



---

NORTH-HOLLAND

## Band-Diagonal Operators

Che Kao Fong  
*Department of Mathematics and Statistics*  
*Carleton University*  
*Ottawa, Ontario K1S 5B6, Canada*

and

Pei Yuan Wu\*  
*Department of Applied Mathematics*  
*National Chiao Tung University*  
*Hsinchu, Taiwan, Republic of China*

Submitted by Richard A. Brualdi

---

### ABSTRACT

We show that there are operators on a five-dimensional Hilbert space which are not tridiagonal, and that there are compact operators and normal operators on separable infinite-dimensional spaces which are not band-diagonal.

---

### 1. INTRODUCTION

A bounded linear operator on a complex separable Hilbert space is *tridiagonal* if it is unitarily equivalent to a (finite or infinite) direct sum of

---

\*The research of this author was partially supported by the National Science Council (Taiwan, China). A part of this was conducted while he was visiting Dalhousie University in the fall term of 1993. He would like to thank his colleagues there, P. Fillmore and H. Radjavi, for the discussions relating to the results herein and the hospitality extended to him during that period.

*LINEAR ALGEBRA AND ITS APPLICATIONS* 248:185–204 (1996)

© Elsevier Science Inc., 1996  
655 Avenue of the Americas, New York, NY 10010

0024-3795/96/\$15.00  
SSDI 0024-3795(95)00168-Q



*In fact, nontridiagonal operators on  $H$  form a dense, second-category subset of  $\mathcal{B}(H)$ , the space of all operators on  $H$ .*

As mentioned in Section 1, the proof is based on Sard's theorem as in [9] and [11] for  $n \geq 6$ . (Our reference for the manifold theory is [3].) Common to such nonconstructive arguments, it has the advantage of showing the topological abundance of the objects under consideration, but the drawback of not yielding one single such object cannot be avoided. We start with the following lemma.

LEMMA 2.2. *Tridiagonal operators on a finite-dimensional space  $H$  form a closed subset of  $\mathcal{B}(H)$ .*

*Proof.* Let  $\{T_k\}$  be a sequence of tridiagonal operators on  $H$  which converges to  $T$ . Assume that  $T_k = U_k^* S_k U_k$ , where  $U_k$  is unitary and  $S_k$  is of the form

$$\begin{bmatrix} a_1 & b_1 & & & 0 \\ c_1 & a_2 & b_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & b_{n-1} \\ 0 & & & c_{n-1} & a_n \end{bmatrix}, \tag{*}$$

and let  $\|\cdot\|$  denote the operator norm. Since the sequence  $\|S_k\| = \|T_k\|$  converges to  $\|T\|$ , it is bounded. Hence there exists a subsequence  $\{S_{k_j}\}$  converging, say, to an operator  $S$ . On the other hand, the corresponding bounded sequence  $\{U_{k_j}\}$  has also a convergent subsequence  $\{U_{k_{j'}}\}$  converging, say, to  $U$ . It is obvious that  $S$  is of the form  $(*)$ ,  $U$  is unitary, and  $\{T_{k_{j'}}$  converges to  $U^* S U$ . It follows that  $T = U^* S U$  is tridiagonal, completing the proof. ■

*Proof of Theorem 2.1.*

(a): This has been proved in [9, Proposition 2.3]. We include an alternative proof here for completeness. Let  $T$  be a  $3 \times 3$  matrix. We may assume

that  $T$  is of the triangular form

$$\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}.$$

If  $U$  is a  $2 \times 2$  unitary matrix such that  $U \begin{bmatrix} c \\ e \end{bmatrix}$  is a vector of the form  $\begin{bmatrix} 0 \\ * \end{bmatrix}$ , then  $(U \oplus 1)T(U^* \oplus 1)$  is a matrix of the form

$$\begin{bmatrix} * & * & 0 \\ * & * & * \\ 0 & 0 & * \end{bmatrix}.$$

This shows that  $T$  is tridiagonal.

(b) To prepare for the ground for applying Sard's theorem, we first represent tridiagonal operators in a special form. Indeed, if  $T$  is tridiagonal, then  $T = U^*SU$ , where  $U$  is unitary and  $S$  is of the form (\*). Let  $\theta_1, \dots, \theta_n$  be real numbers such that  $b_j \exp[i(\theta_{j+1} - \theta_j)] = |b_j|$  for  $j = 1, \dots, n-1$  and  $\exp(i \sum_j \theta_j) = \det U$ . If  $D$  is the diagonal unitary matrix  $\text{diag}(\exp(i\theta_j))$ , then  $T = (D^*U)^*(D^*SD)(D^*U)$ , where  $D^*U$  is unitary with determinant 1 and  $D^*SD$  is a tridiagonal matrix with real superdiagonal entries.

