第二年的精簡報告

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1. 中文摘要

今年我們研究degenerate differential equations describing two-phase flow in porous media 的弱解的問題。我們借用alternative argument 及設法控制degeneracy at two sides 證明此differential equations 的弱解的Hölder continuity。弱解的唯一性可由我們的結果直接推導出。

2. 英文摘要

The degenerate differential equations describing two-phase flow in porous media are considered this year. Hölder continuity of the weak solution of the differential equations with degeneracy at two sides and without mild degeneracy assumption is shown by an alternative argument. Uniqueness of the weak solutions is a direct consequence from this result.

3. 關鍵字

Hölder continuity, Porous media, Two-scale convergence.

4. 前言

此三年的計劃是探討不均勻介質中二相流變化的宏觀數學模式。此研究的主要動機是(1) 描述不均勻介質中二相流變化的微觀模式很難做分析，(2) 用計算的方式去找出微觀模式的近似解也不可行，原因是為計算量太過龐大，(3) 已有的描述多孔介質中二相流變化的宏觀模式，只能描述二相流在單孔介質中的變化，並不適用於不均勻介質。因此有必要回到微觀下二相流在多孔介質中的數學模式去找出正確對應的宏觀模式。流體在不均勻介質中的變化有一特殊的性質，就

是在不同的區塊中流體的變化會有不同的time-scale的情形。同時，不同的區塊形狀所組成的不均勻介質也影響到二相流在區塊中的變化。正確的數學模式應該要能產生以上這些特性。在第一年的計劃中[20]，我們討論了二相流在由小尺寸區塊組成的不均勻介質中的微觀下的數學模式的解的well-posedness，並藉由two-scale的技巧找出其對應的宏觀數學模式。因此，我們想利用類似的觀念去探討由中尺寸區塊或其他形狀區塊所構成的不均勻介質中的二相流的問題。第二、三年的計劃中，我們將回答這個問題。

5. 目的

今年主要討論由中尺寸區塊所構成的不均勻介質的二相流在微觀下的解的well-posedness的問題。這是要去找出其對應的宏觀模式的第一步。若是能得到解的某種uniform bound，則就有可能藉由two-scale的技巧去推導出它相對應的宏觀模式。

6. 文獻探討

在微觀下，二相流在小尺寸與中尺寸區塊所構造的不均勻介質的變化的最大差異是在小尺寸的介質中，重力可忽略不計，而相反的，在中尺寸的介質中，重力是很重要的。因此，他們所對應的數學模式也不同。小尺寸區塊對應到nonlinear degenerate differential equation，而中尺寸的對應到nonlinear degenerate differential system。因此在討論中尺寸下的二相流問題時困難度增加很多。另一種籠統的說法是研究中尺寸的情形等於是解degenerate elliptic-parabolic equations with discontinuous and highly oscillating coefficients and with degeneracy at two sides。在Antontsev etc. [4] 及Chavent etc. [8] 的書中所介紹的二相流問題，主要是針對nondegenerate differential equations 而言。DiBenedetto [3] 也曾研究degenerate elliptic-parabolic equations with degeneracy at two sides 的情形，但他只能證明解的連續性。Chen [9] 也研究過degenerate elliptic-parabolic equations，但結果只限於一端退化的情形。至於其他已發表的討論degenerate parabolic equations的結果[10]，也只有針對一端是退化的情形，而這些都不適用於我們的問題。

7. 研究方法

我們採用alternative argument及設法控制degeneracy at two sides，來得到我們所關心的問題的解的regularity。

8. 結果與討論

We first describe our problem then state main result.

8.1. *Statement of problem*

The degenerate differential equations describing two-phase flow in porous media are concerned. Existence of weak solutions of the equations had been considered in [3, 4, 7, 8, 9, 16, 19] and references therein. However, regularity of the weak solutions has not been well-established. In this paper, we show Hölder continuity of the weak solutions for the differential equations of two-phase flow in porous media. Uniqueness of the weak solutions is a direct consequence from this result. If $\Omega \subset \Re^N$ ($N = 3$ in reality) is a porous medium, equations for the two-phase flow in porous media in global pressure formulation are (see [4, 8]), for $t > 0$,

$$\Phi\partial_t S - \nabla\left(K\Lambda_w(S)\nabla(P - E_w) - K\frac{\Lambda_w\Lambda_o}{\Lambda}\nabla\Upsilon(S)\right) = Q_w, \tag{8.1}$$

$$-\nabla(K\Lambda(S)\nabla P - K\Lambda_w(S)\nabla E_w - K\Lambda_o(S)\nabla E_o) = Q_t, \tag{8.2}$$

Φ is porosity; K is absolute permeability field; $S \in [0, 1]$ is water saturation; Λ_α ($\alpha = w, o$) is phase mobility of α-phase, a nonnegative monotone function of S; $\Lambda(= \Lambda_w + \Lambda_o)$ is the total mobility; P denotes global pressure; Q_w (resp. Q_t) is the water (resp. total) external source; E_α is a function depending on density, gravity, and position; and Υ is capillary pressure, a nonnegative decreasing function of S. From physics, $\frac{\Lambda_w\Lambda_o}{\Lambda}\Upsilon'(0) = \frac{\Lambda_w\Lambda_o}{\Lambda}\Upsilon'(1) = 0$, so equation (8.1) is a degenerate parabolic equation with degeneracy at two end sides, that is, $S = 0, 1$. For the purpose of presentation, we shall set porosity Φ and permeability field K to 1, and neglect external sources Q_w, Q_t.

