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Unified theory of symmetry for two-dimensional complex polynomials using delta discrete-time operator

I.-Hung Khoo · Hari C. Reddy · P. K. Rajan

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Abstract The complexity in the design and implementation of 2-D filters can be reduced considerably if the symmetries that might be present in the frequency responses of these filters are utilized. As the delta operator (γ-domain) formulation of digital filters offers better numerical accuracy and lower coefficient sensitivity in narrow-band filter designs when compared to the traditional shift-operator formulation, it is desirable to have efficient design and implementation techniques in γ-domain which utilize the various symmetries in the filter specifications. Furthermore, with the delta operator formulation, the discrete-time systems and results converge to their continuous-time counterparts as the sampling periods tend to zero. So a unifying theory can be established for both discrete- and continuous-time systems using the delta operator approach. With these motivations, we comprehensively establish the unifying symmetry theory for delta-operator formulated discrete-time complex-coefficient 2-D polynomials and functions, arising out of the many types of symmetries in their magnitude responses. The derived symmetry results merge with the s-domain results when the sampling periods tend to zero, and are more general than the real-coefficient results presented earlier. An example is provided to illustrate the use of the symmetry constraints in the design of a 2-D IIR filter with complex coefficients. For the narrow-band filter in the example, it can be seen that the γ-domain transfer function possesses better sensitivity to coefficient rounding than the z-domain counterpart.

Late Professor N. K. Bose through his fundamental and pioneering contributions as well as through his many writings has influenced the works of a large number of researchers in multidimensional Circuits and Systems theory including the authors of this paper. This paper is thus dedicated to his memory.

I.-H. Khoo (✉) · H. C. Reddy
Department of Electrical Engineering, California State University Long Beach, Long Beach, CA, USA
e-mail: ikhoo@csulb.edu

H. C. Reddy
College of Computer Science, National Chiao-Tung University, HsinChu City, Taiwan

P. K. Rajan
Department of Electrical & Computer Engineering, Tennessee Technological University, Cookeville, TN, USA

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1 Introduction

Two-dimensional (2-D) digital filters find applications in many fields such as image processing and seismic signal processing. The design and implementation of 2-D digital filters is more complicated than 1-D digital filters because the increase in dimension brings about an exponential increase in the number of filter coefficients and multipliers. Fortunately, the frequency responses of 2-D filters can possess various types of symmetries and the presence of these symmetries can be used to reduce the complexity of the design. Symmetry present in the frequency response induces a relation among the filter coefficients and multipliers. This reduces the number of design parameters in an optimization scheme, as well as the number of multipliers in an implementation structure. The usefulness of symmetry relations in the design of 2-D filters in s-domain and z-domain have been studied extensively (Narasimha and Peterson 1978; Rajan and Swamy 1978; Aly and Fahmy 1981; Rajan et al. 1982; Lodge and Fahmy 1983; George and Venetsanopoulos 1984; Pitas and Venetsanopoulos 1986; Swamy and Rajan 1986; Rajaravivarma et al. 1991; Fettweis 1997; Reddy et al. 2002, 2003). However, conventional discrete time filters and systems utilizing the shift operator (q) in z-domain can exhibit unacceptable numerical problems when the filter is narrowband or when the filter poles are clustered near $z = 1$ in the complex z-plane, such as when high sampling rates are needed (Middleton and Goodwin 1990; Goodwin et al. 1992). This translates into very poor coefficient and filter parameter sensitivity. By replacing the conventional shift operator (q) in the z-domain approach with the delta discrete-time operator (δ), one can overcome this numerical ill-conditioning situation (Middleton and Goodwin 1990; Goodwin et al. 1992). Since the introduction by Middleton and Goodwin, delta operator based designs have been studied extensively in the area of digital control systems and signal processing due to their excellent finite wordlength performance under fast sampling (Li and Gevers 1993; Premaratne et al. 1994; Kauraniemi et al. 1998; Wong and Ng 2000). The delta operator is defined as:

$$\delta\left[x\left(nT\right)\right] = \frac{x\left(nT+T\right) - x\left(nT\right)}{T} \tag{1}$$

where T may denote the sampling period or a constant.

It is easy to see that the relationship between the delta operator and the shift operator is given by $\delta = (q-1)/T$. In the transform domain, δ is represented by the transform variable $\gamma = (z-1)/T$. Or, as a causal element: $\gamma^{-1} = T \cdot z^{-1}/\left(1 - z^{-1}\right)$. It can also be seen from (1) that as $T \to 0$, the delta-operator becomes the continuous-time operator d/dt :

$$\lim_{T\to 0} \delta\left[x\left(nT\right)\right] = \lim_{T\to 0} \frac{x\left(nT+T\right) - x\left(nT\right)}{T} = \frac{d}{dt}x\left(t\right). \tag{2}$$

Due to this convergence property, a unifying system theory can be developed using the delta operator approach to cover both continuous and discrete-time systems.

The objective of this paper is to present a unifying symmetry theory for delta-operator formulated discrete-time complex-coefficient 2-D polynomials and functions, arising out of the many types of symmetries in their magnitude responses. We first provide a review of some of the notations and mathematical relations that are used in the study of 2-D symmetry properties. This is followed by the definitions for the various types of symmetries. We then derive the symmetry constraints for complex-coefficient polynomials. The derived symmetry constraints can then be used for the efficient design of complex coefficient 2-D digital

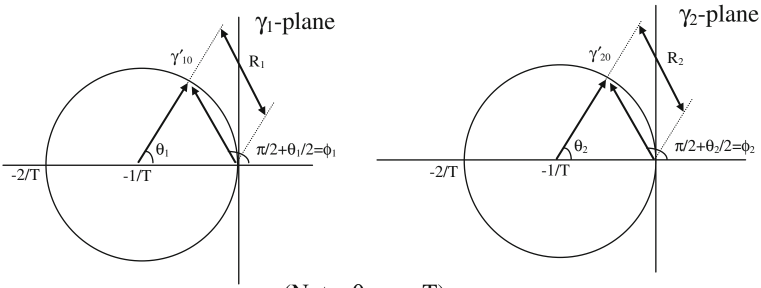

Fig. 1 $\{\gamma_1, \gamma_2\}$ complex bi-plane corresponding to delta-operator formulation

filters. These symmetry results are more in depth and exhaustive than the results presented earlier in (Reddy et al. 1996, 1997, 1998; Khoo et al. 2001, 2003, 2006). The convergence of the delta-operator symmetry results to the continuous-time complex symmetry constraints in Rajaravivarma et al. (1991) is then shown, which demonstrates the unifying property of the delta operator approach. Finally, a design example is provided to illustrate the use of the derived symmetry constraints in the design of a complex-coefficient 2-D IIR filter with symmetry in the specifications. The filter design approach mirrors the real-coefficient one presented in Khoo et al. (2006). The symmetry constraints result in fewer coefficients in the filter transfer function. In addition, it can be seen in the example that the narrow-band filter possesses better coefficient sensitivity to rounding than its z-domain counterpart.

2 Definitions and notations

Following the notations in Khoo et al. (2003, 2006), let $\gamma_i = (z_i - 1)/T_i$ for $i = 1, 2$, represent the delta operator in the transform domain for 2-D systems. Then the transfer functions of a 2-D system $H_z(z_1, z_2)$ in the z-domain and $H_\gamma(\gamma_1, \gamma_2)$ in the γ-domain are related as follows:

$$H_\gamma(\gamma_1, \gamma_2) = H_z(z_1, z_2)|_{z_i=(1+T_i\gamma_i),\quad i=1,2} \tag{3}$$

and

$$H_z(z_1, z_2) = H_\gamma(\gamma_1, \gamma_2)\Big|_{\gamma_i=\frac{(z_i-1)}{T_i},\quad i=1,2} \tag{4}$$

For linear shift invariant discrete time systems, $H_z(z_1, z_2)$ are restricted to be rational functions in z_1 and z_2 and so $H_\gamma(\gamma_1, \gamma_2)$ are also restricted to be rational functions in γ_1 and γ_2. As we will be dealing mostly in γ-domain in this paper, we will omit the subscript "γ" in $H_\gamma(\gamma_1, \gamma_2)$. We will also assume the sampling periods to be the same, i.e., $T_1 = T_2 = T$ (a positive real constant). Assuming the sampling periods to be the same is a necessity in defining some of the symmetries and relating them to one another.

2.1 Frequency response

Let $P(\gamma_1, \gamma_2)$ be a 2-D γ-domain polynomial. Then, its frequency response is given by $P\left(\frac{e^{j\omega_1 T}-1}{T}, \frac{e^{j\omega_2 T}-1}{T}\right)$ where ω_1 and ω_2 are the radian frequencies (Fig. 1).

The magnitude squared function of the frequency response is given by:

$$F(\omega_1,\omega_2) = P\left(\frac{e^{j\omega_1 T}-1}{T}, \frac{e^{j\omega_2 T}-1}{T}\right)\cdot P^*\left(\frac{e^{-j\omega_1 T}-1}{T}, \frac{e^{-j\omega_2 T}-1}{T}\right)$$
$$= P(\gamma_1,\gamma_2)\cdot P^*\left(\frac{-\gamma_1}{1+T\gamma_1}, \frac{-\gamma_2}{1+T\gamma_2}\right)\Bigg|_{\gamma_i=\frac{e^{j\omega_i T}-1}{T},\quad i=1,2} \tag{5}$$

where P^* denotes the polynomial obtained by conjugating the coefficients of P.

Let $\gamma_{i*} = -\gamma_i/(1+T\gamma_i)$ for $i=1,2$. It may be verified that when $\gamma_i = \left(\left(e^{j\omega_i T}-1\right)/T\right)$, $\gamma_{i*} = \left(\left(e^{-j\omega_i T}-1\right)/T\right)$ which means on the unit circle $\gamma_{i*} = (\gamma_i)*$, the complex conjugate of γ_i. Then using analytic continuation, (5) can be written in γ-domain as:

$$F_\gamma(\gamma_1,\gamma_2) = P(\gamma_1,\gamma_2)\cdot P^*\left(\frac{-\gamma_1}{(1+T\gamma_1)}, \frac{-\gamma_2}{(1+T\gamma_2)}\right)$$
$$= P(\gamma_1,\gamma_2)P^*(\gamma_{1*},\gamma_{2*}) \tag{6}$$

We will also use the notation $\hat{\gamma} = (\gamma_1,\gamma_2)$ and $\hat{\gamma}_* = (\gamma_{1*},\gamma_{2*})$. We now study several polynomial operations on the (Υ_1,Υ_2)-variable polynomials.

2.2 Inverse operation on a 2-D Υ-domain polynomial

Let $P(\gamma_1,\gamma_2)$ be a polynomial of degree M in Υ_1 and N in Υ_2.

Definition 1(a) The inverse of $P(\gamma_1,\gamma_2)$ with respect to γ_1 is defined as:

$$P_{\dagger\gamma_1}(\gamma_1,\gamma_2) = P(\gamma_{1*},\gamma_2)\cdot(1+T\gamma_1)^M \tag{7}$$

Definition 1(b) The inverse of $P(\gamma_1,\gamma_2)$ with respect to γ_2 is defined as:

$$P_{\dagger\gamma_2}(\gamma_1,\gamma_2) = P(\gamma_1,\gamma_{2*})\cdot(1+T\gamma_2)^N \tag{8}$$

Definition 2 The inverse of $P(\gamma_1,\gamma_2)$with respect to Υ_1 and Υ_2 is defined as:

$$P_{\dagger}(\gamma_1,\gamma_2) = P(\gamma_{1*},\gamma_{2*})\cdot(1+T\gamma_1)^M\cdot(1+T\gamma_2)^N \tag{9}$$

Here, the factors $(1+T\gamma_1)^M$ and $(1+T\gamma_2)^N$ are introduced to cancel the denominator in $P(\gamma_{1*},\gamma_{2*})$ and make the inverse a polynomial.

Note that we have defined the inverse operation differently from the paraconjugate operation which requires conjugating the complex coefficients. This is to facilitate the derivation of the polynomial factors in this paper. For instance, it may be noted that the inverse of the polynomial $P(\Upsilon_1,\Upsilon_2)$ with respect to Υ_1 defined in (7) induces ω_2–axis reflection symmetry in the frequency response of $P(\gamma_1,\gamma_2)$, i.e., $H_{P_{\dagger\gamma_1}}(\omega_1,\omega_2) = H_P(-\omega_1,\omega_2)$ where $H_P(\omega_1,\omega_2) = P\left(\frac{e^{j\omega_1 T}-1}{T}, \frac{e^{j\omega_2 T}-1}{T}\right)$ is the frequency response of $P(\gamma_1,\gamma_2)$.

