty}^+ - b})$, for $i = 1, -1$, and $(m_{0,\infty}^-, \frac{m_{0,\infty}^- \bar{a}_0}{m_{0,\infty}^- - b})$ for $i = 0$.

From here on, to same notations, we write

$$\ell_{i_\infty},\ i = 1, 0, -1, \text{ as } \ell_{i_\infty} \cap R_i. \tag{2.8}$$

Here R_i are given as in (2.3). For any line or line segment ℓ, we also use the following notation

$$T\ell = \begin{cases} \ell_1, & \text{if } y \geq 1, \\ \ell_0, & \text{if } |y| \leq 1, \\ \ell_{-1}, & \text{if } y \leq -1. \end{cases} \tag{2.9a}$$

In case ℓ is a line segment or ℓ is a horizontal line, ℓ_i, $i = 1, 0, -1$, could be empty depending on the range of y in ℓ. Likewise, we may define $T(\ell_{i_1, i_2, \cdots, i_{n-1}})$ inductively as follows.

$$T\ell = \begin{cases} \ell_{i_1, i_2, \cdots, i_{n-1}, 1}, & \text{if } y \geq 1, \\ \ell_{i_1, i_2, \cdots, i_{n-1}, 0}, & \text{if } |y| \leq 1, \\ \ell_{i_1, i_2, \cdots, i_{n-1}, -1}, & \text{if } y \leq -1. \end{cases} \tag{2.9b}$$

3. Main Results-Boundary Influence on the Spatial Entropy

The following lemma is very useful in determining how we number and order the line segments and half-lines of $T^n \ell_{m,k}$. The proof is trivial and, thus, skipped.

Lemma 3.1. *For fixed y, if $x_1 \geq x_2$, then the y-coordinate of $T(x_1, y)$ is no greater than that of $T(x_2, y)$.*

Using lemma 3.1 and the fact that T is one-to-one, we have the following principle.

Proposition 3.1. *Let ℓ and k be lines or line segments, and $\ell \cap k = \emptyset$. If k is to the right of ℓ. Then so are k_i to ℓ_i, $i = 1, -1$. However, ℓ_0 is to the right of k_0. Here k_i, ℓ_i, $i = 1, 0, -1$ are defined in (2.8).*

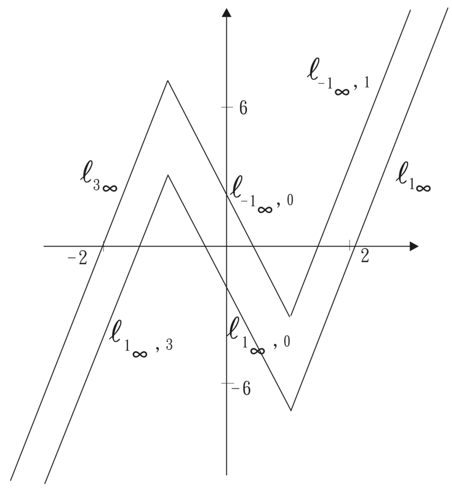

FIGURE 3.1

Note that the reverse of the ordering in k_0 and ℓ_0 is due to the fact that, in R_0, $F(y)$ has a negative slope.

It follows from Proposition 3.1 that the construction of the N-shaped figure with boundaries indicated as in Figure 3.1 makes sense. We shall call the region bounded by two N-shaped lines the N-shaped tunnel of T.