Let  $M(n)$  be the set of all  $n \times n$  matrices,  $SU(n)$  the set of all  $n \times n$  unitary matrices with determinant 1, and  $\Delta(n)$  the set of  $n \times n$  matrices of the form (\*) with all  $b_j$ 's real. Note that these sets form (real  $C^\infty$ ) differential manifolds with dimensions  $2n^2$ ,  $n^2 - 1$ , and  $5n - 3$ , respectively (cf. [3, p. 70]). Define the differentiable mapping  $f$  from  $\Delta(n) \times SU(n)$  to  $M(n)$  by  $f(S, U) = U^*SU$ , where  $S \in \Delta(n)$  and  $U \in SU(n)$ . Since the dimension of  $\Delta(n) \times SU(n)$  is  $(5n - 3) + (n^2 - 1) = n^2 + 5n - 4$ , which is strictly less than  $2n^2$ , the dimension of  $M(n)$ , when  $n \geq 5$ , Sard's theorem [3, p. 167] implies that the image of  $f$  has Lebesgue measure zero in  $M(n)$ . As shown in the first paragraph of the proof, this image coincides with the set of all tridiagonal operators. Hence by Lemma 2.2 tridiagonal operators form a nowhere dense set in  $\mathcal{B}(H)$ . The assertions in (b) follow immediately. ■

Note that in the proof above Sard's theorem is not applicable when  $n = 4$ , since in this case the dimensions of  $\Delta(4) \times SU(4)$  and  $M(4)$  are both equal to 32.

For the remaining part of this section, we consider  $4 \times 4$  matrices. The next result gives a matrix representation closest to the  $4 \times 4$  tridiagonal one.

PROPOSITION 2.3. *Every  $4 \times 4$  matrix is unitarily equivalent to a matrix of the form*

$$\begin{bmatrix} * & * & 0 & 0 \\ * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{bmatrix},$$

where the (2, 4) entry may be chosen to be real.

*Proof.* Let  $T$  be a  $4 \times 4$  matrix. We first show that  $T$  is unitarily equivalent to a matrix of the form

$$\begin{bmatrix} * & * & 0 & 0 \\ * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix}, \tag{**}$$

Indeed, let  $T = T_1 + iT_2$ , where  $T_1$  and  $T_2$  are Hermitian, and let  $x$  be any unit eigenvector for  $T_1$ . Apply the Gram-Schmidt process to the vectors  $x$ ,  $T_2x$ ,  $T_2^2x$ , and  $T_2^3x$ , and let  $e_1, e_2, e_3$ , and  $e_4$  be the resulting orthonormal basis. (For simplicity, we assume that these four vectors are linearly independent, for otherwise an analogous argument, to the one below can be carried through.) Since  $T_2x$  is a linear combination of  $e_1$  and  $e_2$ , the first column of the matrix representation of  $T_2$  with respect to this basis is of the form  $[* \ * \ 0 \ 0]^t$ . Hence the self-adjointness of  $T_2$  implies that its matrix representation is of the form (\*\*). On the other hand, since  $T_1x$  is a scalar multiple of  $e_1$ , the self-adjointness of  $T_1$  implies that the matrix representation of  $T_1$  is of the form

$$\begin{bmatrix} * & 0 & 0 & 0 \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix}.$$

Thus  $T$  can be represented in the form (\*\*), as asserted.

To finish our proof, let  $a$  and  $b$  denote the (3, 2) and (4, 2) entries in the matrix representation of  $T$  of the form (\*\*), and let  $U$  be a  $2 \times 2$  unitary

matrix such that  $U \begin{bmatrix} a \\ b \end{bmatrix}$  is of the form  $\begin{bmatrix} * \\ 0 \end{bmatrix}$ . Then

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & & \\ 0 & 0 & & \end{bmatrix} U \begin{bmatrix} * & * & 0 & 0 \\ * & * & * & * \\ 0 & a & * & * \\ 0 & b & * & * \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & & \\ 0 & 0 & & U^* \end{bmatrix} = \begin{bmatrix} * & * & 0 & 0 \\ * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{bmatrix}.$$

If the (2, 4) entry is required to be real, just multiply the above matrix from the left and the right by a diagonal unitary matrix  $\text{diag}(\exp(i\theta_j))$  and its adjoint, respectively, with some suitable  $\theta_2$  and  $\theta_4$ . ■

Proposition 2.3 can be rephrased in the form: for each  $4 \times 4$  matrix  $T$ , there is a nonzero vector  $x$  for which the subspace generated by the five vectors  $x, Tx, T^2x, T^*x$ , and  $TT^*x$  is of dimension at most three. The next proposition, orally communicated to the second author by H. Radjavi, strengthens this property to a characterization of tridiagonality.