2.3 Self-inverse ("even") and anti-self-inverse ("odd") polynomials

If the inverse of a polynomial is the polynomial itself, we call the polynomial self-inverse or "even" and if the inverse of a polynomial is the negative of the polynomial we call the polynomial anti-self-inverse or "odd". It is noted that the even and odd polynomial definitions look different from the conventional odd and even function definitions. This terminology

Table 1 Definitions of various even and odd polynomials

Type of P	Even and odd nature	Property
$P^{ee}(\gamma_1, \gamma_2)$	Even in γ_1 and even in γ_2	$P^{ee}(\gamma_{1*}, \gamma_2)(1+T\gamma_1)^M = P^{ee}(\gamma_1, \gamma_2)$
		$P^{ee}(\gamma_1, \gamma_{2*})(1+T\gamma_2)^N = P^{ee}(\gamma_1, \gamma_2)$
$P^{eo}(\gamma_1, \gamma_2)$	Even in γ_1 and odd in γ_2	$P^{eo}(\gamma_{1*}, \gamma_2)(1+T\gamma_1)^M = P^{eo}(\gamma_1, \gamma_2)$
		$P^{eo}(\gamma_1, \gamma_{2*})(1+T\gamma_2)^N = -P^{eo}(\gamma_1, \gamma_2)$
$P^{oe}(\gamma_1, \gamma_2)$	Odd in γ_1 and even in γ_2	$P^{oe}(\gamma_{1*}, \gamma_2)(1+T\gamma_1)^M = -P^{oe}(\gamma_1, \gamma_2)$
		$P^{oe}(\gamma_1, \gamma_{2*})(1+T\gamma_2)^N = P^{oe}(\gamma_1, \gamma_2)$
$P^{oo}(\gamma_1, \gamma_2)$	Odd in γ_1 and odd in γ_2	$P^{oo}(\gamma_{1*}, \gamma_2)(1+T\gamma_1)^M = -P^{oo}(\gamma_1, \gamma_2)$
		$P^{oo}(\gamma_1, \gamma_{2*})(1+T\gamma_2)^N = -P^{oo}(\gamma_1, \gamma_2)$

is used because the frequency response of the γ-domain odd(even) polynomial satisfies the conditions for odd(even) functions in the ω-domain. In this sense these definitions may be termed ω-even and ω-odd polynomials. However, for the sake of simplicity of notations, we use the odd and even terms in this paper.

As the inverse operation can be taken with respect to the two variables, there are a number of "odd" and "even" definitions. In Table 1, we define four classes of odd and even polynomials.

2.4 Decomposition of γ-domain polynomials

It can be shown that an odd polynomial in γ_i can be expressed as γ_i times an even polynomial. Then letting superscript 'x' represent a don't care case (may be even, odd or neither), we have

$$P^{ox}(\gamma_1, \gamma_2) = \gamma_1 P^{ex}(\gamma_1, \gamma_2) \tag{10}$$

$$P^{xo}(\gamma_1, \gamma_2) = \gamma_2 P^{xe}(\gamma_1, \gamma_2) \tag{11}$$

$$P^{oo}(\gamma_1, \gamma_2) = \gamma_1 \gamma_2 P^{ee}(\gamma_1, \gamma_2) \tag{12}$$

Further we can show that

$$\begin{aligned} P^{ex}(\gamma_1, \gamma_2) &= P^{ee}(\gamma_1, \gamma_2) + P^{eo}(\gamma_1, \gamma_2) \\ &= P^{ee}(\gamma_1, \gamma_2) + \gamma_2 P_1^{ee}(\gamma_1, \gamma_2) \end{aligned} \tag{13}$$

$$\begin{aligned} P^{xe}(\gamma_1, \gamma_2) &= P^{ee}(\gamma_1, \gamma_2) + P^{oe}(\gamma_1, \gamma_2) \\ &= P^{ee}(\gamma_1, \gamma_2) + \gamma_1 P_1^{ee}(\gamma_1, \gamma_2) \end{aligned} \tag{14}$$

Now combining the above results, we get a general decomposition result as stated in Theorem 1.

Theorem 1 *A polynomial $P(\gamma_1, \gamma_2)$ can always be decomposed into the form:*

$$P(\gamma_1, \gamma_2) = P^{ee}(\gamma_1, \gamma_2) + P^{oo}(\gamma_1, \gamma_2) + P^{oe}(\gamma_1, \gamma_2) + P^{eo}(\gamma_1, \gamma_2) \tag{15}$$

which can also be written as

$$P(\gamma_1, \gamma_2) = P_1^{ee}(\gamma_1, \gamma_2) + \gamma_1 \gamma_2 P_2^{ee}(\gamma_1, \gamma_2) + \gamma_1 P_3^{ee}(\gamma_1, \gamma_2) + \gamma_2 P_4^{ee}(\gamma_1, \gamma_2) \tag{16}$$

where some of the terms may be zero (absent) and each additive term present in the summation has the same degrees in γ_1 and γ_2 as those of the given polynomial $P(\gamma_1, \gamma_2)$. Note also that each term is complex and can be expressed as $P^{ee}_{iR'}(\gamma_1, \gamma_2) + jP^{ee}_{iR''}(\gamma_1, \gamma_2)$ for i = 1..4, where the subscript 'R' denotes a real-coefficient polynomial.

The individual terms can be found using the matrix relation:

$$\begin{bmatrix} P_1^{ee}(\gamma_1, \gamma_2) \\ \gamma_1\gamma_2 P_2^{ee}(\gamma_1, \gamma_2) \\ \gamma_1 P_3^{ee}(\gamma_1, \gamma_2) \\ \gamma_2 P_4^{ee}(\gamma_1, \gamma_2) \end{bmatrix} = \frac{1}{4} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix} \cdot \begin{bmatrix} P(\gamma_1, \gamma_2) \\ P_{\dagger\gamma_1}(\gamma_1, \gamma_2) \\ P_{\dagger\gamma_2}(\gamma_1, \gamma_2) \\ P_{\dagger}(\gamma_1, \gamma_2) \end{bmatrix} \tag{17}$$

A general form of $P^{ee}(\gamma_1, \gamma_2)$ can be written as

$$P^{ee}(\gamma_1, \gamma_2) = \left(1 + \frac{T\gamma_1}{2}\right)^{\alpha} \left(1 + \frac{T\gamma_2}{2}\right)^{\beta} (1 + T\gamma_1)^{\left[\frac{M}{2}\right]} (1 + T\gamma_2)^{\left[\frac{N}{2}\right]} Q\left(\frac{\gamma_1^2}{1 + T\gamma_1}, \frac{\gamma_2^2}{1 + T\gamma_2}\right) \tag{18}$$

where

$$Q\left(\frac{\gamma_1^2}{1 + T\gamma_1}, \frac{\gamma_2^2}{1 + T\gamma_2}\right) = \sum_{m=0}^{\left[\frac{M}{2}\right]} \sum_{n=0}^{\left[\frac{N}{2}\right]} a_{mn} \left(\frac{\gamma_1^2}{1 + T\gamma_1}\right)^m \left(\frac{\gamma_2^2}{1 + T\gamma_2}\right)^n$$

the notation [x] represents the integer part of x and

$\alpha = 0$ when M is even and 1 when M is odd
and $\beta = 0$ when N is even and 1 when N is odd.
and a_{mn} are complex coefficients.

2.5 Symmetric and anti-symmetric polynomials

Definition 3 A polynomial $P(\gamma_1, \gamma_2)$ is said to be symmetric if $P(\gamma_1, \gamma_2) = P(\gamma_2, \gamma_1)$. Such a polynomial is denoted with a subscript "S" as $P_S(\gamma_1, \gamma_2)$.

Definition 4 A polynomial $P(\gamma_1, \gamma_2)$ is said to be anti-symmetric if $P(\gamma_1, \gamma_2) = -P(\gamma_2, \gamma_1)$. Such a polynomial is denoted with a subscript "AS" as $P_{AS}(\gamma_1, \gamma_2)$.

2.6 Factorization of two-variable polynomials

A two-variable polynomial can be factored into a set of irreducible polynomials, and the factors are unique within a multiplying constant (Rajan and Swamy 1978). Using the notation $\hat{\gamma} = (\gamma_1, \gamma_2)$, let $P(\hat{\gamma})$ be factored in two ways as the left-, and right-hand sides of the following identity:

$$K_1 \prod_{i=1}^{I_1} P_i(\hat{\gamma}) \equiv K_2 \prod_{j=1}^{I_2} Q_j(\hat{\gamma}) \tag{19}$$

where all $P_i(\hat{\gamma})$'s and $Q_j(\hat{\gamma})$'s are irreducible polynomials. Then from the unique factorization property, we have $I_1 = I_2 = I$, and for each $P_i(\hat{\gamma}), i = 1, 2, \ldots, I$, there exists a

Table 2 Definitions of various symmetries in the magnitude squared function

	Type of symmetry	Conditions
1)	Reflection symmetry about ω_1 axis	$F(\omega_1,\omega_2) = F(\omega_1,-\omega_2),\ \forall(\omega_1,\omega_2)$
2)	Reflection symmetry about ω_2 axis	$F(\omega_1,\omega_2) = F(-\omega_1,\omega_2),\ \forall(\omega_1,\omega_2)$
3)	Reflection symmetry about $\omega_1 = \omega_2$ diagonal	$F(\omega_1,\omega_2) = F(\omega_2,\omega_1),\ \forall(\omega_1,\omega_2)$
4)	Reflection symmetry about $\omega_1 = -\omega_2$ diagonal	$F(\omega_1,\omega_2) = F(-\omega_2,-\omega_1),\ \forall(\omega_1,\omega_2)$
5)	Centro symmetry	$F(\omega_1,\omega_2) = F(-\omega_1,-\omega_2),\ \forall(\omega_1,\omega_2)$
6)	Quadrantal symmetry	$F(\omega_1,\omega_2) = F(-\omega_1,\omega_2) = F(\omega_1,-\omega_2) = F(-\omega_1,-\omega_2),\ \forall(\omega_1,\omega_2)$
7)	Diagonal symmetry	$F(\omega_1,\omega_2) = F(\omega_2,\omega_1) = F(-\omega_1,-\omega_2) = F(-\omega_2,-\omega_1),\ \forall(\omega_1,\omega_2)$
8)	Four-Fold (90°) rotational symmetry	$F(\omega_1,\omega_2) = F(-\omega_2,\omega_1) = F(-\omega_1,-\omega_2) = F(\omega_2,-\omega_1),\ \forall(\omega_1,\omega_2)$
9)	Octagonal symmetry	$F(\omega_1,\omega_2) = F(\omega_2,\omega_1) = F(-\omega_2,\omega_1) = F(-\omega_1,\omega_2) = F(-\omega_1,-\omega_2) = F(-\omega_2,-\omega_1) = F(\omega_2,-\omega_1) = F(\omega_1,-\omega_2),\ \forall(\omega_1,\omega_2)$

unique $Q_j(\hat{\gamma})(i$ may be equal to $j)$ such that $P_i(\hat{\gamma}) \equiv k_j Q_j(\hat{\gamma})$, where k_j's are constants such that $K_2 = K_1 \prod_{j=1}^{I} k_j$.

3 Symmetry definitions and operations

The various symmetries that could be present in the magnitude squared function of the frequency response are given in Table 2.
From Table 2, we can observe the following properties and relationships among the various symmetries:

- A real-coefficient polynomial always possesses centro symmetry, as can be seen from (5).
- Quadrantal symmetry is a combination of ω_1–axis reflection, ω_2–axis reflection, and centro symmetries. The presence of any two of the three symmetries implies the existence of the third and thus is enough to ensure quadrantal symmetry.
- Diagonal symmetry is a combination of $\omega_1 = \omega_2$ diagonal, $\omega_1 = -\omega_2$ diagonal, and centro symmetries. The presence of any two of the three symmetries implies the existence of the third and thus is enough to ensure diagonal symmetry.
- The condition $F(\omega_1,\omega_2) = F(-\omega_2,\omega_1)$ is enough to ensure four-fold 90° rotational symmetry, as it automatically implies the symmetries in the other quadrants as well.
- Octagonal symmetry is a combination of quadrantal, diagonal and 4-fold rotational symmetries. The presence of any two of the three symmetries implies the existence of the third, and is sufficient to guarantee octagonal symmetry.

It is noted that as $F(\omega_1,\omega_2)$ is a nonlinear function of $\omega_1 T_1$ and $\omega_2 T_2$, $F(\omega_1,\omega_2)$ cannot be equal to $F(\omega_2,\omega_1)$ unless $T_1 = T_2$. Hence in the case of symmetries that involve the interchange of ω_1 and ω_2 such as reflection symmetry about $\omega_1 = \omega_2$ diagonal or $\omega_1 = -\omega_2$ diagonal, diagonal symmetry, rotational symmetry, and octagonal symmetry, the sampling intervals T_1 and T_2 should be identical.