The intersection of the lines/line segments ℓ and k will be denoted by

$$\ell \cap k. \tag{3.1}$$

Lemma 3.2. *Suppose $\bar{a}_0$ and $b > 0$ are sufficiently small, and $a_i > 1+b$, $i = 1, -1$, and $-a_0 > 1 + b$. Then the y-coordinate $(\ell_{-1_\infty,0} \cap \ell_{-1_\infty,1})_y$ of $(\ell_{-1_\infty,0} \cap \ell_{-1_\infty,1})$ is less than -1, and $(\ell_{1_\infty,-1} \cap \ell_{1_\infty,0})_y > 1$.*

Proof. We illustrate only $(\ell_{1_\infty,-1} \cap \ell_{1_\infty,0})_y > 1$. The other assertion is similarly obtained. Note that the equation of the line ℓ_{1_∞} is $y = m^+_{1,\infty}x + k_{1,\infty}$. Letting $y = -1$, we see $x = \frac{-k_{1,\infty}-1}{m^+_{1,\infty}}$. Clearly, $(\ell_{1_\infty,-1} \cap \ell_{1_\infty,0})_y$=the y-coordinate of $T(\frac{-k_{1,\infty}-1}{m^+_{1,\infty}}, -1) =$

$$-a_0 + \bar{a}_0 + \frac{b(k_{1,\infty} + 1)}{m^+_{1,\infty}} =: t > 1, \tag{3.2}$$

whenever a_0 and b are sufficiently small. □

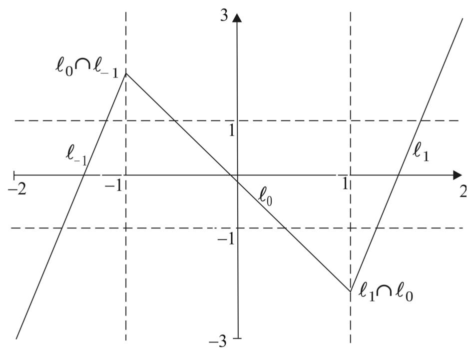

FIGURE 3.2

Lemma 3.3. *Suppose*

$$a_i > 3\ ,\ i = 1, -1,\ -a_0 > 3 \text{ and that } \bar{a}_0 \text{ and } b > 0 \text{ are sufficiently small.} \tag{3.3}$$

Let A be any point in the line segment for which its both endpoints are $\ell_{-1_\infty} \cap \ell_{-1_\infty,0}$ and $\ell_{1_\infty,-1} \cap \ell_{1_\infty,0}$. Then the limit of both coordinates of $T^n(A)$ approaches to $+\infty$.

Proof. We first note that T has a fixed point $B = (\frac{a_1-a_0-\bar{a}_0}{a_1-1-b}, \frac{a_1-a_0-\bar{a}_0}{a_1-1-b})$ for which its stable (resp., unstable) direction is $(1, \frac{a_1-\sqrt{a_1^2-4b}}{2})$. (resp., $(1, \frac{a_1+\sqrt{a_1^2-4b}}{2})$). Since $(\ell_{-1_\infty} \cap \ell_{-1_\infty,0})_y > (\ell_{1_\infty,-1} \cap \ell_{1_\infty,0})_y > 1$, as showed in (3.2), it suffices to show that $T^n(\ell_{1_\infty,-1} \cap \ell_{1_\infty,0}) \to (+\infty, +\infty)$ as $n \to \infty$. To this end, we need to show that $T(\ell_{1_\infty,-1} \cap \ell_{1_\infty,0}) = T(-1,t)$, t as given in (3.2), lies on the upper half of the stable line

$$(y - \frac{a_1 - a_0 - \bar{a}_0}{a_1 - 1 - b}) = m^-_{1,\infty}(x - \frac{a_1 - a_0 - \bar{a}_0}{a_1 - 1 - b}),$$

or, equivalently,

$$F(t) + b - \frac{a_1 - a_0 - \bar{a}_0}{a_1 - 1 - b} - m^-_{1,\infty} t + m^-_{1,\infty} \frac{a_1 - a_0 - \bar{a}_0}{a_1 - 1 - b} =: h(b, \bar{a}_0) > 0.$$

Now,

$$h(0,0) = \frac{-a_0 a_1^2 + 2a_0 a_1 - a_1^2}{a_1 - 1} = \frac{a_1[(-a_0 - 1)(a_1 - 2) - 2]}{a_1 - 1} > 0.$$

We thus complete the proof of the lemma. □

For any non horizontal line $\ell_{m,k}$, $m \neq 0$, we have that $T\ell_{m,k} = \ell_{-1} \cup \ell_0 \cup \ell_1$, see Figure 3.2 and (2.9), is an N-shaped graph with $(\ell_1 \cap \ell_0)_y < -1$ and $(\ell_0 \cap \ell_{-1})_y > 1$ provided T satisfies the assumptions in Lemma 3.2.

