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Finite Time Blow Up of the Willmore Flow

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摘要

假設 M_0 為 3-維歐式空間上之一曲面。此報告首先重新建立 Willmore 演化方程藉以導出其相關幾何量之演化方程。其次證明:若始初曲面 M_0 為旋轉曲面時，則其解亦為旋轉曲面。並導出此情形下對應之方程。最後分析有限時間有奇異現象之解的可能形式，及建立有助於觀察此現象之通量公式。

關鍵詞 Willmore 曲面、奇異解、旋轉曲面

Abstract

Let M_0 be a Willmore surface in the 3 dimensional Euclidean space. For establishing evolution equations for the related geometric quantity we first derive the first vaiational formula of the Willmore functional. Next we show that if the initial surface is a surface of revolution then the solution is also a surface of revolution. In

this case we establish the corresponding equations for the generated curve. Then we construct a function which expect to be a finite time blow up solution of a generalized flow. Finally, using the Mobious geometry, we obtain some flux formulas.

Keywords: Willmore flow, singularity, surface of revolution.

1. Introduction

Let X:M→R^3 be a compact immersed surface in the 3-dimensional Euclidean space R^3. Denote by [h_{ij}] the second fundamental form of M, and the mean curvature of M by $H=\sum h_{ii}$. Let $\phi_{ij} = h_{ij} - \frac{H}{2}\delta_{ij}$ and $\Phi = \sum \phi_{ij}^{2}$ the square length of the trace free tensor. Then the Willmore functional of X is given by

$$W(X) = \int_M \Phi.$$

This functional W(X) is invariant under conformal transformations of R^3.

For a given immersion X_0: M → R^3, we consider a variation $X:[0,T)\times M \to R^3$ of X_0. The L^2 gradient flow for the Willmore functional is a system of fourth order quasilinear geometry evolution equations

$$X_t = - \tau N,$$

where $\tau = \Delta H + \Phi H$, X_t is the variational field of X. This flow is called the Willmore flow. A surface in R^3 is called a Willmore surface if it is a critical surface of the Willmore functional; A surface M in R^3 is a Willmore surface if and only if

$$\Delta H + \Phi H = 0.$$

The short time existence is standard. The question whether a solution always exists long time has been asked by many authors(see {KS1}). Kuwer and Schatzle in their paper have shown the following: There exists $\varepsilon > 0$ such that if at time t = 0 we have $W(X_0) < \varepsilon$, then the Willmore flow exists smoothly for all times and converges to a round sphere (see [KS2]). It is not known whether or not the fourth order geometric evolution equation can develop singularities in finite time. In accordance with numerical simulation of Mayer and Simonett, the existence of such a surface is possible (see [MS]). The purpose of this project is to study their numerical example which is finite time blow up.

Our first work is to establish evolution equations for the elementary geometric quantity, such as the area element, the mean curvature, the Gaussian curvature and the energy (see section 2). Since the numerical example of Mayer and Simonett is a family of surfaces of revolution, we need to show that if the initial surface is a surface of revolution then the solution is also a surface of revolution for each existence time t. We make it in section 3. In this case the Willmore flow is given by

$$h_t = \frac{2}{\sqrt{(h')^2 + (v')^2}} v',$$
$$v_t = -\frac{2}{\sqrt{(h')^2 + (v')^2}} h',$$

where (h, v) is the parametrization of the generated curve. In additional, if we assume the initial surface is given by rotation about the z-axis and reflection about the xy-plane of this curve then

$$h(t, \cdot\,) = h(t, ® - \cdot\,), v(t, \cdot\,) = -v(t, ® - \cdot\,).$$

In this case the boundary conditions will be

$$h(0) = 0, v'(0) = 0 \quad \text{and} \quad h'(\frac{®}{2}) = 0, v(\frac{®}{2}) = 0.$$

Palmer consider the conformal Gauss map $Y : M \to S_1^4$, where S_1^4 is the deSitter space, and prove that a Willmore surface of disc type which has its boundary on a circle and which intersects the plane of the circle in a constant angle is a spherical cap or a flat disc (see [P]). The flux formulas and a holomorphic quartic differential play crucial roles in his proof. If we truncate the solution of the surface of revolution, then its boundary is a circle which intersects the plane of the circle in an angle depending only on time t. In the final section we establish the flux formulas to a Willmore flow with symmetric initial surface, which we believe will be an important tool for finding finite time blow up solution.

2. The Willmore Flow

In this section we state briefly the first variational formula of the Willmore functional, and establish evolution equations for the area element, the mean curvature, the Gaussian curvature and the energy.

Let M be a compact surface in the 3-dimensional Euclidean space R^3. Denote by e_1, e_2, X and N, where e_1, e_2 are tangent to M, X is the position vector of M and N is the unit normal of M in R^3. Let ù $_1$ and ù $_2$ be the dual coframe. Then the structure equations are

$$
\begin{aligned}
& dx = \Sigma \mu_i e_i, \\
& de_i = \Sigma \mu_{ij} e_j + h_{ij} \mu_j N, \\
& dN = -\Sigma h_{ij} \mu_j e_i, \\
& d\mu_i = \Sigma \mu_{ij} \wedge \mu_j, \\
& d\mu_{ij} = \Sigma \mu_{ik} \wedge \mu_{kj} - \frac{1}{2} R_{ijkl} \mu_k \wedge \mu_l, \\
& R_{ijkl} = h_{ik} h_{jl} - h_{il} h_{jk}.
\end{aligned}
$$

Let $X : [0, T) \times M \to R^3$ be a smooth variation of an initial surface X_0 . Then the first variation of the Willmore functional is given by

$$(\int_M \Phi)_t = \int_M (\Delta H + \Phi H) X_t \cdot N,$$

where Δ is the Laplacian of X (for details see [Lee]). The L^2 gradient flow for the Willmore functional is a system of fourth order quasilinear geometric evolution quations

$$X_t = - \tau N,$$

where $^2 = \Delta H + \Phi H.$ This flow is briefly called the Willmore flow.