Boundary $\partial\Omega$ of the porous medium Ω includes Γ_1 and Γ_2 satisfying $\Gamma_1 \cap \Gamma_2 = \emptyset$ and $\overline{\Gamma}_1 \cup \overline{\Gamma}_2 = \partial\Omega$. The initial and boundry conditions are given by

$$\begin{cases} (\Lambda_w(S)\nabla(P - E_w) - \frac{\Lambda_w\Lambda_o}{\Lambda}\nabla\Upsilon(S)) \cdot \vec{n} = 0, & \text{for } x \in \Gamma_1, \\ (\Lambda(S)\nabla P - \Lambda_w(S)\nabla E_w - \Lambda_o(S)\nabla E_o) \cdot \vec{n} = 0, & \text{for } x \in \Gamma_1, \\ S = S_b, & \text{for } x \in \Gamma_2, \\ P = P_b, & \text{for } x \in \Gamma_2, \\ S(0, x) = S_{init}(x), & \text{for } x \in \Omega, \end{cases} \tag{8.3}$$

where $\vec{n}$ is the unit vector outward normal to Γ_1.

8.2. *Assumption and main result*

Definition 8.1 *Boundary $\partial\Omega$ of the bounded domain Ω belongs to $\mathbf{H}_*^m$, $m \geq 1$, if (1) in the neighborhood $U(x)$ of each boundary point $x \notin \overline{\Gamma}_1 \cap \overline{\Gamma}_2$ there exists a homeomorphic transformation $x'(x) = (x'_1(x), x'_2(x), \cdots, x'_N(x)) \in C^m$, $\left|\frac{dx'}{dx}\right| \geq M > 0$ ($\frac{dx'}{dx}$ is the Jacobian of the transformation) such that $x'_N(\partial\Omega \cap U(x)) = 0, x'_N(\Omega \cap U(x)) > 0$, i.e., $\Gamma_i (i = 1, 2)$ can*

be locally straightened, (2) in the neighborhood of each point $x \in \overline{\Gamma}_1 \cap \overline{\Gamma}_2$ there exists a transformation $x' = x'(x)$ with the same properties mapping it at the neighbor of the edge(vertex) of a cube in variable x'.
We make the following assumptions:

A1. $\partial\Omega \in \mathbf{H}^1_*$,

A2. Λ_w (*resp.* Λ_o) $: [0,1] \to [0,1]$ is continuous and increasing (*resp.* decreasing), $\Lambda_w(0) = \Lambda_o(1) = 0$, $\Lambda_w\Lambda_o(z)|_{z\in(0,1)} \neq 0$, $\inf_{z\in[0,1)} \Lambda(z) > 0$,

A3. $\Upsilon : (0,1] \to \Re_0^+$ is onto, decreasing, and a locally Lipschitz continuous function, and $\inf_{z\in[0,1)} \left|\frac{d\Upsilon}{dS}(z)\right| > 0$, $\frac{\Lambda_w\Lambda_o}{\Lambda}\Upsilon'(z) \in L^\infty((0,1))$,

A4. $E_w, E_o \in L^\infty(0,T;W^{1,\infty}(\Omega))$, $P_b \in L^2(0,T;H^1(\Omega))$,

A5. $S_b, S_{init} \in L^2(0,T;H^1(\Omega)) \cap C^{0,\mathbf{k}_1}(\overline{\Omega}^T)$, $\partial_t\Upsilon(S_b) \in L^1(\Omega^T)$, $S_b, S_{init} \in (\mathbf{k}_2, 1-\mathbf{k}_2)$,

where $\mathbf{k}_1 > 2, \mathbf{k}_2$ are positive constants. Introduce the following notation:

$$\begin{cases} \mathcal{U} \equiv \{\zeta \in H^1(\Omega) : \zeta|_{\Gamma_2} = 0\}, \\ dual\ X \equiv \text{ dual space of } X, \\ \mathcal{J}(z) \equiv \frac{-\Lambda_w\Lambda_o}{\Lambda}\Upsilon'(z), \quad z \in (0,1), \\ \mathcal{R}(z) \equiv \int_0^z \mathcal{J}(\xi)d\xi. \end{cases} \tag{8.4}$$