Table 3 Symmetry operations

Symmetry operation	$\Psi_\omega(\hat{\omega})$	$\Psi(\hat{\gamma})$
ω_1-axis reflection	$(\omega_1, -\omega_2)$	(γ_1, γ_{2*})
ω_2-axis reflection	$(-\omega_1, \omega_2)$	(γ_{1*}, γ_2)
Centro symmetry	$(-\omega_1, -\omega_2)$	$(\gamma_{1*}, \gamma_{2*})$
$\omega_1 = \omega_2$ diagonal reflection	(ω_2, ω_1)	(γ_2, γ_1)
$\omega_1 = -\omega_2$ diagonal reflection	$(-\omega_2, -\omega_1)$	$(\gamma_{2*}, \gamma_{1*})$
90° rotation about the origin	$(-\omega_2, \omega_1)$	(γ_{2*}, γ_1)

The frequency domain symmetry definitions can be stated in terms of basic symmetry operations, such as reflection about an axis, rotation about the origin, etc. Using the notation $\hat{\omega} = (\omega_1, \omega_2)$, let $\Psi_\omega(\hat{\omega})$ represent such an operation. Then $\Psi_\omega(\hat{\omega})$ is said to be a 2-cyclic operation if $\Psi_\omega^2(\hat{\omega}) = \Psi_\omega(\Psi_\omega(\hat{\omega})) = \hat{\omega}$. Similarly, $\Psi_\omega(\hat{\omega})$ is a 4-cyclic operation if $\Psi_\omega^4(\hat{\omega}) = \hat{\omega}$. In general, a $\Psi_\omega(\hat{\omega})$ operation is said to be n-cyclic if $\Psi_\omega^n(\hat{\omega}) = \hat{\omega}$.

As ω_i and γ_i are related by $\gamma_i = \frac{(e^{j\omega_i T}-1)}{T}$ and $\gamma_{i*} = \frac{(e^{-j\omega_i T}-1)}{T}$, $\Psi_\omega(\hat{\omega})$ can be translated to $\Psi(\hat{\gamma})$ using the correspondence $\omega_i \rightarrow \gamma_i$ and $-\omega_i \rightarrow \gamma_{i*}$ through analytic continuation. It is easy to see that $\Psi(\hat{\gamma})$ retains the cyclic properties of $\Psi_\omega(\hat{\omega})$, i.e., if Ψ_ω is 2-cyclic, then $\Psi(\hat{\gamma})$ is also 2-cyclic and if Ψ_ω is 4-cyclic, then $\Psi(\hat{\gamma})$ is also 4-cyclic. Table 3 shows some of the symmetry operations and the corresponding $\Psi_\omega(\hat{\omega})$ and $\Psi(\hat{\gamma})$.

Using the symmetry operations, a magnitude squared function $F(\hat{\omega})$ is said to possess a Ψ_ω-symmetry if

$$F(\hat{\omega}) = F(\Psi_\omega(\hat{\omega})) \tag{20}$$

Alternatively, using (6), the symmetry condition can be expressed in terms of the polynomials P:

$$P(\hat{\gamma}) \cdot P^*(\hat{\gamma}_*) = P(\Psi(\hat{\gamma})) \cdot P^*(\Psi(\hat{\gamma}_*)) \tag{21}$$

This equation can be made a polynomial equation by multiplying appropriate factors of $(1+T\gamma_1)$ and $(1+T\gamma_2)$ as

$$P(\hat{\gamma}) \cdot \left\{P^*(\hat{\gamma}_*)(1+T\gamma_1)^M(1+T\gamma_2)^N\right\} = \left\{P(\Psi[\hat{\gamma}])(1+T\gamma_1)^{M_1}(1+T\gamma_2)^{N_1}\right\} \cdot \left\{P^*(\Psi[\hat{\gamma}_*])(1+T\gamma_1)^{M_2}(1+T\gamma_2)^{N_2}\right\} \tag{22}$$

Depending on the Ψ operation, M_1, M_2, N_1, and N_2 are appropriate integers such that $M_1 + M_2 = M$, $N_1 + N_2 = N$ and the denominator factors are eliminated.

To aid the derivation of the symmetry constraints in the next section, the various symmetries in Table 2 are classified according to its number of cycles as follows:

- ω_1–axis reflection, ω_2–axis reflection, $\omega_1 = \omega_2$ diagonal, $\omega_1 = -\omega_2$ diagonal, and centro symmetries are all 2-cyclic symmetries.
- Four-fold 90° rotational symmetry is a 4-cyclic symmetry.
- Diagonal symmetry is a double 2-cyclic symmetry. The reason it is called "double" is that only *two* of the three 2-cyclic symmetries $\{\omega_1 = \omega_2$ diagonal, $\omega_1 = -\omega_2$ diagonal, and centro symmetries$\}$ are needed to guarantee the presence of diagonal symmetry.

Using the same reasoning, we can see that quadrantal symmetry is also a double 2-cyclic symmetry.

- Octagonal symmetry is either a triple 2-cyclic symmetry, or a combination of 4-cyclic and 2-cyclic symmetries. Recall that the presence of any two of the three symmetries {diagonal, quadrantal, and 4-fold rotational} is sufficient to guarantee octagonal symmetry. So, if diagonal and quadrantal symmetries are present, then the resulting octagonal symmetry is a tripe 2-cyclic symmetry. On the other hand, if 4-fold rotational and either diagonal or quadrantal symmetries are present, then the resulting octagonal symmetry is a combination of 4-cyclic and 2-cyclic symmetries.

4 Symmetry constraints for complex-coefficient 2-D polynomials in γ-domain

In this section, we derive the symmetry constraints for complex polynomials, starting with 2-cyclic symmetry.

4.1 Conditions for 2–cyclic symmetry in complex polynomials

We derive first the conditions for irreducible polynomials, followed by reducible polynomials.

4.1.1 Irreducible polynomials

From (22), a complex polynomial $P(\hat{\gamma})$ possesses a Ψ-symmetry in its magnitude squared response if:

$$P(\hat{\gamma}) \cdot \left\{P^*\left(\hat{\gamma}_*\right)(1+T\gamma_1)^M(1+T\gamma_2)^N\right\} = \left\{P\left(\Psi\left[\hat{\gamma}\right]\right)(1+T\gamma_1)^{M_1}(1+T\gamma_2)^{N_1}\right\} \cdot \left\{P^*\left(\Psi\left[\hat{\gamma}_*\right]\right)(1+T\gamma_1)^{M_2}(1+T\gamma_2)^{N_2}\right\} \tag{23}$$

where $\hat{\gamma} = (\gamma_1, \gamma_2)$ and $\hat{\gamma}_* = (\gamma_{1*}, \gamma_{2*})$.

As $P(\hat{\gamma})$ is irreducible, the unique factorization property of 2-variable polynomials states that $P(\hat{\gamma})$ should satisfy one of the following two conditions:

$$\text{(i)} \quad P(\hat{\gamma}) = k_1 P\left(\Psi\left[\hat{\gamma}\right]\right)(1+T\gamma_1)^{M_1}(1+T\gamma_2)^{N_1} \tag{24}$$

$$\text{(ii)} \quad P(\hat{\gamma}) = k_2 P^*\left(\Psi\left[\hat{\gamma}_*\right]\right)(1+T\gamma_1)^{M_2}(1+T\gamma_2)^{N_2} \tag{25}$$

where k_1 and k_2 are constants.

We first consider case (i) above. Letting $\hat{\gamma} = \Psi\left[\hat{\gamma}\right]$ in (24) and multiplying by the appropriate $(1+T\gamma_1)$ and $(1+T\gamma_2)$ factors, and realizing that $\Psi^2\left[\hat{\gamma}\right] = \hat{\gamma}$ for 2-cyclic symmetry, we obtain $P\left(\Psi\left[\hat{\gamma}\right]\right)(1+T\gamma_1)^{M_1}(1+T\gamma_2)^{N_1} = k_1 P(\hat{\gamma})$. Substituting this back into (24) we get $P(\hat{\gamma}) = k_1^2 P(\hat{\gamma})$, i.e. $k_1 = +1$ or -1. For case (ii), substituting $\hat{\gamma} = \Psi\left[\hat{\gamma}_*\right]$ into (25) and conjugating the equation, we get $P^*\left(\Psi\left[\hat{\gamma}_*\right]\right)(1+T\gamma_1)^{M_2}(1+T\gamma_2)^{N_2} = k_2^* P\left(\Psi^2\left[\hat{\gamma}\right]\right) = k_2^* P(\hat{\gamma})$. Substitute this back into (25), we obtain $k_2 \cdot k_2^* = 1$, i.e. $|k_2|^2 = 1$ or $k_2 = e^{j\theta}$ where θ is a real constant.

Thus, an irreducible $P(\hat{\gamma})$ should satisfy one of the following three conditions to possess 2-cyclic symmetry:

$$\begin{aligned}
&1. \quad P(\hat{\gamma}) = P\left(\Psi\left[\hat{\gamma}\right]\right)(1+T\gamma_1)^{M_1}(1+T\gamma_2)^{N_1}\\
&2. \quad P(\hat{\gamma}) = -P\left(\Psi\left[\hat{\gamma}\right]\right)(1+T\gamma_1)^{M_1}(1+T\gamma_2)^{N_1}\\
&3. \quad P(\hat{\gamma}) = e^{j\theta}\cdot P^*\left(\Psi\left[\hat{\gamma}_*\right]\right)(1+T\gamma_1)^{M_2}(1+T\gamma_2)^{N_2}
\end{aligned} \tag{26}$$

4.1.2 Reducible polynomials

Now in (23) if $P(\hat{\gamma})$ is reducible, then it can be expressed as a product of irreducible factors:

$$P(\hat{\gamma}) = k\cdot\prod_{i=1}^{N} P_i(\hat{\gamma}) \tag{27}$$

Let $P_i(\hat{\gamma})$ be one such irreducible factor in $P(\hat{\gamma})$. Using the unique factorization property, we can see that $P_i(\hat{\gamma})$ must be present in either $P\left(\Psi\left[\hat{\gamma}\right]\right)(1+T\gamma_1)^{M_1}(1+T\gamma_2)^{N_1}$ or $P^*\left(\Psi\left[\hat{\gamma}_*\right]\right)(1+T\gamma_1)^{M_2}(1+T\gamma_2)^{N_2}$ and thus must satisfy one of the following four conditions:

$$\begin{aligned}
&\text{(i)} \quad P_i(\hat{\gamma}) = k_3\cdot P_i\left(\Psi\left[\hat{\gamma}\right]\right)(1+T\gamma_1)^{M_3}(1+T\gamma_2)^{N_3}\\
&\text{(ii)} \quad P_i(\hat{\gamma}) = k_4\cdot P_j\left(\Psi\left[\hat{\gamma}\right]\right)(1+T\gamma_1)^{M_4}(1+T\gamma_2)^{N_4} \quad \text{where } i\neq j\\
&\text{(iii)} \quad P_i(\hat{\gamma}) = k_5\cdot P_i^*\left(\Psi\left[\hat{\gamma}_*\right]\right)(1+T\gamma_1)^{M_5}(1+T\gamma_2)^{N_5}\\
&\text{(iv)} \quad P_i(\hat{\gamma}) = k_6\cdot P_j^*\left(\Psi\left[\hat{\gamma}_*\right]\right)(1+T\gamma_1)^{M_6}(1+T\gamma_2)^{N_6} \quad \text{where } i\neq j
\end{aligned} \tag{28}$$

Cases (i) and (iii) are identical to the ones for irreducible polynomials, so the previous results apply. We investigate cases (ii) and (iv) here.

From case (ii), $P_i(\hat{\gamma}) = k_4\cdot P_j\left(\Psi\left[\hat{\gamma}\right]\right)(1+T\gamma_1)^{M_4}(1+T\gamma_2)^{N_4}$. We solve for $P_j(\hat{\gamma})$ by multiplying both sides by $1/k_4$ and letting $\hat{\gamma} = \Psi\left[\hat{\gamma}\right]$ (noting that $\Psi^2(\hat{\gamma}) = \hat{\gamma}$ since it is a 2-cyclic symmetry). This gives $P_j(\hat{\gamma}) = (1/k_4)\cdot P_i\left(\Psi\left[\hat{\gamma}\right]\right)(1+T\gamma_1)^{M_{4a}}(1+T\gamma_2)^{N_{4a}}$. Multiplying both sides of this equation by $P_i(\hat{\gamma})$, we get:

$$P_i(\hat{\gamma})\cdot P_j(\hat{\gamma}) = P_i(\hat{\gamma})\cdot\frac{1}{k_4}\cdot\left\{P_i\left(\Psi\left[\hat{\gamma}\right]\right)(1+T\gamma_1)^{M_{4a}}(1+T\gamma_2)^{N_{4a}}\right\} \tag{29}$$

Similarly for case (iv), multiplying by $1/k_6$, using the $\hat{\gamma} = \Psi\left[\hat{\gamma}_*\right]$ substitution, and conjugating, we can get the product of $P_i(\hat{\gamma})\cdot P_j(\hat{\gamma})$ as:

$$P_i(\hat{\gamma})\cdot P_j(\hat{\gamma}) = P_i(\hat{\gamma})\cdot\frac{1}{k_6^*}\cdot\left\{P_i^*\left(\Psi\left[\hat{\gamma}_*\right]\right)(1+T\gamma_1)^{M_{6a}}(1+T\gamma_2)^{N_{6a}}\right\} \tag{30}$$

So, (29) and (30) give the two new classes of polynomials.