Moreover, $T^2\ell_{m,k} \cap R_1 = \ell_{-1,1} \cup \ell_{0,1} \cup \ell_{1,1}$. Note that $\ell_{i,1}$, $i = 1, 0, -1$, are obtained by applying the action of T on the portion of ℓ_i for which their y coordinates

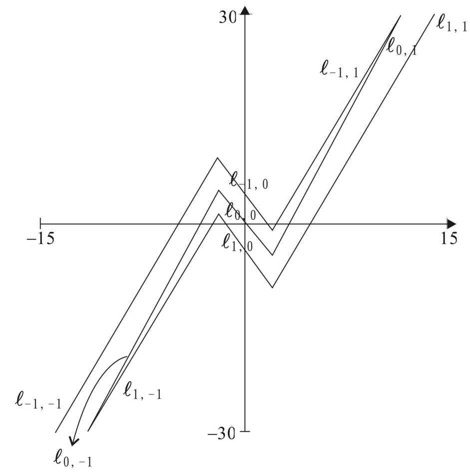

FIGURE 3.3

are greater or equal than 1. By Proposition 3.1, we see that the ordering of $\ell_{i,1}$, $i = 1, 0, -1$, going from left to right, is $\ell_{-1,1}, \ell_{0,1}$ and $\ell_{1,1}$.

Likewise, we define $\ell_{i,j}$ $i = 1, 0, -1$, $j = 0, -1$, accordingly so that

$$T^2\ell_{m,k} \cap R_0 = \ell_{1,0} \cup \ell_{0,0} \cup \ell_{-1,0},$$

and

$$T^2\ell_{m,k} \cap R_{-1} = \ell_{-1,-1} \cup \ell_{0,-1} \cup \ell_{1,-1}.$$

Note that the ordering of $\ell_{i,0}$ (resp., $\ell_{i,-1}$), $i = 1, 0, -1$, going from left to right is $\ell_{1,0}$, $\ell_{0,0}$ and $\ell_{-1,0}$ (resp., $\ell_{-1,-1}$, $\ell_{0,-1}$ and $\ell_{1,-1}$).

In general, $T^n\ell_{m,k}$ consists of 3^n line segments/half lines that can be labelled as $\ell_{i_1,i_2,\cdots,i_n}$, $i_j = 1, 0, -1$, $j = 1, 2, \cdots, n$. See Figure 3.3.

Our first main result is to characterize how $T^n\ell_1$behaves for all n.

Theorem 3.1. *Suppose (3.3) holds. Then for any n, all $\ell_{i_1,i_2,\cdots,i_n}$ lie in the N-shaped tunnel of T except possibly $\ell_{1,1,\cdots,1}$, $\ell_{-1,-1,\cdots,-1}$, $\ell_{1,1,\cdots,1,0}$ and $\ell_{-1,-1,\cdots,-1,0}$.*

