行政院國家科學委員會專題研究計畫　成果報告

虧格為零之 SL_2(Z)同餘子群

計畫類別：個別型計畫
計畫編號：NSC92-2119-M-009-001-
執行期間：92 年 10 月 01 日至 93 年 07 月 31 日
執行單位：國立交通大學應用數學研究所

計畫主持人：楊一帆

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中文摘要.在本計畫中我們決定所有虧格為零之模同餘子群.我們也同時決定了對應於這些同餘子群的函數體的生成函數.

關鍵詞：模群，同餘子群，虧格，模函數.

Abstract. In this project we determine all genus zero congruence subgroups of SL(2,Z). It is shown that there are 132 conjugacy classes of such subgroups. In addition, the Hauptmoduls for those groups are also determined using the generalized Dedekind eta functions.

Keywords: modular group, congruence subgroups, genus zero, modular functions, Hauptmoduls.

1. Background and the main goal

Congruence subgroups of PSL(2,R) are subgroups of PSL(2,R) that contain the principal congruence subgroups Γ(N) consisting of all matrices in SL(2,Z) congruent to the identity matrix modulo N. Let G be a congruence subgroup of PSL(2,R). The genus of G is defined to be the genus of the compact Riemann surface G\H*. It is known that the genus of a congruence subgroup G is finite. Among all the congruence subgroups of PSL(2,R), the collection of all congruence subgroups of genus zero are of special interest due to their mysterious connection with the Monster simple group (see [B], [CN] for examples). It is an easy matter to see that there are infinitely many such groups. However, sub-collections of congruence subgroups that are contained in PSL(2,Z) are conjectured to be finite. The finiteness of such groups was first conjectured by Rademacher. Affirmative answer was provided by J. Dennin ([D1], [D2]) a few years later. It is not very clear that a complete list of all such groups is in reach or not as proofs provided by Thompson and Dennin are not constructive. This obstruction was finally removed by P. Zograf [Z], where bounds are given (see [Z] for more detail). His results, together with results of H. Larcher [Lar], D. McQuillan [M], and W. W. Stothers [St] provide us with possible levels. (If G is a congruence subgroup of genus zero, then the level of G is divisible by 2, 3, 5, 7, 11 or 13 and an upper bound of the index [PSL(2,Z) :G] < 128. Moreover the level of G is at most 127). It is now clear that a complete list of such groups is within reach and a thorough study of the Riemann-Hurwitz formula of all the intermediate groups between Γ(N) and PSL(2,Z) is all we need.

The main goal of our proposal is the determination of a complete list of congruence subgroups of PSL(2,Z) of genus zero. As the function field K(G\H*) of the Riemann surface associated to such a group G is known to be generated by a single function, this function can be viewed as a building block of the function field and its importance is needless to say. This makes our second goal, the determination of a generator of K(G\H), adequate and un-avoidable.

2. Results

In this project we successfully achieve the above goal using the following methodology, and the results have appeared in the Journal of Algebra [CLY].

The first step towards our study is the study of the Riemann-Hurwitz formula. Although the formula is well know, to our best knowledge, a version suitable for GAP (a software known as *Groups, Algorithms, and Programming*) is not available in the literature. In order to achieve such a version, we worked on the finite quotient PSL(2,Z)/Γ(n). This reduces the study of the Riemann-Hurwitz formula to a finite group. While the finite version of the Riemann-Hurwitz formula is easier to dealt with, it is still far from being perfect as matrices are usually less machine friendly than permutation. Thus, in this work, we first constructed a faithful representation of PSL(2,Z)/ Γ(n) in symmetry groups.

In the second step we developed a program in GAP that calculates all the geometrical invariants involved in the Riemann-Hurwitz formula, in particular, the genus of the surfaces associated with congruence subgroup. It turns out that are totally 132 congruence subgroups of PSL(2,Z), up to conjugation. We note that, in addition to determining all such groups, our program is also able to determine the independent generators and a fundamental domain for such a group.

The third and last step is the determination of generators of function fields K(G\H*). As the Dedekind eta functions fail to be the building blocks of the Hauptmoduls (Generators of the function fields with the additional properties that the only pole of the functions is at infinity and the residue is 1.), we introduce the generalized Dedekind eta functions. In our earlier work [Ya], the transformation formulas for these functions are derived, and the behaviors of these functions at cusps are also determined. From these results, we successfully construct Hauptmoduls of all genus zero congruence subgroups of PSL(2,Z).

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