Combining the results for both irreducible and reducible polynomials, we have the following theorem:

Theorem 2 *For a complex polynomial satisfying 2-cyclic symmetry, each of its factors must either alone or jointly satisfy one of the following five conditions:*

$$\begin{aligned}
&a)\quad P(\hat{\gamma}) = P\left(\Psi\left[\hat{\gamma}\right]\right)(1+T\gamma_1)^{M_1}(1+T\gamma_2)^{N_1}\\
&b)\quad P(\hat{\gamma}) = -P\left(\Psi\left[\hat{\gamma}\right]\right)(1+T\gamma_1)^{M_1}(1+T\gamma_2)^{N_1}\\
&c)\quad P(\hat{\gamma}) = e^{j\theta}\cdot P^*\left(\Psi\left[\hat{\gamma}_*\right]\right)(1+T\gamma_1)^{M_2}(1+T\gamma_2)^{N_2}\\
&d)\quad P(\hat{\gamma}) = Q(\hat{\gamma})Q\left(\Psi\left[\hat{\gamma}\right]\right)(1+T\gamma_1)^{M_7}(1+T\gamma_2)^{N_7}\\
&e)\quad P(\hat{\gamma}) = Q(\hat{\gamma})Q^*\left(\Psi\left[\hat{\gamma}_*\right]\right)(1+T\gamma_1)^{M_8}(1+T\gamma_2)^{N_8}
\end{aligned} \tag{31}$$

where $Q(\hat{\gamma})$ is any arbitrary complex polynomial. Note that conditions d) and e) above are (29) and (30) without the constants. The reader is reminded again that appropriate $(1+T\gamma_i)^k$ factors were introduced above to cancel the denominator that may result from the variable change to γ_{i*}.

With Theorem 2, we will now state the conditions for the various 2-cyclic symmetries: ω_1–axis reflection, ω_2–axis reflection, $\omega_1 = \omega_2$ diagonal, $\omega_1 = -\omega_2$ diagonal, and centro symmetries.

4.2 Reflection symmetry about ω_1-axis for complex polynomials

From Table 3, for ω_1–axis reflection symmetry, we use $\Psi\left[(\gamma_1,\gamma_2)\right] = (\gamma_1,\gamma_{2*})$. So, substituting this in Theorem 2, we get the following theorem:

Theorem 3 *A complex polynomial $P(\gamma_1,\gamma_2)$ possesses reflection symmetry with respect to ω_1-axis in its magnitude response if its factors either alone or jointly satisfy one of the following conditions:*

$$\begin{aligned}
&a)\quad P(\gamma_1,\gamma_2) = P(\gamma_1,\gamma_{2*})(1+T\gamma_2)^{N}\\
&b)\quad P(\gamma_1,\gamma_2) = -P(\gamma_1,\gamma_{2*})(1+T\gamma_2)^{N}\\
&c)\quad P(\gamma_1,\gamma_2) = e^{j\theta}\cdot P^*(\gamma_{1*},\gamma_2)(1+T\gamma_1)^{M}\\
&d)\quad P(\gamma_1,\gamma_2) = Q(\gamma_1,\gamma_2)Q_{\dagger\gamma_2}(\gamma_1,\gamma_2)\\
&e)\quad P(\gamma_1,\gamma_2) = Q(\gamma_1,\gamma_2)Q^*_{\dagger\gamma_1}(\gamma_1,\gamma_2)
\end{aligned} \tag{32}$$

We can obtain polynomial factors that satisfy conditions (32)(a–c) as follows.

Applying the decomposition of polynomials as in (16), we can write (32)(a) as:

$$\begin{aligned}
&P_1^{ee}(\gamma_1,\gamma_2) + \gamma_1\gamma_2 P_2^{ee}(\gamma_1,\gamma_2) + \gamma_1 P_3^{ee}(\gamma_1,\gamma_2) + \gamma_2 P_4^{ee}(\gamma_1,\gamma_2)\\
&\quad = P_1^{ee}(\gamma_1,\gamma_2) - \gamma_1\gamma_2 P_2^{ee}(\gamma_1,\gamma_2) + \gamma_1 P_3^{ee}(\gamma_1,\gamma_2) - \gamma_2 P_4^{ee}(\gamma_1,\gamma_2)
\end{aligned}$$

$$\Rightarrow \gamma_1\gamma_2 P_2^{ee}(\gamma_1,\gamma_2) + \gamma_2 P_4^{ee}(\gamma_1,\gamma_2) = -\gamma_1\gamma_2 P_2^{ee}(\gamma_1,\gamma_2) - \gamma_2 P_4^{ee}(\gamma_1,\gamma_2)$$

This implies $P_2^{ee}(\gamma_1,\gamma_2) = 0$ and $P_4^{ee}(\gamma_1,\gamma_2) = 0$. Thus, the following polynomial factor satisfies condition (32)(a):

$$P(\gamma_1,\gamma_2) = P_1^{ee}(\gamma_1,\gamma_2) + \gamma_1 P_3^{ee}(\gamma_1,\gamma_2) = Q^{xe}(\gamma_1,\gamma_2) \tag{33}$$

Following the same procedure, it can be shown that (32)(b) yields the polynomial factor:

$$P(\gamma_1,\gamma_2) = \gamma_2\cdot Q^{xe}(\gamma_1,\gamma_2) \tag{34}$$

We now work on condition (32)(c). If we multiply both sides of (32)(c) by $e^{-j\theta/2}$, we get:

$$e^{-j\theta/2} \cdot P(\gamma_1, \gamma_2) = e^{j\theta/2} \cdot P^*(\gamma_{1*}, \gamma_2)(1 + T\gamma_1)^M$$

Letting $\tilde{P}(\gamma_1, \gamma_2) = e^{-j\theta/2} \cdot P(\gamma_1, \gamma_2)$, the above equation becomes:

$$\tilde{P}(\gamma_1, \gamma_2) = \tilde{P}^*(\gamma_{1*}, \gamma_2)(1 + T\gamma_1)^M$$

Applying the decomposition of polynomials to the above, we get the following:

$$\begin{aligned} &P_1^{ee}(\gamma_1, \gamma_2) + \gamma_1\gamma_2 P_2^{ee}(\gamma_1, \gamma_2) + \gamma_1 P_3^{ee}(\gamma_1, \gamma_2) + \gamma_2 P_4^{ee}(\gamma_1, \gamma_2) \\ &\quad = P_1^{ee*}(\gamma_1, \gamma_2) - \gamma_1\gamma_2 P_2^{ee*}(\gamma_1, \gamma_2) - \gamma_1 P_3^{ee*}(\gamma_1, \gamma_2) + \gamma_2 P_4^{ee*}(\gamma_1, \gamma_2) \end{aligned}$$

Equating the terms on both sides of the equation:

$P_1^{ee}(\gamma_1, \gamma_2) = P_1^{ee*}(\gamma_1, \gamma_2)$ implies that P_1^{ee} is real, i.e. $P_1^{ee} = P_{1R}^{ee}(\gamma_1, \gamma_2)$, where the subscript 'R' denotes a polynomial with real coefficients.

$$\begin{aligned} P_2^{ee}(\gamma_1, \gamma_2) &= -P_2^{ee*}(\gamma_1, \gamma_2) \text{ implies } P_2^{ee} = j \cdot P_{2R}^{ee}(\gamma_1, \gamma_2) \\ P_3^{ee}(\gamma_1, \gamma_2) &= -P_3^{ee*}(\gamma_1, \gamma_2) \text{ implies } P_3^{ee} = j \cdot P_{3R}^{ee}(\gamma_1, \gamma_2) \\ P_4^{ee}(\gamma_1, \gamma_2) &= P_4^{ee*}(\gamma_1, \gamma_2) \text{ implies } P_4^{ee} = P_{4R}^{ee}(\gamma_1, \gamma_2) \end{aligned}$$

So,

$$\tilde{P}(\gamma_1, \gamma_2) = P_{1R}^{ee}(\gamma_1, \gamma_2) + \gamma_2 P_{4R}^{ee}(\gamma_1, \gamma_2) + j\left\{\gamma_1\gamma_2 P_{2R}^{ee}(\gamma_1, \gamma_2) + \gamma_1 P_{3R}^{ee}(\gamma_1, \gamma_2)\right\}$$

Using (13) and (14), we can rewrite the above as:

$$\tilde{P}(\gamma_1, \gamma_2) = P_{5R}^{ex}(\gamma_1, \gamma_2) + j \cdot \gamma_1 \cdot P_{6R}^{ex}(\gamma_1, \gamma_2)$$

Converting back to $P(\gamma_1, \gamma_2)$,

$$\begin{aligned} P(\gamma_1, \gamma_2) &= e^{j\theta/2} \cdot \tilde{P}(\gamma_1, \gamma_2) \\ &= e^{j\theta/2} \cdot \left\{P_{5R}^{ex}(\gamma_1, \gamma_2) + j \cdot \gamma_1 \cdot P_{6R}^{ex}(\gamma_1, \gamma_2)\right\} \end{aligned} \quad (35)$$

With this understanding, the polynomial factors satisfying Theorem 3 are listed in Table 4. Note that (34) is omitted there as it can be expressed as the product of (33) and (35).

4.3 Reflection symmetry about ω2-axis for complex polynomials

For ω_2–axis reflection symmetry, we substitute $\Psi\left[(\gamma_1, \gamma_2)\right] = (\gamma_{1*}, \gamma_2)$ into Theorem 2 to obtain the following:

Theorem 4 *A complex polynomial $P(\gamma_1, \gamma_2)$ possesses reflection symmetry with respect to ω_2-axis in its magnitude response if its factors either alone or jointly satisfy one of the following conditions:*

$$\begin{aligned} &a)\quad P(\gamma_1, \gamma_2) = P(\gamma_{1*}, \gamma_2)(1 + T\gamma_1)^M \\ &b)\quad P(\gamma_1, \gamma_2) = -P(\gamma_{1*}, \gamma_2)(1 + T\gamma_1)^M \\ &c)\quad P(\gamma_1, \gamma_2) = e^{j\theta} \cdot P^*(\gamma_1, \gamma_{2*})(1 + T\gamma_2)^N \\ &d)\quad P(\gamma_1, \gamma_2) = Q(\gamma_1, \gamma_2) Q_{\dagger\gamma_1}(\gamma_1, \gamma_2) \\ &e)\quad P(\gamma_1, \gamma_2) = Q(\gamma_1, \gamma_2) Q^*_{\dagger\gamma_2}(\gamma_1, \gamma_2) \end{aligned} \quad (36)$$