Proof. Since T takes a horizontal line into a vertical line, we may assume that $\ell_{m,k}$ is a non horizontal line. Set $T\ell_{m,k} = \ell_{-1} \cup \ell_0 \cup \ell_1$, see Figure 3.2. It is clear that ℓ_{-1} and ℓ_0 are to the left of ℓ_{1_∞} and ℓ_0 and ℓ_1 are to the right of ℓ_{-1_∞}. Using proposition 3.1 inductively, we conclude that all $\ell_{i_1,i_2,\cdots,i_{n-1}1}$ (resp., $\ell_{i_1,i_2,\cdots,i_{n-1}-1}$) except possibly $\ell_{1,1,\cdots,1}$ (resp.,$\ell_{-1,-1,\cdots,-1}$) must lie in the region bounded by $\ell_{-1_\infty,1}$, ℓ_{1_∞} and $x = 1$ (resp., $\ell_{-1_\infty}, \ell_{1_\infty,-1}$ and $x = -1$). Since the above are true for all n, we see immediately, via Proposition 3.1, that all $\ell_{i_1,i_2,\cdots,i_{n-1},0}$ except possibly $\ell_{1,1,\cdots,1,0}$

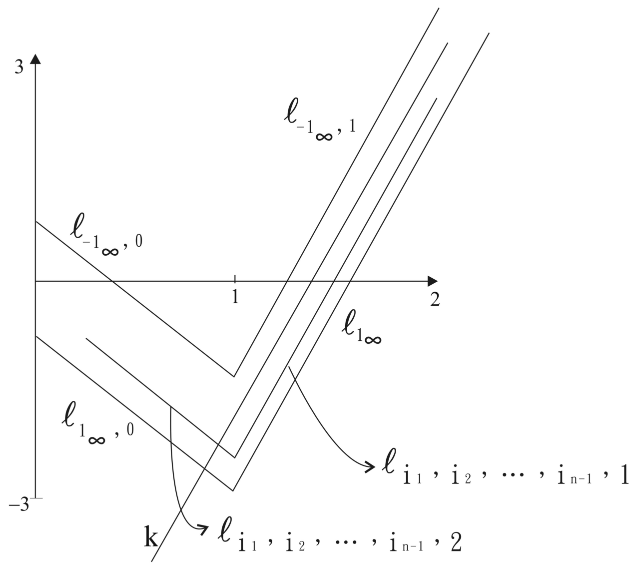

FIGURE 3.4

and $\ell_{-1,-1,\cdots,-1,0}$, must lie in the region bounded by $x=-1$, $x=1$, $\ell_{1_\infty,0}$ and $\ell_{-1_\infty,0}$. □

We note that the boundary points of the N-shaped tunnel are ω-limit points $\omega(\ell_1;T)$ of ℓ_1 for T. That is if $y \in \omega(\ell_1;T)$, then there exists a $x \in \ell_1$, and a sequence $\{n_k\}_{k=1}^{\infty}$, $n_k \in \mathbb{N}$, such that $T^{n_k}(x) \to y$ as $k \to \infty$.

The second main results are stated in the following.

Theorem 3.2. *Suppose (3.3) holds. (i) Then $h_{\ell_1,\ell_2}(T)$ is independent of the choice of ℓ_1. We then write $h_{\ell_1,\ell_2}(T)$ as $h_{\ell_2}(T)$. (ii) If $a_1 > a_{-1}$, (resp., $a_1 < a_{-1}$), let ℓ_2 be a line passing through $\ell_{1_\infty} \cap \ell_{1_\infty,0}$ (resp.,$\ell_{-1_\infty} \cap \ell_{-1_\infty,0}$) with slope m satisfying $a_{-1} \le m \le a_1$ (resp.,$a_1 \le m \le a_{-1}$), then $h_{\ell_2}(T)=0$; otherwise, $h_{\ell_2}(T)=\ln 3$. (iii) If $a_1 = a_{-1}$, let ℓ_2 be a line with slope $m=a_1$ and y-intercept k satisfying $k \ge k_{-1,\infty}$ or $k \le k_{1,\infty}$, then $h_{\ell_2}(T)=0$; otherwise, $h_{\ell_2}(T)=\ln 3$.*