**PROPOSITION 2.4.** *A  $4 \times 4$  matrix  $T$  is tridiagonal if and only if there is a nonzero vector  $x$  for which the subspace generated by the seven vectors  $x, Tx, T^2x, T^*x, T^{*2}x, TT^*x$ , and  $T^*Tx$  is of dimension at most three.*

*Proof.* If  $T = U^*SU$  is tridiagonal, where  $U$  is unitary and  $S$  is of the form (\*), then

$$x = U^* \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

is the required vector.

To prove the converse, let  $x$  be a unit vector satisfying the stated condition. Below we only give the proof for the case when the subspace generated by the seven vectors is of dimension exactly three. The other cases can be handled in a similar (even simpler) fashion.

(1) Assume that  $x$ ,  $Tx$ , and  $T^2x$  are linearly independent. Apply the Gram-Schmidt process to these vectors, and then extend the resulting vectors to an orthonormal basis. Then  $x$ ,  $Tx$ , and  $T^2x$  expressed in terms of this basis are of the forms

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} * \\ a \\ 0 \\ 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} * \\ * \\ b \\ 0 \end{bmatrix},$$

respectively, where  $a$  and  $b$  are nonzero scalars. Since  $T^*x$  is a linear combination of these vectors, it is of the form

$$\begin{bmatrix} * \\ * \\ c \\ 0 \end{bmatrix}.$$

In particular,  $T$  has the matrix representation

$$\begin{bmatrix} * & * & \bar{c} & 0 \\ a & * & * & e \\ 0 & b & * & f \\ 0 & 0 & d & * \end{bmatrix},$$

where  $d$ ,  $e$ , and  $f$  are some scalars. Since  $T^*Tx$  is a linear combination of  $x$ ,  $Tx$ , and  $T^2x$ , its fourth component, which equals  $\bar{c}a$ , must be zero, and hence  $e = 0$ . A similar argument applied to  $T^{*2}x$  yields that  $\bar{f}c = 0$ . If  $c = 0$ , then

$$T = \begin{bmatrix} * & * & 0 & 0 \\ a & * & * & 0 \\ 0 & b & * & f \\ 0 & 0 & d & * \end{bmatrix}$$

is tridiagonal. On the other hand, if  $c \neq 0$ , then  $f = 0$ . Since  $TT^*x$  is a linear combination of  $x$ ,  $Tx$ , and  $T^2x$ , its fourth component  $dc$  is zero, and hence

$d = 0$ , In this case,

$$T = \begin{bmatrix} * & * & \bar{c} & 0 \\ a & * & * & 0 \\ 0 & b & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix}.$$

Since the  $3 \times 3$  principal submatrix

$$\begin{bmatrix} * & * & \bar{c} \\ a & * & * \\ 0 & b & * \end{bmatrix}$$

of  $T$  is unitarily equivalent to a tridiagonal matrix by Theorem 2.1(a),  $T$  itself is tridiagonal.

(2) Assume that  $T^2x$  is dependent on the linearly independent  $x$  and  $Tx$ . We have two subcases to consider.

(2a) Assume that  $x$ ,  $Tx$ , and  $T^*x$  are independent. Apply the Gram-Schmidt process to these vectors, and then extend the resulting ones to an orthonormal basis. Then the matrix representation of  $T$  with respect to this basis has the form

$$\begin{bmatrix} * & * & b & 0 \\ a & * & c & d \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix},$$

where  $a$  and  $b$  are nonzero. Since  $T^*Tx$  is a linear combination of  $x$ ,  $Tx$ , and  $T^*x$ , its fourth component, which equals  $\bar{d}a$ , is zero, and hence  $d = 0$ . Let  $U$  be a  $2 \times 2$  unitary matrix such that  $U \begin{bmatrix} b \\ c \end{bmatrix}$  is of the form  $\begin{bmatrix} 0 \\ * \end{bmatrix}$ . Then

$$\begin{aligned} & \begin{bmatrix} U & & 0 & 0 \\ & & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} * & * & b & 0 \\ a & * & c & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix} \begin{bmatrix} U^* & & 0 & 0 \\ & & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} * & * & 0 & 0 \\ * & * & * & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix}. \end{aligned}$$

This proves the tridiagonality of  $T$ .



(2b) Assume that  $T^*x$  is dependent on  $x$  and  $Tx$ . In this case, apply the Gram-Schmidt process to  $x$  and  $Tx$ , and extend the resulting vectors to an orthonormal basis. With respect to this basis  $T$  has the matrix representation

$$\begin{bmatrix} * & * & 0 & 0 \\ a & * & b & c \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix}.$$

A similar trick to the above (a  $2 \times 2$  unitary matrix  $U$  such that  $[b \ c]U$  is of the form  $[* \ 0]$ ) yields that  $T$  is tridiagonal.