Table 4 γ-Domain complex polynomial factors possessing symmetry

Reflection symmetry w.r.t ω_1 -axis	Reflection symmetry w.r.t ω_2-axis
a) $Q^{xe}(\gamma_1,\gamma_2)$	a) $Q^{ex}(\gamma_1,\gamma_2)$
b) $Q^{ex}_{1R}(\gamma_1,\gamma_2)+j\gamma_1 Q^{ex}_{2R}(\gamma_1,\gamma_2)$	b) $Q^{xe}_{1R}(\gamma_1,\gamma_2)+j\gamma_2 Q^{xe}_{2R}(\gamma_1,\gamma_2)$
c) $Q(\gamma_1,\gamma_2)\cdot Q_{\dagger\gamma_2}(\gamma_1,\gamma_2)$	c) $Q(\gamma_1,\gamma_2)\cdot Q_{\dagger\gamma_1}(\gamma_1,\gamma_2)$
d) $Q(\gamma_1,\gamma_2)\cdot Q^{*}_{\dagger\gamma_1}(\gamma_1,\gamma_2)$	d) $Q(\gamma_1,\gamma_2)\cdot Q^{*}_{\dagger\gamma_2}(\gamma_1,\gamma_2)$
Reflection symmetry w.r.t. $\omega_1=\omega_2$ *diagonal*	*Reflection symmetry w.r.t.* $\omega_1=-\omega_2$ *diagonal*
a) $Q_S(\gamma_1,\gamma_2)$	a) $Q^{ee}_{1S}(\gamma_1,\gamma_2)+\gamma_1\gamma_2 Q^{ee}_{2S}(\gamma_1,\gamma_2)+\gamma_1 Q^{ee}_{3}(\gamma_1,\gamma_2)-\gamma_2 Q^{ee}_{3}(\gamma_2,\gamma_1)$
b) $Q^{ee}_{1RS}(\gamma_1,\gamma_2)+\gamma_1\gamma_2 Q^{ee}_{2RS}(\gamma_1,\gamma_2)+\gamma_1 Q^{ee}_{3R}(\gamma_1,\gamma_2)-\gamma_2 Q^{ee}_{3R}(\gamma_2,\gamma_1)+j\cdot\left\{\begin{array}{l}Q^{ee}_{4RAS}(\gamma_1,\gamma_2)\\+\gamma_1\gamma_2 Q^{ee}_{5RAS}(\gamma_1,\gamma_2)\\+\gamma_1 Q^{ee}_{6R}(\gamma_1,\gamma_2)+\gamma_2 Q^{ee}_{6R}(\gamma_2,\gamma_1)\end{array}\right\}$	b) $Q^{ee}_{1RS}(\gamma_1,\gamma_2)+\gamma_1\gamma_2 Q^{ee}_{2RS}(\gamma_1,\gamma_2)+\gamma_1 Q^{ee}_{3R}(\gamma_1,\gamma_2)+\gamma_2 Q^{ee}_{3R}(\gamma_2,\gamma_1)+j\cdot\left\{\begin{array}{l}Q^{ee}_{4RAS}(\gamma_1,\gamma_2)\\+\gamma_1\gamma_2 Q^{ee}_{5RAS}(\gamma_1,\gamma_2)\\+\gamma_1 Q^{ee}_{6R}(\gamma_1,\gamma_2)-\gamma_2 Q^{ee}_{6R}(\gamma_2,\gamma_1)\end{array}\right\}$
c) $Q(\gamma_1,\gamma_2)\cdot Q(\gamma_2,\gamma_1)$	c) $Q(\gamma_1,\gamma_2)\cdot Q_{\dagger}(\gamma_2,\gamma_1)$
d) $Q(\gamma_1,\gamma_2)\cdot Q^{*}_{\dagger}(\gamma_2,\gamma_1)$	d) $Q(\gamma_1,\gamma_2)\cdot Q^{*}(\gamma_2,\gamma_1)$
Octagonal symmetry	
a) $Q^{ee}_{S}(\gamma_1,\gamma_2)$	
b) $Q^{ee}_{1RS}(\gamma_1,\gamma_2)+jQ^{ee}_{2RAS}(\gamma_1,\gamma_2)$	
c) $Q^{ee}_{1RS}(\gamma_1,\gamma_2)+j\gamma_1\gamma_2 Q^{ee}_{2RAS}(\gamma_1,\gamma_2)$	
d) $Q^{ee}_{1RAS}(\gamma_1,\gamma_2)+j\gamma_1\gamma_2 Q^{ee}_{2RS}(\gamma_1,\gamma_2)$	
e) $Q^{ee}_{1RS}(\gamma_1,\gamma_2)+j\gamma_1\gamma_2 Q^{ee}_{2RS}(\gamma_1,\gamma_2)$	
f) $\gamma_1 Q^{ee}_{R}(\gamma_1,\gamma_2)\pm j\gamma_2 Q^{ee}_{R}(\gamma_2,\gamma_1)$	

The polynomial factors satisfying Theorem 4 can be derived in the same manner as previously shown for ω_1*–axis reflection symmetry, using the polynomial decomposition theorem. They are listed in Table 4.*

4.4 Centro symmetry for complex polynomials

For centro symmetry, we substitute $\Psi\left[(\gamma_1,\gamma_2)\right]=(\gamma_{1*},\gamma_{2*})$ into Theorem 2 to obtain the following:

Theorem 5 *A complex polynomial* $P(\gamma_1,\gamma_2)$ *possesses centro symmetry in its magnitude response if its factors either alone or jointly satisfy one of the following conditions:*

$$\begin{aligned}
&a)\quad P(\gamma_1,\gamma_2)=P(\gamma_{1*},\gamma_{2*})(1+T\gamma_1)^M(1+T\gamma_2)^N\\
&b)\quad P(\gamma_1,\gamma_2)=-P(\gamma_{1*},\gamma_{2*})(1+T\gamma_1)^M(1+T\gamma_2)^N\\
&c)\quad P(\gamma_1,\gamma_2)=e^{j\theta}\cdot P^{*}(\gamma_1,\gamma_2)\\
&d)\quad P(\gamma_1,\gamma_2)=Q(\gamma_1,\gamma_2)\cdot Q_{\dagger}(\gamma_1,\gamma_2)\\
&e)\quad P(\gamma_1,\gamma_2)=Q(\gamma_1,\gamma_2)\cdot Q^{*}(\gamma_1,\gamma_2)
\end{aligned}\tag{37}$$

The polynomial factors satisfying Theorem 5 can be derived in the same manner and are listed in Table 5.

Table 5 γ-Domain, s-domain, z-domain complex polynomial factors possessing symmetry

γ-Domain	s-Domain	z-Domain (Note: $x_i = z_i + z_i^{-1}$, $y_i = z_i - z_i^{-1}$, $i = 1, 2$)
Centro symmetry		
a) $Q_1^{ee}(\gamma_1,\gamma_2) + \gamma_1\gamma_2 Q_2^{ee}(\gamma_1,\gamma_2)$	a) $Q_1\left(s_1^2, s_2^2\right) + s_1 s_2 Q_2\left(s_1^2, s_2^2\right)$	a) $Q_1(x_1,x_2) + y_1 y_2 Q_2(x_1,x_2)$
b) $\gamma_1 Q_1^{ee}(\gamma_1,\gamma_2) + \gamma_2 Q_2^{ee}(\gamma_1,\gamma_2)$	b) $s_1 Q_1\left(s_1^2, s_2^2\right) + s_2 Q_2\left(s_1^2, s_2^2\right)$	b) $y_1 Q_1(x_1,x_2) + y_2 Q_2(x_1,x_2)$
c) $Q_R(\gamma_1,\gamma_2)$	c) $Q_R(s_1,s_2)$	c) $Q_R(z_1,z_2)$
d) $Q(\gamma_1,\gamma_2)\cdot Q_{\dagger}(\gamma_1,\gamma_2)$	d) $Q(s_1,s_2)\cdot Q(-s_1,-s_2)$	d) $Q(z_1,z_2)\cdot Q(z_1^{-1},z_2^{-1})$
e) $Q(\gamma_1,\gamma_2)\cdot Q^*(\gamma_1,\gamma_2)$	e) $Q(s_1,s_2)\cdot Q^*(s_1,s_2)$	e) $Q(z_1,z_2)\cdot Q^*(z_1,z_2)$
Four-fold rotational symmetry		
a) $Q_{1S}^{ee}(\gamma_1,\gamma_2) + \gamma_1\gamma_2 Q_{2AS}^{ee}(\gamma_1,\gamma_2)$	a) $Q_{1S}\left(s_1^2, s_2^2\right) + s_1 s_2 Q_{2AS}\left(s_1^2, s_2^2\right)$	a) $Q_{1S}(x_1,x_2) + y_1 y_2 Q_{2AS}(x_1,x_2)$
b) $Q_{1AS}^{ee}(\gamma_1,\gamma_2) + \gamma_1\gamma_2 Q_{2S}^{ee}(\gamma_1,\gamma_2)$	b) $Q_{1AS}\left(s_1^2, s_2^2\right) + s_1 s_2 Q_{2S}\left(s_1^2, s_2^2\right)$	b) $Q_{1AS}(x_1,x_2) + y_1 y_2 Q_{2S}(x_1,x_2)$
c) $\gamma_1 Q^{ee}(\gamma_1,\gamma_2) \pm j\gamma_2 Q^{ee}(\gamma_2,\gamma_1)$	c) $s_1 Q\left(s_1^2, s_2^2\right) \pm j s_2 Q\left(s_2^2, s_1^2\right)$	c) $y_1 Q(x_1,x_2) \pm j y_2 Q(x_2,x_1)$
d) $\begin{matrix} Q_{1RS}^{ee}(\gamma_1,\gamma_2) + \gamma_1\gamma_2 Q_{2RAS}^{ee}(\gamma_1,\gamma_2) \\ +j\left\{\begin{matrix} Q_{3RAS}^{ee}(\gamma_1,\gamma_2) \\ +\gamma_1\gamma_2 Q_{4RS}^{ee}(\gamma_1,\gamma_2)\end{matrix}\right\}\end{matrix}$	d) $\begin{matrix} Q_{1RS}\left(s_1^2, s_2^2\right) + s_1 s_2 Q_{2RAS}\left(s_1^2, s_2^2\right) \\ +j\left\{\begin{matrix} Q_{3RAS}\left(s_1^2, s_2^2\right) \\ +s_1 s_2 Q_{4RS}\left(s_1^2, s_2^2\right)\end{matrix}\right\}\end{matrix}$	d) $\begin{matrix} Q_{1RS}(x_1,x_2) + y_1 y_2 Q_{2RAS}(x_1,x_2) \\ +j\left\{\begin{matrix} Q_{3RAS}(x_1,x_2) \\ +y_1 y_2 Q_{4RS}(x_1,x_2)\end{matrix}\right\}\end{matrix}$
Quadrantal symmetry		
a) $Q^{ee}(\gamma_1,\gamma_2)$	a) $Q\left(s_1^2, s_2^2\right)$	a) $Q(x_1,x_2)$
b) $Q_R^{xe}(\gamma_1,\gamma_2)$	b) $Q_R\left(s_1, s_2^2\right)$	b) $Q_R(z_1,x_2)$
c) $Q_R^{ex}(\gamma_1,\gamma_2)$	c) $Q_R\left(s_1^2, s_2\right)$	c) $Q_R(x_1,z_2)$
d) $Q_{1R}^{ee}(\gamma_1,\gamma_2) + j\gamma_1\gamma_2 Q_{2R}^{ee}(\gamma_1,\gamma_2)$	d) $Q_{1R}\left(s_1^2, s_2^2\right) + j s_1 s_2 Q_{2R}\left(s_1^2, s_2^2\right)$	d) $Q_{1R}(x_1,x_2) + j y_1 y_2 Q_{2R}(x_1,x_2)$
e) $\gamma_2 Q_{1R}^{ee}(\gamma_1,\gamma_2) + j\gamma_1 Q_{2R}^{ee}(\gamma_1,\gamma_2)$	e) $s_2 Q_{1R}\left(s_1^2, s_2^2\right) + j s_1 Q_{2R}\left(s_1^2, s_2^2\right)$	e) $y_2 Q_{1R}(x_1,x_2) + j y_1 Q_{2R}(x_1,x_2)$

Table 5 continued

γ-domain	s-domain	z-domain (Note: $x_i = z_i + z_i^{-1}$, $y_i = z_i - z_i^{-1}$, $i = 1, 2$)
Diagonal symmetry		
a) $Q_{1S}^{ee}(\gamma_1, \gamma_2) + \gamma_1\gamma_2 Q_{2S}^{ee}(\gamma_1, \gamma_2)$	a) $Q_{1S}\left(s_1^2, s_2^2\right) + s_1 s_2 Q_{2S}\left(s_1^2, s_2^2\right)$	a) $Q_{1S}(x_1, x_2) + y_1 y_2 Q_{2S}(x_1, x_2)$
b) $Q_{RS}(\gamma_1, \gamma_2)$	b) $Q_{RS}(s_1, s_2)$	b) $Q_{RS}(z_1, z_2)$
c) $Q_{1RS}^{ee}(\gamma_1, \gamma_2) + \gamma_1\gamma_2 Q_{2RS}^{ee}(\gamma_1, \gamma_2) + j\left\{ \begin{array}{l} Q_{3RAS}^{ee}(\gamma_1, \gamma_2) \\ +\gamma_1\gamma_2 Q_{4RAS}^{ee}(\gamma_1, \gamma_2) \end{array} \right\}$	c) $Q_{1RS}\left(s_1^2, s_2^2\right) + s_1 s_2 Q_{2RS}\left(s_1^2, s_2^2\right) + j\left\{ \begin{array}{l} Q_{3RAS}\left(s_1^2, s_2^2\right) \\ +s_1 s_2 Q_{4RAS}\left(s_1^2, s_2^2\right) \end{array} \right\}$	c) $Q_{1RS}(x_1, x_2) + y_1 y_2 Q_{2RS}(x_1, x_2) + j\left\{ \begin{array}{l} Q_{3RAS}(x_1, x_2) \\ +y_1 y_2 Q_{4RAS}(x_1, x_2) \end{array} \right\}$
d) $\gamma_1 Q_{1R}^{ee}(\gamma_1, \gamma_2) - \gamma_2 Q_{1R}^{ee}(\gamma_2, \gamma_1) + j\left\{ \begin{array}{l} \gamma_1 Q_{2R}^{ee}(\gamma_1, \gamma_2) \\ +\gamma_2 Q_{2R}^{ee}(\gamma_2, \gamma_1) \end{array} \right\}$	d) $s_1 Q_{1R}\left(s_1^2, s_2^2\right) - s_2 Q_{1R}\left(s_2^2, s_1^2\right) + j\left\{ \begin{array}{l} s_1 Q_{2R}\left(s_1^2, s_2^2\right) \\ +s_2 Q_{2R}\left(s_2^2, s_1^2\right) \end{array} \right\}$	d) $y_1 Q_{1R}(x_1, x_2) - y_2 Q_{1R}(x_2, x_1) + j\left\{ \begin{array}{l} y_1 Q_{2R}(x_1, x_2) \\ +y_2 Q_{2R}(x_2, x_1) \end{array} \right\}$
e) $Q_{1RS}^{ee}(\gamma_1, \gamma_2) + \gamma_1\gamma_2 Q_{2RS}^{ee}(\gamma_1, \gamma_2) + \gamma_1 Q_{3R}^{ee}(\gamma_1, \gamma_2) - \gamma_2 Q_{3R}^{ee}(\gamma_2, \gamma_1)$	e) $Q_{1RS}\left(s_1^2, s_2^2\right) + s_1 s_2 Q_{2RS}\left(s_1^2, s_2^2\right) + s_1 Q_{3R}\left(s_1^2, s_2^2\right) - s_2 Q_{3R}\left(s_2^2, s_1^2\right)$	e) $Q_{1RS}(x_1, x_2) + y_1 y_2 Q_{2RS}(x_1, x_2) + y_1 Q_{3R}(x_1, x_2) - y_2 Q_{3R}(x_2, x_1)$