Proof. Let $\ell_2 = k$ be a line in between $\ell_{-1_\infty,1}$ and ℓ_{1_∞}. Denote by k_y (resp., $(\ell_{1_\infty})_y$) the y-coordinate of $k \cap \{x=1\}$ (resp., $\ell_{1_\infty} \cap \{x=1\}$). Since $\lim_{n\to\infty} T^n \ell_{0,1_\infty} = \ell_{1_\infty}$, there exists an N such that

$$k_y > (T^n \ell_{1_\infty,0})_y > (\ell_{1_\infty})_y \text{ whenever } n \ge N. \tag{3.4}$$

Using Theorem 3.1, we have that all

$$\ell_{i_1,i_2,\cdots,i_{N-1}}, \text{ where } i_1,i_2,\cdots,i_{N-2}\in\{1,0,-1\}, \text{ and } i_{N-1}=0, \tag{3.5}$$

lie in between $\ell_{1_\infty,0}$ and $\ell_{-1_\infty,0}$. It then follows from (3.4) and Proposition 3.1 that for $n \geq N$

$$k_y > (T^n_{\ell_{1_\infty},0})_y > (\ell_{i_1,i_2,\cdots,i_{n-1},1})_y > (\ell_{1_\infty})_y,$$

where $i_1,i_2,\cdots,i_{N-1}$ satisfy (3.5). Consequently, k must intersect $\ell_{i_1,i_2,\cdots,i_{n-1},2}$, where $i_1,i_2,\cdots,i_{N-1}$ satisfy (3.5). See Figure 3.4.

Hence, for $n \geq N$, the number $\mathcal{N}(n,\ell_1,k,T)$ of intersections of $T^n\ell_1\cap k$ satisfies

$$3^{n-N} \leq \mathcal{N}(n,\ell_1,k,T) \leq 3^n.$$

Thus $h_{\ell_1,\ell_2}(T)=\ln 3$. The cases that k lies between $\ell_{-1_\infty,0}$ and $\ell_{1_\infty,0}$ or ℓ_{-1_∞} and $\ell_{1_\infty,-1}$ are similar. The other remaining nontrivial case is $k=\{(x,y): y=d, |d| \text{ is large}\}$. However, using Lemma 3.3, we see similarly that there exists an $M\in\mathbb{N}$, for all n sufficiently large, we have

$$3^{n-M} \leq \mathcal{N}(n,\ell_1,k,T) \leq 3^n.$$

Hence, $h_{\ell_1,\ell_2}(T)=\ln 3$. The remaining part of the theorem is trivial and thus omitted. $\square$

Proposition 3.2. *Suppose (3.3) holds. Then there exists a $p>1$ such that the following holds.*

$$F(p)-bp>p, \tag{3.6a}$$

$$F(1)+bp<-p, \tag{3.6b}$$

$$F(-1)-bp>p, \tag{3.6c}$$

and

$$F(-p)+bp<-p. \tag{3.6d}$$

Proof. Equations (3.6) are equivalent to

$$\min\{\frac{-a_0+\bar{a}_0}{1+b},\frac{-a_0-\bar{a}_0}{1+b}\} > p > \max\{\frac{a_1-a_0-\bar{a}_0}{a_1-1-b},\frac{a_{-1}-a_0+\bar{a}_0}{a_{-1}-1-b}\}. \tag{3.7}$$

Letting $b=\bar{a}_0=0$, (3.7) reduces to

$$-a_0 > p > \max\{\frac{a_1-a_0}{a_1-1},\frac{a_{-1}-a_0}{a_{-1}-1}\}. \tag{3.8}$$

However, under condition (3.3), (2.3) holds and $\frac{a_i-a_0}{a_i-1}>1$, $i=1,-1$. We thus complete the proof of proposition. $\square$