(3) Assume that  $Tx$  is dependent on  $x$ . If  $T^*x$  is also dependent on  $x$ , then the matrix representation of  $T$  with respect to any orthonormal basis whose first element is  $x$  is of the form

$$\begin{bmatrix} * & 0 & 0 & 0 \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix}.$$

Theorem 2.1(a) applied to its reducible  $3 \times 3$  principal submatrix shows that  $T$  is itself tridiagonal. On the other hand, if  $x$  and  $T^*x$  are linearly independent, then we may proceed as in (1) and (2) replacing  $T$  with  $T^*$  to derive that  $T^*$  is tridiagonal. Thus the same is true for  $T$ . ■

As an application of the above criterion for tridiagonality, we have the following

**PROPOSITION 2.5.** *Every  $4 \times 4$  weighted permutation matrix is unitarily equivalent to a tridiagonal one.*

A *weighted permutation matrix* is one which has at most one nonzero entry on each row and each column.

*Proof.* If such a matrix has any nonzero diagonal entry, then it is the direct sum of a  $1 \times 1$  and a  $3 \times 3$  matrix, Theorem 2.1(a) implies our assertion. Among those which have only zero diagonal entries, there are two

different types. Those of the form

$$\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix},$$

where  $A$  and  $B$  are  $2 \times 2$  weighted permutation matrices, are easily seen to be unitarily equivalent to a tridiagonal one by interchanging a certain pair of rows and the corresponding columns. The remaining ones are all unitarily equivalent to a matrix of the form

$$\begin{bmatrix} 0 & 0 & 0 & d \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \end{bmatrix}$$

through an appropriate permutation of rows and columns. To complete the proof, we need only restrict our considerations to such matrices. If  $d = 0$ , then  $T$  is already tridiagonal. If  $a = 0$ , then

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & d & 0 \end{bmatrix},$$

which shows that  $T$  is unitarily equivalent to a tridiagonal one. Similar arguments apply to the cases  $b = 0$  and  $c = 0$ . Hence in the following we may assume that  $a, b, c$ , and  $d$  are all nonzero.

Let  $\alpha = (c\bar{b}/a\bar{d})^{1/2}$  and  $x = [\alpha \ 0 \ 1 \ 0]^t$ . In view of Proposition 2.4, we only need to check that any four vectors among

$$x = \begin{bmatrix} \alpha \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad Tx = \begin{bmatrix} 0 \\ a\alpha \\ 0 \\ c \end{bmatrix}, \quad T^2x = \begin{bmatrix} cd \\ 0 \\ ab\alpha \\ 0 \end{bmatrix}, \quad T^*x = \begin{bmatrix} 0 \\ \bar{b} \\ 0 \\ \bar{d}\alpha \end{bmatrix},$$

$$T^{*2}x = \begin{bmatrix} \overline{ab} \\ 0 \\ \overline{cd}\alpha \\ 0 \end{bmatrix}, \quad TT^*x = \begin{bmatrix} |d|^2\alpha \\ 0 \\ |b|^2 \\ 0 \end{bmatrix}, \quad \text{and} \quad T^*Tx = \begin{bmatrix} |a|^2\alpha \\ 0 \\ |c|^2 \\ 0 \end{bmatrix}$$

are linearly dependent. We consider three cases separately.

(1) If the four vectors include both  $Tx$  and  $T^*x$ , then they must be linearly dependent, since, by our choice of  $\alpha$ ,  $Tx$  and  $T^*x$  are.

(2) If the four vectors include  $Tx$  but not  $T^*x$ , then consider the  $4 \times 4$  matrix, call it  $A$ , formed by taking these four vectors as columns. Its second and fourth rows consist of three zeros and one nonzero component ( $a\alpha$  or  $c$ ) and hence are scalar multiples of each other. Thus  $A$  is singular and hence has dependent column vectors. An analogous argument works for the case when the four vectors include  $T^*x$  but not  $Tx$ .

(3) If the four vectors include neither  $Tx$  nor  $T^*x$ , then the matrix formed by these vectors as columns has two zero rows and hence is singular. Such a set of four vectors must be linearly dependent. ■

We conclude this section with the following result, saying that a non-tridiagonal  $4 \times 4$  matrix can be found, if it ever exists, among the (dense set of) cyclic invertible operators. Recall that an operator  $T$  on  $H$  is *cyclic* if there is a vector  $x$  (called *cyclic vector* of  $T$ ) such that  $H$  is generated by the vectors  $x, Tx, T^2x, \dots$ .