4.5 Reflection symmetry about $\omega_1 = \omega_2$ diagonal for complex polynomials

For $\omega_1 = \omega_2$ diagonal reflection symmetry, we substitute $\Psi\left[(\gamma_1, \gamma_2)\right] = (\gamma_2, \gamma_1)$ into Theorem 2 to obtain the following:

Theorem 6 *A complex polynomial $P(\gamma_1, \gamma_2)$ possesses reflection symmetry with respect to $\omega_1 = \omega_2$ diagonal in its magnitude response if its factors either alone or jointly satisfy one of the following conditions:*

$$\begin{aligned}
&a)\quad P(\gamma_1,\gamma_2) = P(\gamma_2,\gamma_1)\\
&b)\quad P(\gamma_1,\gamma_2) = -P(\gamma_2,\gamma_1)\\
&c)\quad P(\gamma_1,\gamma_2) = e^{j\theta}\cdot P^*(\gamma_{2*},\gamma_{1*})(1+T\gamma_1)^N(1+T\gamma_2)^M\\
&d)\quad P(\gamma_1,\gamma_2) = Q(\gamma_1,\gamma_2)\cdot Q(\gamma_2,\gamma_1)\\
&e)\quad P(\gamma_1,\gamma_2) = Q(\gamma_1,\gamma_2)\cdot Q^*_\dagger(\gamma_2,\gamma_1)
\end{aligned} \tag{38}$$

The polynomial factors satisfying Theorem 6 can be derived in the same manner and are listed in Table 4.

4.6 Reflection symmetry about $\omega_1 = -\,\omega_2$ diagonal for complex polynomials

For $\omega_1 = -\,\omega_2$ diagonal reflection symmetry, we substitute $\Psi\left[(\gamma_1, \gamma_2)\right] = (\gamma_{2*}, \gamma_{1*})$ into Theorem 2 to obtain the following:

Theorem 7 *A complex polynomial $P(\gamma_1, \gamma_2)$ possesses reflection symmetry with respect to $\omega_1 = -\,\omega_2$ diagonal in its magnitude response if its factors either alone or jointly satisfy one of the following conditions:*

$$\begin{aligned}
&a)\quad P(\gamma_1,\gamma_2) = P(\gamma_{2*},\gamma_{1*})(1+T\gamma_1)^N(1+T\gamma_2)^M\\
&b)\quad P(\gamma_1,\gamma_2) = -P(\gamma_{2*},\gamma_{1*})(1+T\gamma_1)^N(1+T\gamma_2)^M\\
&c)\quad P(\gamma_1,\gamma_2) = e^{j\theta}\cdot P^*(\gamma_2,\gamma_1)\\
&d)\quad P(\gamma_1,\gamma_2) = Q(\gamma_1,\gamma_2)\cdot Q_\dagger(\gamma_2,\gamma_1)\\
&e)\quad P(\gamma_1,\gamma_2) = Q(\gamma_1,\gamma_2)\cdot Q^*(\gamma_2,\gamma_1)
\end{aligned} \tag{39}$$

The polynomial factors satisfying Theorem 7 can be derived in the same manner using the polynomial decomposition theorem. They are shown in Table 4.

4.7 Four-fold (90°) rotational symmetry for complex polynomials

Fourfold (90°) rotational symmetry is a 4-cyclic symmetry and has to be handled differently. Its conditions are shown in Theorem 8. Because of space constraints, the derivation is omitted.

Theorem 8 *A complex polynomial $P(\gamma_1, \gamma_2)$ possesses 4-fold (90°) rotational symmetry in its magnitude response if its factors either alone or jointly satisfy one of the following conditions:*

$$
\begin{aligned}
&a) \quad P(\gamma_1, \gamma_2) = P(\gamma_{2*}, \gamma_1)(1 + T\gamma_2)^M \\
&b) \quad P(\gamma_1, \gamma_2) = -P(\gamma_{2*}, \gamma_1)(1 + T\gamma_2)^M \\
&c) \quad P(\gamma_1, \gamma_2) = j \cdot P(\gamma_{2*}, \gamma_1)(1 + T\gamma_2)^M \\
&d) \quad P(\gamma_1, \gamma_2) = -j \cdot P(\gamma_{2*}, \gamma_1)(1 + T\gamma_2)^M \\
&e) \quad P(\gamma_1, \gamma_2) = e^{j\theta} \cdot P^*(\gamma_2, \gamma_{1*})(1 + T\gamma_1)^N \\
&f) \quad P(\gamma_1, \gamma_2) = Q(\gamma_1, \gamma_2) Q_{\dagger\gamma_2}(\gamma_2, \gamma_1) \\
&g) \quad P(\gamma_1, \gamma_2) = Q(\gamma_1, \gamma_2) Q^*_{\dagger\gamma_1}(\gamma_2, \gamma_1)
\end{aligned} \tag{40}
$$

The polynomial factors satisfying (a–e) of Theorem 8 can be derived in the same manner as previously shown for the other symmetries, using the polynomial decomposition theorem. They are listed in Table 5.

4.8 Conditions for double 2-cyclic symmetry for complex polynomials

We now derive the conditions for double 2-cyclic symmetry which can be applied to quadrantal and diagonal symmetries. Since there are numerous conditions, we will just focus on those for irreducible polynomials.

Suppose $P(\hat{\gamma})$ is irreducible and possesses two 2-cyclic symmetries Ψ_1 and Ψ_2. Then according to (31)(a–c), $P(\hat{\gamma})$ needs to satisfy the following two sets of conditions:

Ψ_1 2-cyclic symmetry	Ψ_2 2-cyclic symmetry
(i) $P(\hat{\gamma}) = P\left(\Psi_1\left[\hat{\gamma}\right]\right) \cdot (1+T\gamma_1)^{M_1}(1+T\gamma_2)^{N_1}$	(i) $P(\hat{\gamma}) = P\left(\Psi_2\left[\hat{\gamma}\right]\right) \cdot (1+T\gamma_1)^{M_3}(1+T\gamma_2)^{N_3}$
(ii) $P(\hat{\gamma}) = -P\left(\Psi_1\left[\hat{\gamma}\right]\right) \cdot (1+T\gamma_1)^{M_1}(1+T\gamma_2)^{N_1}$	(ii) $P(\hat{\gamma}) = -P\left(\Psi_2\left[\hat{\gamma}\right]\right) \cdot (1+T\gamma_1)^{M_3}(1+T\gamma_2)^{N_3}$
(iii) $P(\hat{\gamma}) = e^{j\theta_1} P^*\left(\Psi_1\left[\hat{\gamma}_*\right]\right) \cdot (1+T\gamma_1)^{M_2}(1+T\gamma_2)^{N_2}$	(iii) $P(\hat{\gamma}) = e^{j\theta_2} P^*\left(\Psi_2\left[\hat{\gamma}_*\right]\right) \cdot (1+T\gamma_1)^{M_4}(1+T\gamma_2)^{N_4}$

Combining these two sets of conditions, we get the following theorem.

Theorem 9 *For a complex polynomial satisfying double 2-cyclic symmetry, each of its factors must either alone or jointly satisfy one of the following nine conditions:*

$$
\begin{aligned}
&\text{a)} \quad P(\hat{\gamma}) = P\left(\Psi_1\left[\hat{\gamma}\right]\right)(1+T\gamma_1)^{M_1}(1+T\gamma_2)^{N_1} = P\left(\Psi_2\left[\hat{\gamma}\right]\right)(1+T\gamma_1)^{M_3}(1+T\gamma_2)^{N_3} \\
&\text{b)} \quad P(\hat{\gamma}) = P\left(\Psi_1\left[\hat{\gamma}\right]\right)(1+T\gamma_1)^{M_1}(1+T\gamma_2)^{N_1} = -P\left(\Psi_2\left[\hat{\gamma}\right]\right)(1+T\gamma_1)^{M_3}(1+T\gamma_2)^{N_3} \\
&\text{c)} \quad P(\hat{\gamma}) = P\left(\Psi_1\left[\hat{\gamma}\right]\right)(1+T\gamma_1)^{M_1}(1+T\gamma_2)^{N_1} = e^{j\theta_2} P^*\left(\Psi_2\left[\hat{\gamma}_*\right]\right)(1+T\gamma_1)^{M_4}(1+T\gamma_2)^{N_4} \\
&\text{d)} \quad P(\hat{\gamma}) = -P\left(\Psi_1\left[\hat{\gamma}\right]\right)(1+T\gamma_1)^{M_1}(1+T\gamma_2)^{N_1} = P\left(\Psi_2\left[\hat{\gamma}\right]\right)(1+T\gamma_1)^{M_3}(1+T\gamma_2)^{N_3} \\
&\text{e)} \quad P(\hat{\gamma}) = -P\left(\Psi_1\left[\hat{\gamma}\right]\right)(1+T\gamma_1)^{M_1}(1+T\gamma_2)^{N_1} = -P\left(\Psi_2\left[\hat{\gamma}\right]\right)(1+T\gamma_1)^{M_3}(1+T\gamma_2)^{N_3} \\
&\text{f)} \quad P(\hat{\gamma}) = -P\left(\Psi_1\left[\hat{\gamma}\right]\right)(1+T\gamma_1)^{M_1}(1+T\gamma_2)^{N_1} = e^{j\theta_2} P^*\left(\Psi_2\left[\hat{\gamma}_*\right]\right)(1+T\gamma_1)^{M_4}(1+T\gamma_2)^{N_4} \\
&\text{g)} \quad P(\hat{\gamma}) = e^{j\theta_1} P^*\left(\Psi_1\left[\hat{\gamma}_*\right]\right)(1+T\gamma_1)^{M_2}(1+T\gamma_2)^{N_2} = P\left(\Psi_2\left[\hat{\gamma}\right]\right)(1+T\gamma_1)^{M_3}(1+T\gamma_2)^{N_3} \\
&\text{h)} \quad P(\hat{\gamma}) = e^{j\theta_1} P^*\left(\Psi_1\left[\hat{\gamma}_*\right]\right)(1+T\gamma_1)^{M_2}(1+T\gamma_2)^{N_2} = -P\left(\Psi_2\left[\hat{\gamma}\right]\right)(1+T\gamma_1)^{M_3}(1+T\gamma_2)^{N_3} \\
&\text{i)} \quad P(\hat{\gamma}) = e^{j\theta_1} P^*\left(\Psi_1\left[\hat{\gamma}_*\right]\right)(1+T\gamma_1)^{M_2}(1+T\gamma_2)^{N_2} = e^{j\theta_2} P^*\left(\Psi_2\left[\hat{\gamma}_*\right]\right)(1+T\gamma_1)^{M_4}(1+T\gamma_2)^{N_4}
\end{aligned} \tag{41}
$$

4.9 Quadrantal symmetry for complex polynomials

Recall that quadrantal symmetry is a combination of three symmetries: ω_1-axis reflection, ω_2-axis reflection, and centro symmetry. The presence of any two of the three symmetries is sufficient to guarantee quadrantal symmetry. So we apply $\Psi_1(\gamma_1, \gamma_2) = (\gamma_1, \gamma_{2*})$, ω_1-axis reflection symmetry, and $\Psi_2(\gamma_1, \gamma_2) = (\gamma_{1*}, \gamma_{2*})$, centro symmetry, to (41)(a–i) of Theorem 9 to get the following.