We consider the following problem: Find $\{S, P\}$ satisfying

$$\partial_t S \in dual\ L^2(0,T;\mathcal{U}), \tag{8.5}$$
$$0 \le S \le 1, \tag{8.6}$$
$$\mathcal{R}(S) - \mathcal{R}(S_b),\ P - P_b \in L^2(0,T;\mathcal{U}), \tag{8.7}$$
$$\int_{\Omega^T} \partial_t S\ \zeta + \int_{\Omega^T} (\Lambda_w(S)\nabla(P - E_w) + \nabla\mathcal{R}(S))\nabla\zeta = 0, \tag{8.8}$$
$$\int_{\Omega^T} \Lambda(S)\nabla P\nabla\xi - \sum_{i\in\{w,o\}} \int_{\Omega^T} \Lambda_i(S)\nabla E_i\nabla\xi = 0, \tag{8.9}$$
$$S(x,0) = S_{init}, \tag{8.10}$$

for any $\zeta, \xi \in L^2(0,T;\mathcal{U})$. By [2, 9, 19], we have
Lemma 8.1 *Under* (A1)-(A5)*,* (8.5–8.10) *has a weak solution.*
Let us define $S_o \equiv 1 - S$. It is easy to see that Λ_o is an increasing function of S_o, and that equations (8.5–8.10) are equivalent to the following equations: Find $\{S_o, P\}$ satisfying

$$\partial_t S_o \in dual\ L^2(0,T;\mathcal{U}), \tag{8.11}$$

$$0 \le S_o \le 1, \tag{8.12}$$

$$\mathcal{R}(S_o) - \mathcal{R}(1-S_b),\ P - P_b \in L^2(0,T;\mathcal{U}), \tag{8.13}$$

$$\int_{\Omega^T} \partial_t S_o \zeta + \int_{\Omega^T} (\Lambda_o(S_o)\nabla(P - E_o) - \nabla\mathcal{R}(1-S_o))\nabla\zeta = 0, \tag{8.14}$$

$$\int_{\Omega^T} \Lambda(1-S_o)\nabla P \nabla\xi - \sum_{i\in\{w,o\}} \int_{\Omega^T} \Lambda_i(1-S_o)\nabla E_i \nabla\xi = 0, \tag{8.15}$$

$$S_o(x,0) = 1 - S_{init}, \tag{8.16}$$

for any $\zeta, \xi \in L^2(0,T;\mathcal{U})$. Let $\vartheta \in (0, 1/8)$ such that $\mathcal{J}$ is increasing (resp. decreasing) in $(0,\vartheta)$ (resp. $(1-\vartheta, 1)$). Next we give more assumptions:

A6. $P_b \in L^\infty(0,T;W^{1,\mathbf{k}}(\Omega))$ with $\mathbf{k} > \max\{N, \mathbf{k}_1\}$,

A7. $\max_{z\in[0,1]} |\Lambda(z) - 1| < \mathbf{b}$, where $\mathbf{b} < 1$ is a small number,

A8. $\Lambda_w(z) \le \mathbf{k}_3 z\sqrt{\mathcal{J}(z)}$ and $\mathcal{J}(z) = \mathbf{k}_4 z^{\mathbf{m}}$ for $z \in (0,\vartheta)$,

A9. $\Lambda_o(1-z) \le \mathbf{k}_3|1-z|\sqrt{\mathcal{J}(1-z)}$ and $\mathcal{J}(1-z) = \mathbf{k}_5|1-z|^{\mathbf{m}_1}$ for $z \in (1-\vartheta, 1)$,

where $\mathbf{k}_3, \mathbf{k}_4, \mathbf{k}_5, \mathbf{m}, \mathbf{m}_1$ are positive constants. By [3, 4], we have
Lemma 8.2 *Under* (A1)-(A9),

$$\|P\|_{L^\infty(0,T;W^{1,\mathbf{k}_1}(\Omega))} \le c(\sum_\alpha \|E_\alpha\|_{L^\infty(0,T;W^{1,\mathbf{k}}(\Omega))} + \|P_b\|_{L^\infty(0,T;W^{1,\mathbf{k}}(\Omega))})^{\mathbf{k}_1}.$$

Our main result [21] is:
Theorem 8.1 *Under* (A1)-(A9), *S of* (8.1–8.3) *is Hölder continuous in $\overline{\Omega}^T$.*
If $P_b, E_w, E_o \in L^\infty(0,T;C^{1,\mathbf{k}}(\overline{\Omega}))$, then $P \in L^\infty(0,T;W^{1,\infty}(\Omega))$ by **Theorem 8.1** and **Corollary 8.35** of [13]. Uniqueness of weak solution of (8.1–8.3) is a direct result of **Theorem 8.1** by [19].

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10. 成果自評

Hölder continuity of saturation for two-phase flows in fractured media is shown. It is an improvement of previous results in literatures. Our result also implies that global pressure is uniformly bounded in $W^{1,k}$ space. However, this is not good enough to employ two-scale method to show the convergence of the microscopic model for two-phase flows in fractured media. To get that convergence, a stronger bound (e.g. $W^{1,\infty}$) is required, and this is our next target.