**PROPOSITION 2.6.** *Let  $T$  be a  $4 \times 4$  matrix. If  $T$  has rank at most two or is noncyclic, then it is tridiagonal.*

*Proof.* Since  $T$  has the block matrix representation

$$\begin{bmatrix} 0 & A \\ 0 & B \end{bmatrix}$$

with respect to the decomposition  $H = \ker T \oplus \text{ran } T^*$ , by our hypothesis that  $\text{rank } T \leq 2$  we may assume that the four operator entries,  $0, 0, A,$  and  $B$  are all acting on two-dimensional spaces. Let  $U$  and  $V$  be unitary operators such that

$$UAV = \begin{bmatrix} 0 & b \\ a & 0 \end{bmatrix}.$$

Then

$$\begin{bmatrix} U & 0 \\ 0 & V^* \end{bmatrix} \begin{bmatrix} 0 & A \\ 0 & B \end{bmatrix} \begin{bmatrix} U^* & 0 \\ 0 & V \end{bmatrix} = \left[ \begin{array}{c|cc} 0 & 0 & b \\ \hline & a & 0 \\ \hline 0 & & * \end{array} \right].$$

This latter matrix is unitarily equivalent to

$$\begin{bmatrix} 0 & a & 0 & 0 \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ 0 & 0 & b & 0 \end{bmatrix}$$

by a permutation of the rows and columns. Hence  $T$  is tridiagonal.

An equivalent condition for the noncyclicity of  $T$  is the existence of some scalar  $\lambda$  such that  $\dim \ker(T - \lambda I) \geq 2$ . This latter condition is equivalent to  $\text{rank}(T - \lambda I) \leq 2$ . Hence the first part of our proof implies that  $T - \lambda I$  is tridiagonal. Thus a noncyclic  $T$  is tridiagonal, as asserted. ■

### 3. INFINITE DIMENSION

Recall that on an infinite-dimensional space a tridiagonal operator is one which is unitarily equivalent to a (finite or infinite) direct sum of (finite or infinite) tridiagonal matrices. We start our discussion with Hermitian and unitary operators.

PROPOSITION 3.1.

- (a) *Every Hermitian operator is tridiagonal.*
- (b) *A unitary operator is tridiagonal if and only if it is diagonal, i.e., it is unitarily equivalent to a diagonal matrix.*

Part (a) of the proposition is a well-known result; part (b) can be verified as in [9, Proposition 3.1].

Note that not every Hermitian operator can be represented as a single tridiagonal matrix. (This is the reason why we define the tridiagonal operator as the direct sum of tridiagonal matrices instead of one single such matrix.) Indeed, if  $T$  is unitarily equivalent to a matrix of the form

$$\begin{bmatrix} a_1 & b_1 & & 0 \\ c_1 & a_2 & b_2 & \\ & c_2 & a_3 & \ddots \\ 0 & & \ddots & \ddots \end{bmatrix},$$



This is proved by first reducing to the finite-dimensional case via the following lemma and then invoking Sard's theorem. For any  $m$  ( $0 \leq m \leq \infty$ ), let  $0_m$  denote the zero operator on an  $m$ -dimensional space.

**LEMMA 3.4.** *Let  $T$  be a rank- $k$  operator on an  $n$ -dimensional space ( $1 \leq n < \infty$ ). Then  $T \oplus 0_\infty$  is tridiagonal if and only if  $T \oplus 0_{3k-n}$  is. (In case  $3k \leq n$ ,  $T \oplus 0_{3k-n}$  is interpreted as  $T$ .)*

*Proof.* Assume that  $T \oplus 0_\infty$  is tridiagonal. Since  $\text{rank}(T \oplus 0_\infty) = k$ , it follows from the special pattern of nonzero entries in a tridiagonal matrix that there can be at most  $3k$  nonzero rows and columns in the tridiagonal matrix representation of  $T \oplus 0_\infty$ . In particular,  $T \oplus 0_\infty$  is unitarily equivalent to an operator of the form  $S \oplus 0_\infty$ , where  $S$  is tridiagonal on a  $3k$ -dimensional space. Let  $A = T \oplus 0_\infty$  and  $B = S \oplus 0_\infty$ , and let  $U$  be a unitary operator such that  $UA = BU$ . Assume that  $T$  and  $S$  act on the spaces  $H$  and  $K$ , respectively, and  $H_1 = \text{ran } A \vee \text{ran } A^*$  and  $K_1 = \text{ran } B \vee \text{ran } B^*$ . We have  $U(\text{ran } A) \subseteq \text{ran } B$  and  $U(\text{ran } A^*) \subseteq \text{ran } B^*$ , and hence  $UH_1 \subseteq K_1$ . Similarly,  $U^*K_1 \subseteq H_1$ . Thus  $U_1 = U|_{H_1}$  is a unitary operator from  $H_1$  onto  $K_1$ . If  $T = T_1 \oplus 0$  on  $H_1 \oplus (H \ominus H_1)$  and  $S = S_1 \oplus 0$  on  $K_1 \oplus (K \ominus K_1)$ , then  $U_1 T_1 = S_1 U_1$  and hence  $T_1$  is unitarily equivalent to  $S_1$ . This implies that  $T \oplus 0_{3k-n}$  is unitarily equivalent to  $S$  in case  $3k \geq n$ , and  $T$  is unitarily equivalent to  $S \oplus 0_{n-3k}$  in case  $3k < n$ . Our assertion follows immediately.  $\blacksquare$

*Proof of Theorem 3.3.* In view of the preceding lemma, we need only show that there are rank-9 operators on a space of dimension 27 which are not tridiagonal.