Theorem 10 *A complex polynomial $P(\gamma_1, \gamma_2)$ possesses quadrantal symmetry in its magnitude response if its factors either alone or jointly satisfy one of the following conditions:*

$$
\begin{aligned}
&a)\ \ P(\hat{\gamma}) = P(\gamma_1, \gamma_{2*})(1 + T\gamma_2)^N = P(\gamma_{1*}, \gamma_{2*})(1 + T\gamma_1)^M(1 + T\gamma_2)^N\\
&b)\ \ P(\hat{\gamma}) = P(\gamma_1, \gamma_{2*})(1 + T\gamma_2)^N = -P(\gamma_{1*}, \gamma_{2*})(1 + T\gamma_1)^M(1 + T\gamma_2)^N\\
&c)\ \ P(\hat{\gamma}) = P(\gamma_1, \gamma_{2*})(1 + T\gamma_2)^N = e^{j\theta_2}P^*(\gamma_1, \gamma_2)\\
&d)\ \ P(\hat{\gamma}) = -P(\gamma_1, \gamma_{2*})(1 + T\gamma_2)^N = P(\gamma_{1*}, \gamma_{2*})(1 + T\gamma_1)^M(1 + T\gamma_2)^N\\
&e)\ \ P(\hat{\gamma}) = -P(\gamma_1, \gamma_{2*})(1 + T\gamma_2)^N = -P(\gamma_{1*}, \gamma_{2*})(1 + T\gamma_1)^M(1 + T\gamma_2)^N\\
&f)\ \ P(\hat{\gamma}) = -P(\gamma_1, \gamma_{2*})(1 + T\gamma_2)^N = e^{j\theta_2}P^*(\gamma_1, \gamma_2)\\
&g)\ \ P(\hat{\gamma}) = e^{j\theta_1}P^*(\gamma_{1*}, \gamma_2)(1 + T\gamma_1)^M = P(\gamma_{1*}, \gamma_{2*})(1 + T\gamma_1)^M(1 + T\gamma_2)^N\\
&h)\ \ P(\hat{\gamma}) = e^{j\theta_1}P^*(\gamma_{1*}, \gamma_2)(1 + T\gamma_1)^M = -P(\gamma_{1*}, \gamma_{2*})(1 + T\gamma_1)^M(1 + T\gamma_2)^N\\
&i)\ \ P(\hat{\gamma}) = e^{j\theta_1}P^*(\gamma_{1*}, \gamma_2)(1 + T\gamma_1)^M = e^{j\theta_2}P^*(\gamma_1, \gamma_2)
\end{aligned}
\tag{42}
$$

The polynomial factors satisfying Theorem 10 can be derived as before using the polynomial decomposition theorem. For example, the first part of condition (a), $P(\gamma_1, \gamma_2) = P(\gamma_1, \gamma_{2*})(1 + T\gamma_2)^N$, is from ω_1-axis reflection symmetry. So we know the result as: $P(\gamma_1, \gamma_2) = Q_1^{ee}(\gamma_1, \gamma_2) + \gamma_1 Q_3^{ee}(\gamma_1, \gamma_2)$. If we apply to this the second part of condition (a), $P(\gamma_1, \gamma_2) = P(\gamma_{1*}, \gamma_{2*})(1 + T\gamma_1)^M(1 + T\gamma_2)^N$, we get the polynomial factor as the result: $P(\gamma_1, \gamma_2) = Q_1^{ee}(\gamma_1, \gamma_2)$.

The remaining polynomial factors satisfying the other conditions can be derived in the same manner. They are summarized in Table 5.

4.10 Diagonal symmetry for complex polynomials

Diagonal symmetry is a combination of three symmetries: $\omega_1 = \omega_2$ diagonal reflection symmetry, $\omega_1 = -\omega_2$ diagonal reflection symmetry, and centro symmetry. The presence of any two of the three symmetries is sufficient to guarantee diagonal symmetry. So we apply $\Psi_1(\gamma_1, \gamma_2) = (\gamma_2, \gamma_1)$, $\omega_1 = \omega_2$ diagonal reflection symmetry, and $\Psi_2(\gamma_1, \gamma_2) = (\gamma_{1*}, \gamma_{2*})$, centro symmetry, to (41)(a–i) of Theorem 9 to get the following.

Theorem 11 *A complex polynomial $P(\gamma_1, \gamma_2)$ possesses diagonal symmetry in its magnitude response if its factors either alone or jointly satisfy one of the following conditions:*

$$
\begin{aligned}
&a)\quad P(\hat{\gamma}) = P(\gamma_2, \gamma_1) = P(\gamma_{1*}, \gamma_{2*})(1+T\gamma_1)^M (1+T\gamma_2)^N \\
&b)\quad P(\hat{\gamma}) = P(\gamma_2, \gamma_1) = -P(\gamma_{1*}, \gamma_{2*})(1+T\gamma_1)^M (1+T\gamma_2)^N \\
&c)\quad P(\hat{\gamma}) = P(\gamma_2, \gamma_1) = e^{j\theta_2} P^*(\gamma_1, \gamma_2) \\
&d)\quad P(\hat{\gamma}) = -P(\gamma_2, \gamma_1) = P(\gamma_{1*}, \gamma_{2*})(1+T\gamma_1)^M (1+T\gamma_2)^N \\
&e)\quad P(\hat{\gamma}) = -P(\gamma_2, \gamma_1) = -P(\gamma_{1*}, \gamma_{2*})(1+T\gamma_1)^M (1+T\gamma_2)^N \\
&f)\quad P(\hat{\gamma}) = -P(\gamma_2, \gamma_1) = e^{j\theta_2} P^*(\gamma_1, \gamma_2) \\
&g)\quad P(\hat{\gamma}) = e^{j\theta_1} P^*(\gamma_{2*}, \gamma_{1*})(1+T\gamma_1)^N (1+T\gamma_2)^M = P(\gamma_{1*}, \gamma_{2*})(1+T\gamma_1)^M (1+T\gamma_2)^N \\
&h)\quad P(\hat{\gamma}) = e^{j\theta_1} P^*(\gamma_{2*}, \gamma_{1*})(1+T\gamma_1)^N (1+T\gamma_2)^M = -P(\gamma_{1*}, \gamma_{2*})(1+T\gamma_1)^M (1+T\gamma_2)^N \\
&i)\quad P(\hat{\gamma}) = e^{j\theta_1} P^*(\gamma_{2*}, \gamma_{1*})(1+T\gamma_1)^N (1+T\gamma_2)^M = e^{j\theta_2} P^*(\gamma_1, \gamma_2)
\end{aligned}
\tag{43}
$$

The polynomial factors satisfying Theorem 11 can be derived in the same manner as before using the polynomial decomposition theorem. They are listed in Table 5.

4.11 Octagonal symmetry for complex polynomials

Octagonal symmetry is a combination of three symmetries: diagonal symmetry, quadrantal symmetry, and rotational symmetry. The presence of any two of the three symmetries is sufficient to guarantee octagonal symmetry. Since there are so many possible cases, we shall consider only those for irreducible polynomials. We can derive these polynomial factors by applying the rotational symmetry conditions (40)(a–e) to each of the quadrantal symmetry polynomials in Table 5. For instance, $P(\gamma_1, \gamma_2) = Q^{ee}(\gamma_1, \gamma_2)$ possesses quadrantal symmetry as per Table 5. Applying to this the rotational symmetry condition $P(\gamma_1, \gamma_2) = P(\gamma_{2*}, \gamma_1)(1+T\gamma_2)^M$ from (40)(a) yields the polynomial factor: $P(\gamma_1, \gamma_2) = Q_S^{ee}(\gamma_1, \gamma_2)$ which possesses octagonal symmetry. Continuing this manner, the rest of the polynomials can be derived and are summarized in Table 4.

4.12 Summary of complex polynomial factors possessing symmetry

The polynomial factors possessing the various symmetries are summarized in Tables 4, 5. In Table 5, the corresponding irreducible polynomial factors in s-domain and z-domain (Rajaravivarma et al. 1991) are also listed for comparison.

4.13 Convergence of γ-domain results to s-domain results

One major advantage of using the delta operator is that as the sampling period $T \to 0$, the γ-domain discrete-time system resembles that of a continuous-time system. As seen in (2), when $T \to 0$, $\delta[x(nT)] \Rightarrow \frac{d}{dt}x(t)$ in time domain. Alternatively, in the frequency or transform domain, we can observe that:

$$\lim_{T\to 0} \frac{e^{j\omega_i T} - 1}{T} = j\omega_i$$

Recall that $\gamma_i = \left(e^{j\omega_i T} - 1\right)/T$ and $s_i = j\omega_i$. So as $T \to 0$,

$$\gamma_i \Rightarrow s_i.$$

From the above we also have:

$$\lim_{T\to 0}\frac{\gamma_i^2}{1+T\gamma_i}\Rightarrow s_i^2$$

Thus, as T $\to$ 0, the γ-domain even-even polynomial $Q^{ee}(\gamma_1,\gamma_2)$ in (18) converges to its s-domain counterpart $Q\left(s_1^2, s_2^2\right)$, i.e.:

$$\begin{aligned}&Q^{ee}(\gamma_1,\gamma_2)\\&=\left(1+\frac{T\gamma_1}{2}\right)^{\alpha}\left(1+\frac{T\gamma_2}{2}\right)^{\beta}(1+T\gamma_1)^{\left[\frac{M}{2}\right]}(1+T\gamma_2)^{\left[\frac{N}{2}\right]}Q_1\left(\frac{\gamma_1^2}{1+T\gamma_1},\frac{\gamma_2^2}{1+T\gamma_2}\right)\\&\Rightarrow Q\left(s_1^2,s_2^2\right)\end{aligned}$$

With this understanding, it is easy to see that the γ-domain symmetry results in Table 5 converge to the s-domain ones when $T = 0$. Such convergence is not possible with the traditional z-domain results. It may be noted that the above implies that we can get s-domain results from γ-domain results. However, it is not easy to generate γ-domain results from s-domain results.

5 Filter design example

We now utilize the symmetry constraints to design a 2-D γ-domain IIR filter with complex coefficients. The optimization based procedure is adapted from Khoo et al. (2001, 2006) to obtain the filter coefficients such that the filter specifications are satisfied. Because of the symmetry constraints, the number of parameters to optimize is greatly reduced.

The magnitude specification of the filter is shown in Fig. 2a. It is clear that the magnitude response possesses $\omega_1 = \omega_2$ diagonal reflection symmetry. (Note that $\theta_i = \omega_i T$). So from Table 4, we select the numerator to be $N_S(\gamma_1,\gamma_2)$, which is case (a). Recall that the subscript ‘s’ means that the polynomial is symmetric, i.e. $N_S(\gamma_1,\gamma_2) = N_S(\gamma_2,\gamma_1)$. We then select the denominator to be $D_R(\gamma_1)\cdot D_R(\gamma_2)$, which is a subset of case (c) in Table 4. The denominator is chosen to be separable so that its stability can be easily assured. In addition, the denominator coefficients are chosen to be real to reduce the number of parameters. The numerator coefficients, however, are complex. The transfer function to be optimized and the forms for the numerator and denominator are shown in Fig. 2b. We pick the order of the filter to be (4,4). Because of the $\omega_1 = \omega_2$ diagonal reflection symmetry constraints, the number of real parameters to optimize is reduced from 58 to 34. Since the numerator coefficients are complex, each a_{ij} has two parameters to be optimized. The denominator coefficients b_i's are real, so there is only one parameter per coefficient.

Because of symmetry, we need only specify the desired response in a reduced region (180° sector) in the frequency plane. In this case, we optimize only in the region $\theta_2 > \theta_1$ and specify the sample points using rectangular grids. We then use the “lsqnonlin” routine in Matlab's Optimization Toolbox to minimize the objective function: $J=\sum_k\sum_l\left[F(\theta_{1k},\theta_{2l})-F_d(\theta_{1k},\theta_{2l})\right]^2$. Here, F is the magnitude squared response of the transfer function following the definition in (5), F_d is the desired magnitude square response from the filter specification, and $(\theta_{1k},\theta_{2l})$ are the selected frequency points to perform the optimization. Once the optimal filter coefficients are obtained, the stability is checked by solving for the poles. This is easy to do as the denominator is separable. Any unstable pole can be stabilized by replacing it with its inverse pole $\gamma_i \to -\gamma_i/(1+T\gamma_i)$ without affecting the magnitude response.