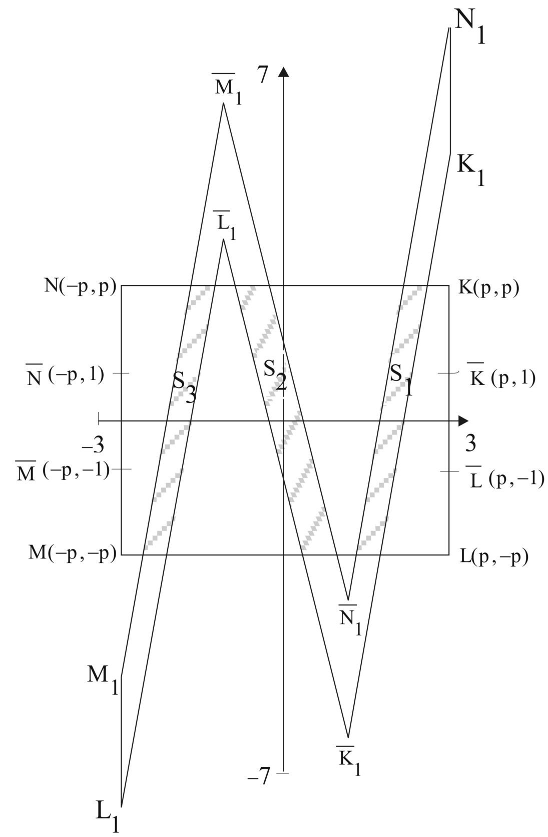

FIGURE 3.5. Here we denote by $K_1 = T(K)$. We use similar notations to denote points under the first iteration of T.

Let S be a square defined as

$$S = \{(x, y) \in \mathbb{R}^2 : |x| \le p,\ |y| \le p\}, \text{ where}$$

p satisfies (3.6). Then $T(S) \cap S = S_3 \cup S_2 \cup S_1$. See Figure 3.5.

Inductively, we see that $T^n(S) \cap S$ consists of 3^n nested pieces of $S_{i_1, i_2, \cdots, i_n}$, $i_j = 1, 0, -1,\ j = 1, 2, \cdots, n$. Likewise, backward iterations: $T^{-n}(S) \cap S$ will produce 3^n nested pieces of $\bar{S}_{i_1, i_2, \cdots, i_n}$, $i_j = 1, 0, -1$, $j = 1, 2, \cdots, n$ with each piece $\bar{S}_{i_1, i_2, \cdots, i_n}$ cross the east and west side of the rectangle S. Using Theorem 2.1, we see that the size of $\bar{S}_{i_1, i_2, \cdots, i_n}$ and $S_{i_1, i_2, \cdots, i_n}$ shrinks to zero as $n \to \infty$. Thus, $\bigcap\limits_{n=-\infty}^{\infty} T^n(S) \cap S =: \Lambda$ is a cantor set of infinite points. Using standard argument in symbolic dynamics, one shows that the dynamics of T on the invariant set Λ is conjugate to the shift map with three symbols. Thus, any trajectory of T in Λ is a bounded steady state of (2). Hence, we have the following theorem.

Theorem 3.3. *Suppose (3.3) holds. Then* $h(T) = \ln 3 = h_D(T) = h_N(T)$.

We conclude this paper with the following remarks.

(1) We have shown that Dirichlet and Neuman boundary conditions have no effect on the entropy of T under the circumstances and that certain Robbin's boundary conditions do influence the entropy of T.

(2) In the language of CNNs, condition (3.3) means that the slopes r and l of the output function f are small and so is the bias term z. The self-interaction weight a has to be strong. However, the right-side (forward) interaction weight β has to be much stronger than that of the left-side (backward) interaction weight α.

(3) The cases when $1 \leq a_1, -a_0, a_{-1} \leq 3$ are complicated as well as interesting.

(4) It is also of interest to see if our techniques developed here can be applied to the cases when $F(y)$ is a cubic polynomial, such as those in p.163 of Afraimovich and Hsu [2003] or a quadratic map for which the resulting T is a Henon map.

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Department of Applied Mathematics, National Chiao Tung University, Hsin Chu, Taiwan, R.O.C.