Indeed, let  $M'(9)$ ,  $SU(27)$ , and  $\Delta'(9)$  be the sets of  $27 \times 27$  matrices with rank 9,  $27 \times 27$  unitary matrices with determinant 1, and  $27 \times 27$  tridiagonal matrices with rank 9 and with real superdiagonal entries, respectively. Note that  $M'(9)$  and  $SU(27)$  form (real  $C^\infty$ ) differential manifolds with dimensions 810 and 728 (cf. [3, p. 171, Problem 4] and [3, p. 70]). On the other hand,  $\Delta'(9)$  is also a differential manifold having finitely many connected components each with dimension no more than 81. (This can be verified by noting that a tridiagonal matrix with rank 9 can have at most 9 nonzero superdiagonal and 9 nonzero subdiagonal entries.) As in the proof of Theorem 2.1(b), every rank-9 tridiagonal operator on a 27-dimensional space can be expressed in the form  $U^*SU$ , where  $S$  and  $U$  are in  $\Delta'(9)$  and  $SU(27)$ , respectively. Hence the image of the differentiable mapping  $f$  from  $\Delta'(9) \times SU(27)$  to  $M'(9)$  given by  $f(S, U) = U^*SU$  coincides with the set of rank-9 tridiagonal operators. We infer from Sard's theorem that this latter set has

Lebesgue measure zero in  $M'(9)$ . This shows the existence of nontridiagonal rank-9 operators on a 27-dimensional space and hence that of nontridiagonal rank-9 operators on an infinite-dimensional space. ■

For the remaining part of this section, we consider the more general band-diagonal operators. An operator  $T$  is *band-diagonal* with band width  $n$  ( $n = 2k - 1$ ,  $k \geq 1$ ) if  $T$  is unitarily equivalent to a direct sum of matrices of the form  $[a_{ij}]_{i,j=1}^{\infty}$  with  $a_{ij} = 0$  for all  $i$  and  $j$  satisfying  $|i - j| \geq k$ . In particular, the cases  $n = 1, 3$ , and  $5$  correspond to the diagonal, tridiagonal, and pentadiagonal operators, respectively. The next proposition shows that every unitary operator is pentadiagonal although, as we have seen before, not every one is tridiagonal.

PROPOSITION 3.5. *Every isometry is pentadiagonal.*

*Proof.* By the Wold decomposition, every isometry can be decomposed as the direct sum of a unitary operator and a unilateral shift. Since the latter is obviously tridiagonal, we need only prove that every unitary operator is pentadiagonal. Hence let  $T$  be unitary on  $H$ , and  $x$  be any nonzero vector in  $H$ . Consider the subspace  $K$  generated by the vectors  $x, Tx, T^*x, T^2x, T^{*2}x, \dots$ . Evidently,  $K$  reduces  $T$ . After deleting those vectors from the above sequence which are linearly dependent on the one preceding them, we denote the resulting sequence by  $\{x_1, x_2, \dots\}$ . Apply the Gram-Schmidt process to the  $x_n$ 's to obtain an orthonormal basis  $\{e_n\}$  for  $K$ . Since

$$Tx_n \in \vee \{x_1, \dots, x_{n-2}\} \quad \text{for } n \geq 3 \text{ odd,}$$

$$Tx_n \in \vee \{x_1, \dots, x_{n+2}\} \quad \text{for } n \text{ even,}$$

$$T^*x_n \in \vee \{x_1, \dots, x_{n+2}\} \quad \text{for } n \text{ odd,}$$

$$T^*x_n \in \vee \{x_1, \dots, x_{n-2}\} \quad \text{for } n \geq 4 \text{ even,}$$

the matrix representation of  $T|_K$  with respect to  $\{e_n\}$  is easily seen to be pentadiagonal. If  $K \neq H$ , repeat the above process with  $H \ominus K$ . ■

In light of the above results, it seems natural to ask whether every normal operator and compact operator is band-diagonal. As we will see later, the answers to both questions are negative. These are shown by making use of commutators and zero-diagonal operators, which we are going to define now.

An operator is *zero-diagonal* if it is unitarily equivalent to a matrix with all diagonal entries zero. Such operators were first studied by P. Fan [4]. The next result is due to him; it gives a condition for zero-diagonality which is easier to verify than the condition in the original definition.

**PROPOSITION 3.6.** *An operator is zero-diagonal if (and only if) it is unitarily equivalent to a matrix  $[a_{ij}]_{i,j=1}^{\infty}$  for which the sequence  $\{\sum_{i=1}^n a_{ii}\}_{n=1}^{\infty}$  has 0 as a limit point.*

As a consequence, a trace-class operator is zero-diagonal if and only if its trace is zero. The next proposition gives a necessary condition for band-diagonality.