The contour and 3-D magnitude plots of the optimized filter are shown in Fig. 3.

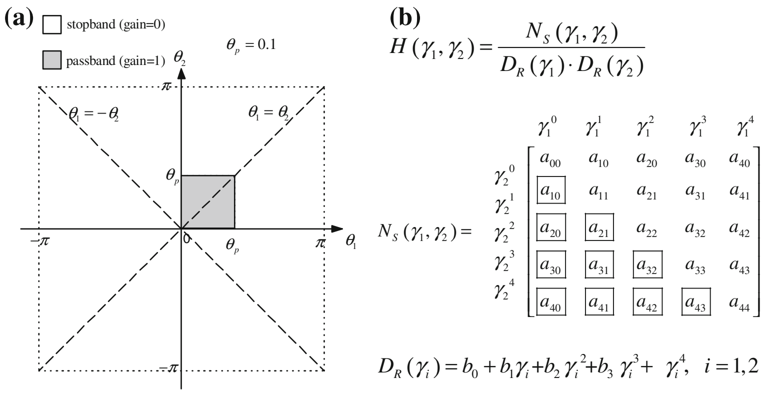

Fig. 2 **a** Specification of a lowpass filter with $\omega_1 = \omega_2$ diagonal reflection symmetry (Note that $\theta_i = \omega_i T$); **b** filter transfer function to be optimized and the forms for the numerator and denominator

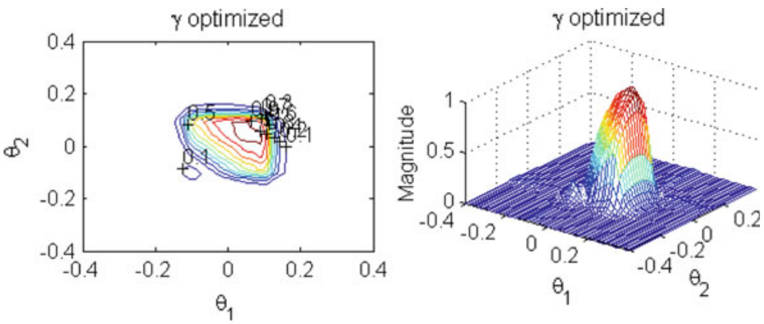

Fig. 3 Contour and 3-D magnitude plots of the optimized lowpass filter in γ-domain

5.1 Coefficient sensitivity advantage of γ-domain design

From the filter specifications, it can be seen that the lowpass filter has a narrow passband. For such filters, the γ-domain design has lower coefficient sensitivity to rounding errors compared to the z-domain design. The plots in Fig. 4 show the filter responses when the coefficients are rounded to 2 significant digits. It can be seen that the filter response is still very similar to the original. The magnitude response only starts to deviate from the original at 1 significant digit.

We next optimize for a z-domain filter satisfying the same filter specifications. The optimized response is shown in Fig. 5. The response when the coefficients are rounded to 4 significant digits is shown in Fig. 6. It can be seen that the response has deviated significantly at 4 significant digits. In fact, the filter has also become unstable.

So, in this example, the z-domain design requires 5 significant digits of precision in order to maintain the correct magnitude response, while the γ-domain design only needs 2 significant digits. This illustrates the sensitivity advantage of the γ-domain design over the z-domain design for narrowband filters.

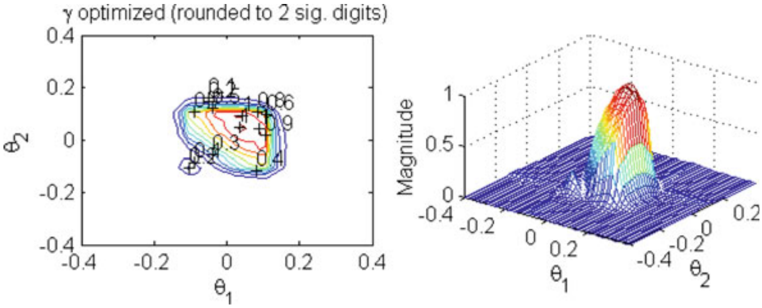

Fig. 4 γ-Domain lowpass filter with filter coefficients rounded to 2 significant digits

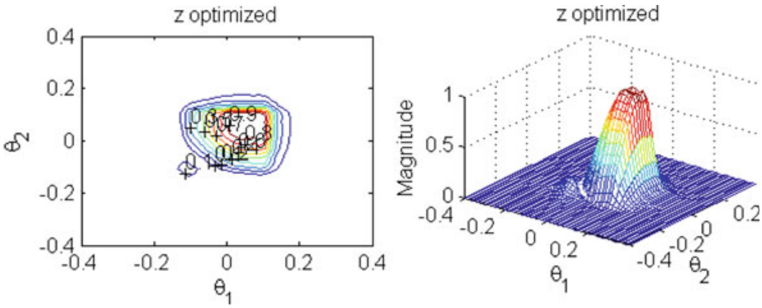

Fig. 5 Original (unperturbed) z-domain lowpass filter magnitude response

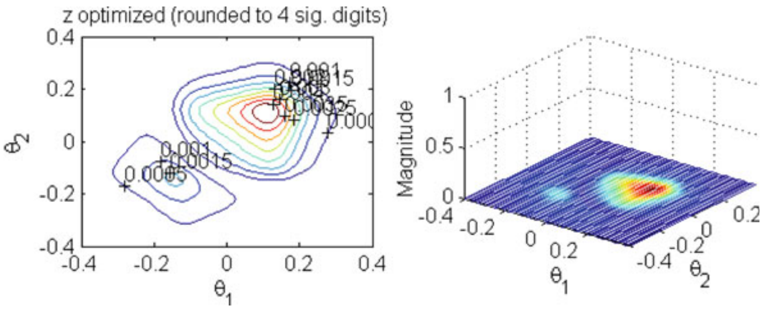

Fig. 6 z-Domain lowpass filter with filter coefficients rounded to 4 significant digits

6 Conclusions

The delta operator formulated discrete-time system (γ-domain) offers advantages over the traditional shift-operator formulated discrete-time system (z-domain) such as better numerical accuracy and lower coefficient sensitivity when high sampling rates are involved. In addition, it provides convergence to the underlying continuous-time system when the sampling period tends to zero. With these motivations, a unifying symmetry theory for γ-domain complex-coefficient 2-D polynomials and functions is established that can be used to aid the design and implementation of 2-D filters. Symmetry present in the 2-D frequency responses of a filter induces a relation among the filter coefficients and multipliers which can reduce the complexity of the design. In this paper, the symmetry constraints are identified and corresponding polynomial factors derived for 9 different types of symmetries. These symmetry results using the γ-domain approach merge with the continuous-time results when the sampling period tends to zero, which is not possible with the traditional z-domain approach. It may be noted that the symmetry results for real coefficient γ-domain polynomials presented in Khoo et al. (2006) are subsets of the results for complex polynomials presented here.

An example is then provided to illustrate the use of the symmetry constraints to design a 2-D IIR complex-coefficient filter, where the reduction in the number of independent filter coefficients is shown. For the narrowband filter in the example, it is also shown that the γ-domain design possesses better coefficient sensitivity than the z-domain counterpart.

Some of the future research could be focused on the application of 2-D symmetry theory to 2-D FIR/IIR filter bank design based on the works of Charoenlarpnopparut and Bose (1999, 2001).

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Author Biographies

I.-Hung Khoo received his Ph.D. in Electrical and Computer Engineering from the University of California Irvine in 2002. He is currently an Assistant Professor in the Department of Electrical Engineering, California State University Long Beach. Previously, he was a Senior I.C. Design Engineer at Agilent/Avago Technologies. Dr. Khoo's research interests include high speed & mixed signal circuit design, and analog & digital signal processing. He is a member of the IEEE.

Haranatha (Hari) C. Reddy (IEEE: Member 1977- Senior Member 1982- Fellow 1992) received B.E. degree in Electrical Engineering from Sri Venkateswara University, Tirupati, India; the M.E. degree in Electrical Engineering from M.S. University of Baroda, Baroda, India; and the Ph.D. degree in ECE, from Osmania University, Hyderabad, India. Dr. Reddy has been with the Department of Electrical Engineering, California State University, Long Beach, as a tenured Full-Professor since 1987. He was with the EE Department at Tennessee Technological University, Cookeville, Tennessee as an Associate Professor/Professor from 1980–1987. Before joining Tennessee Tech, Dr. Reddy held teaching and research positions at the State University of New York College at Buffalo; Concordia University, Montreal, Canada; Osmania University, Hyderabad, India; and S.G.S. Institute of Science and Technology, Indore, India. Professor Reddy was appointed in 2007 as Chair Professor at National Chiao Tung University (NCTU) in Taiwan where he is also serving as Founding Director of the Center for Circuit Theory, Communications and Signal Processing. From 1993–1999 he was a Guest Professor at the Institutes for Signal and Information Processing, and Integrated Systems, Swiss Federal Institute of Technology (ETH), Zurich, Switzerland; and as Visiting Researcher from 1994 to 1999 in the Department of Electrical and Computer Engineering at the University of California, Irvine, California. In 2003, he was awarded Honorary Professorship of the Technical University of Iasi, Iasi, Romania Dr. Reddy's teaching and research interests are in the broad areas of Circuit Design, Filters, Systems and Signal Processing. His research is concerned with the theory and applications of Multidimensional Circuits, Systems and Digital Filters. He has also been working on the development of filter theory for high speed signal processing and other related problems using delta discrete time operator. He has co-authored about one hundred and forty research papers in journals/conferences and four book chapters. He attended and presented papers in over fifty IEEE international conferences. Dr. Reddy was a guest editor for the special issue (Research and Education: Circuits and Systems) of the IEEE Transactions on Education published in August 1989. He served as the Principal Associate Editor for the IEEE Transactions on Education (1984–1990). From 1993 to 1996, he was also an Associate Editor for IEEE Transactions on Circuits and Systems (Part I) and also served as an Associate Editor for the Journal of Circuits, Systems and Computers. In 1999, he co-organized and served as General Chair of the IEEE CAS society's 3rd Emerging Technologies Workshop on Mixed Signal Integrated Circuit Design (MSICD) held in Long Beach, California. Professor Reddy served as the President-elect, President and past President of the IEEE Circuits and Systems Society in 2000, 2001 & 2002 respectively. Earlier, he has served as CAS Society's Vice President for Conferences for three years (1996, 1997 & 1999). From 1993 to 1995, he was a member of the IEEE CAS Society's Board of Governors. He has served as Chair of the IEEE CAS Society's Fellows Evaluation Committee in 2003 and 2004. Prof. Reddy is a recipient of the 2003 IEEE Circuits and Systems Society's Meritorious Service Award. Also, in 2003, Prof. Reddy received the diploma of honorary member of the Senate of the Technical University of Iasi, Romania and the "Gh. Asachi" University Medal for outstanding contributions in the activities of cooperation with the faculties of the Technical University. Prof. Reddy is a recipient of the Distinguished Faculty Scholarly and Creative Achievement Award (1990) and the Outstanding Professor Award (1993) from the California State University at Long Beach. He has also received in 1990 & 1992 the TRW Excellence in Research and Teaching Awards.

P. K. Rajan was born on September 20, 1942. He received the B.E. degree in electrical engineering from the University of Madras, India, in 1966, and M.Tech. and Ph.D. degrees in electrical engineering from the Indian Institute of Technology, Madras in 1969 and 1975, respectively. After having taught in various universities including Regional Engineering College, Tiruchirappalli, India, Indian Institute of Technology, Madras, Concordia University Montreal, Canada, SUNY College at Buffalo, North Dakota State University, Fargo, he joined Tennessee Tech University, Cookeville in 1983 as an Associate Professor of Electrical Engineering. He was promoted to Professor in 1985 and served as the Chair of the Department from 1991 to 2006. He has conducted research at Concordia University, Montreal, Canada, as a post-doctoral fellow, and at the Wright Patterson Air Force Base, Dayton, Ohio and at the NASA Johnson Space Center, Houston, Texas as a summer faculty fellow. He has published more than 100 articles on circuits and signal processing in various journals and conference proceedings. He has served as an associate editor of IEEE Transactions on Circuits and Systems for a number of terms and on the Technical Program Committees of a number of conferences. He is presently serving as an associate editor for Multidimensional Systems and Signal Processing. He recently served as the General Chair of the Southeastern Symposium on System Theory (SSST 2006). He has served as the President of Technical Committee on Digital Signal Processing of IEEE Circuits and Systems Society and the Southeastern EE Department Heads Association. He received the Caplenor Faculty Research Award from the Tennessee Tech University in 1990. His teaching and research interests are in the areas of circuits, digital signal processing, multidimensional signal processing and image processing. Dr. Rajan is a Fellow of IEEE and a member of ASEE and Sigma Xi.