**PROPOSITION 3.7.**

(a) *If  $A$  is band-diagonal and  $B$  is compact, then  $AB - BA$  is zero-diagonal.*

(b) *If  $A$  is band-diagonal and compact and  $B$  is arbitrary, then  $AB - BA$  is zero-diagonal.*

*Proof.* We only prove (a) and omit the analogous proof of (b). There is no loss of generality in assuming that  $A = [a_{ij}]$  with  $a_{ij} = 0$  for  $|i - j| \geq k$ , and  $B = [b_{ij}]$ . A little computation yields that the accumulated sums of the diagonal entries of  $AB - BA = [c_{ij}]$  are given, for  $n \geq k$ , by,

$$\begin{aligned} \sum_{i=1}^n c_{ii} &= \sum_{i=1}^n \sum_{j=1}^{\infty} (a_{ij}b_{ji} - b_{ij}a_{ji}) \\ &= \sum_{i=n-k+1}^n \sum_{j=n+1}^{n+k} (a_{ij}b_{ji} - b_{ij}a_{ji}). \end{aligned}$$

Hence

$$\left| \sum_{i=1}^n c_{ii} \right| \leq \|A\| \sum_{i=n-k+1}^n \sum_{j=n+1}^{n+k} (|b_{ji}| + |b_{ij}|).$$

The sum on the right-hand side has  $2k^2$  terms, all converging to zero as  $n$  approaches infinity due to the compactness of  $B$ . Therefore, the sequence  $\{\sum_{i=1}^n c_{ii}\}_{n=1}^{\infty}$  converges to zero. Our assertion then follows from Proposition 3.6. ■



COROLLARY 3.8.

- (a) *If  $A$  is Hermitian and  $B$  is compact, then  $AB - BA$  is zero-diagonal.*
- (b) *If  $A$  is an isometry and  $B$  is compact, then  $AB - BA$  is zero-diagonal.*

Note that part (a) of the preceding corollary generalizes [6, Lemma 1.3], and part (b) generalizes [8, Theorem 5].

Now we are ready for examples of non-band-diagonal operators among normal and compact ones.

THEOREM 3.9.

(a) *If a normal operator is the sum of a band-diagonal operator and a Schatten  $p$ -class operator for some  $p$ ,  $1 \leq p < 2$ , then its spectral measure is supported on a set of planar measure zero. In particular, every normal operator with spectral measure not supported on a set of planar measure zero is not band-diagonal.*

(b) *There is an operator in the intersection of all the Schatten  $p$ -classes,  $p > 2$ , which is not band-diagonal.*

*Proof.* Using results in [2], it is easy to show that any normal operator  $N$  with spectral measure supported on a set of positive planar measure contains, as a direct summand, the operator  $M$  of multiplication by  $z$  on the space  $L^2(\mu)$ , where the compactly supported positive regular Borel measure  $\mu$  on the plane is such that its associated Cauchy transform

$$(Xf)(z) = \mu(\mathbb{C})^{-1} \int_{\mathbb{C}} f(w)(z - w)^{-1} d\mu(w) \quad \text{for } f \in L^2(\mu)$$

is in every Schatten  $p$ -class,  $p > 2$ . Since

$$(MX - XM)(f) = \mu(\mathbb{C})^{-1} \int_{\mathbb{C}} f(w) d\mu(w) \quad \text{for any } f \in L^2(\mu),$$

$MX - XM$  is a rank-one projection. If  $N = M \oplus L$ , then  $Y \equiv X \oplus 0$  is in the intersection of the Schatten  $p$ -classes,  $p > 2$ , and  $NY - YN$  is a rank-one projection. Proposition 3.7 implies that  $N$  cannot be the sum as specified in (a) and  $X$  (or  $Y$ ) is not band-diagonal. ■

Note that the preceding theorem is more general than [10, Corollary 4], as the latter concerns only the representation of one band-diagonal matrix (instead of direct sum of such matrices).

We remark that the existence of a non-band-diagonal operator in the Schatten  $p$ -class  $\mathcal{E}_p$  for each  $p > 2$  can also be deduced from [1, Theorem 3]. Indeed, it was shown therein that for any  $p > 2$  there are operators  $A$  in  $\mathcal{E}_p$  and  $B$  in  $\mathcal{E}_r$  for some large  $r$  such that  $AB - BA$  is a rank-one projection. Such  $A$  and  $B$  were constructed as matrices of the form

$$\begin{bmatrix} 0 & C_{12} & & & \\ C_{21} & 0 & C_{23} & & \\ & C_{32} & 0 & \ddots & \\ & & \ddots & \ddots & \\ & & & \ddots & \ddots \end{bmatrix},$$

where, for each pair of  $i$  and  $j$ ,  $C_{ij}$  is an  $i \times j$  matrix; this is completely different from the above measure-theoretic construction.

Knowing that there are indeed non-band-diagonal operators, we now proceed to show their abundance with the following theorem.

**THEOREM 3.10.** *The set of non-band-diagonal operators on  $H$  is norm-dense in  $\mathcal{B}(H)$ .*

Recall that a property of operators is referred to as *bad* (according to Herrero) if it satisfies the following conditions:

- (1) if  $T$  is bad, then  $aT + bI$  is bad for any scalars  $a \neq 0$  and  $b$ ;
- (2) if  $T$  is bad and  $S$  is similar to  $T$ , then  $S$  is also bad;
- (3) if  $T$  is bad and  $S$  and  $T$  have disjoint spectra, then  $T \oplus S$  is bad.

The following result is due to Herrero [7, Theorem 3.51].

**PROPOSITION 3.11.** *Every set of bad operators on  $H$  is either empty or norm-dense in  $\mathcal{B}(H)$ .*

*Proof of Theorem 3.10.* Define an operator  $T$  to be bad if there is a compact operator  $K$  such that  $TK - KT$  is a finite-rank operator with nonzero trace. By Proposition 3.7(a), a bad operator cannot be band-diagonal. Our assertion then follows from Theorem 3.9 and Proposition 3.11.  $\blacksquare$

## 4. OPEN QUESTIONS

One question left unsolved from Section 2 is the following

QUESTION 4.1. Is there any  $4 \times 4$  nontridiagonal matrix? More elaborately, is the set of nontridiagonal operators on a four-dimensional  $H$  dense and of the second category in  $\mathcal{B}(H)$ ?

As seen in Section 3, there are tridiagonal (pentadiagonal) operators which are not diagonal (tridiagonal). In fact, examples of such operators are provided by Hermitian (unitary) operators. These lead naturally to the following

QUESTION 4.2. Is there any operator band-diagonal with width  $n + 2$  but not band-diagonal with width  $n$  for every odd integer  $n \geq 5$ ? Can such operators be found among normal ones?

The examples of non-band-diagonal compact operators we found are in the class  $\mathcal{E}_p$  with  $p > 2$ . How about  $1 \leq p \leq 2$ ?

QUESTION 4.3. Is there any operator in the class  $\mathcal{E}_p$  ( $1 \leq p \leq 2$ ) which is not band-diagonal?

The next two questions concern topological properties of the set of band-diagonal operators. Theorem 3.10 says that the complement of this set is dense. This does not rule out the possibility of itself being dense.

QUESTION 4.4. Is the set of band-diagonal operators on  $H$  norm-dense in  $\mathcal{B}(H)$ ? Can this be true for tridiagonal operators?

Note that both sets are dense in the strong operator topology. Indeed, since an operator with norm strictly less than one can be extended to some backward shift [5, Problem 152], every operator has a tridiagonal extension. Our assertions then follow from [5, p. 124].

QUESTION 4.5. Is the set of non-band-diagonal operators on  $H$  of the second category in  $\mathcal{B}(H)$ ? More precisely, is the set of band-diagonal operators with width  $n$  nowhere dense for every odd integer  $n \geq 1$ ?

Note that the last question has a positive answer when  $n = 1$ : the norm closure of diagonal operators is the set of normal operators, which is indeed nowhere dense.

Although it is always risky to speculate on answers, we conjecture that the answers to the above questions are “yes,” “yes,” “yes,” “no,” and “yes.”

## REFERENCES

- 1 J. H. Anderson, Commutators of compact operators, *J. Reine Angew. Math.* 291:128–132 (1977).
- 2 M. Sh. Birman and M. Z. Solomyak, Estimates of singular numbers of integral operators, *Russian Math. Surveys* 32:15–89 (1977).
- 3 J. Dieudonné, *Treatise on Analysis*, Vol. III, Academic, New York, 1972.
- 4 P. Fan, On the diagonal of an operator, *Trans. Amer. Math. Soc.* 283:239–251 (1984).
- 5 P. R. Halmos, *A Hilbert Space Problem Book*, 2nd ed., Springer-Verlag, New York, 1982.
- 6 J. W. Helton and R. E. Howe, Traces of commutators of integral operators, *Acta Math.* 135:271–305 (1975).
- 7 D. A. Herrero, *Approximation of Hilbert Space Operators*, Vol. I, Pitman, Boston, 1982.
- 8 F. Kittaneh, On zero-trace commutators, *Bull. Austral. Math. Soc.* 34:119–126 (1986).
- 9 W. E. Longstaff, On tridiagonalization of matrices, *Linear Algebra Appl.* 109:153–163 (1988).
- 10 V. S. Shul’man, Multiplication operators and traces of commutators, *J. Soviet Math.* 31:2749–2757 (1985).
- 11 B. Sturmfels, Tridiagonalization of complex matrices and a problem of Longstaff, *Linear Algebra Appl.* 109:165–166 (1988).

*Received 30 September 1994; final manuscript accepted 21 February 1995*