We then obtain the evolution equations for the area element, the mean curvature H, the Gaussian curvature K and the energy Φ as follows

$$(\mu_1 \wedge \mu_2)_t = ²H\mu_1 \wedge \mu_2,$$
$$H_t = -\Delta ² - (\Phi + \frac{H^2}{2})²,$$
$$K_t = ¤_{ij}²_{ij} - \frac{H}{2}\Delta ² + \frac{H}{2}(\Phi - \frac{H^2}{2})²,$$
$$\Phi_t = -2¤_{ij}²_{ij} - 2\Phi H².$$

3. Solutions with Symmetric Initial Values.

In this section we show that if the initial surface is a surface of revolution then the solution is also a surface of revolution for each time t. More precisely, if the initial surface M_0 is a surface of revolution which is parametrized by $X_0 : S^2 \to R^3$ with

$$X_0(\sin \cdot \, \cos ¯, \sin \cdot \, \sin ¯, \cos \cdot)$$
$$= (h_0(\cdot)\cos ¯, h_0(\cdot)\sin ¯, v_0(\cdot)),$$

Then the solution X is also parametrized by $X : S^2 \times [0, T) \to R^3$, and has the form

$$X(t, \sin \cdot \, \cos ¯, \sin \cdot \, \sin ¯, \cos \cdot)$$
$$= (h(t, \cdot)\cos ¯, h(t, \cdot)\sin ¯, v(t, \cdot)).$$

Furthermore we show that if X_0 is symmetric with respect to the xy-plane then X is also symmetric with respect to the xy-plane. Thus

$$h(t, \cdot) = h(t, ® - \cdot), v(t, \cdot) = -v(t, ® - \cdot)$$
$$if \quad h_0(\cdot) = h_0(® - \cdot), v_0(\cdot) = -v_0(® - \cdot).$$

<u>Outline of the proof.</u> (1). Let X be the solution of the Willmore flow. For fixed real α, let

$$Y(t, \sin \cdot \, \cos ¯, \sin \cdot \, \sin ¯, \cos \cdot)$$
$$= T_{\sim}^{-1} X(t, e^{-i(\sim + ¯)} \sin \cdot, \cos \cdot),$$

here T_α is the operator of rotating the angle α in the xy-plane if the xy-plane is identify with the complex plane. It is easy to see that $Y = X_0$ when t = 0. From the the structure equations stated in section 1, the first and second fundament forms of Y and X are related by α. It follows that Y also satisfies the equation of Willmore flow. By the uniqueness of solution, we have Y=X.

(2). Let x_3 be the third component of R^3. Then

$$x_3(t, e^{i(\sim + ¯)} \sin \cdot, \cos \cdot) = x_3(t, e^{i¯} \sin \cdot, \cos \cdot),$$

for all α. This implies the third component x_3 is independent with θ. Thus $x_3 = v(t, \phi)$ for some function v. On the other hand, in terms of polar coordinate, let

$(x_1 + ix_2)(t,\theta,\phi) = h(t,\theta,\phi)e^{i\Theta(t,\theta,\phi)}$,

then the relation of the first two components gives

$h(t,\theta+\alpha,\phi)e^{i(\Theta(t,\theta+\alpha,\phi)-\alpha)} = h(t,\theta,\phi)e^{i\Theta(t,\theta,\phi)}$, for all α. This implies

$h(t,\theta+\alpha,\phi) = h(t,\theta,\phi)$

and

$\Theta(t,\theta+\alpha,\phi) = \Theta(t,\theta,\phi)+\alpha$

for all α.It follows that

$h(t,\theta,\phi) = h(t,\phi)$, $\Theta(t,\theta,\phi) = \theta + \eta(t,\phi)$ for some function η.

(3). Now we can assume that the solution X is given by

$X(t,\theta,\phi) = (h(t,\phi)e^{i(\theta+\eta(t,\phi))}, v(t,\phi))$.

Since $X_t = -\tau N$, we have

$h_t h' + v_t v' + \eta_t h^2 \eta' = 0,\quad \eta_t h^2 = 0$ and

$h_t v' - h' v_t = \sqrt{(h')^2+(v')^2}\,\tau$. From the second equation the function η vanishes identically since $\eta(0,\phi) = 0$. The first and third equations imply that

$$h_t = \frac{\tau}{\sqrt{(h')^2+(v')^2}}\,v',$$

$$v_t = -\frac{\tau}{\sqrt{(h')^2+(v')^2}}\,h'.$$

(4). For prove that X is also symmetric with respect to the xy-plane, let

$Y(t,\sin\phi\cos\theta,\sin\phi\sin\theta,\cos\phi)$
$=(h(t,\pi-\phi)\cos\theta, h(t,\pi-\phi)\sin\theta, -v(t,\pi-\phi))$. Then Y is also a solution of the Willmore flow. By the uniqueness of solution, we have

$h(t,\phi) = h(t,\pi-\phi), v(t,\phi) = -v(t,\pi-\phi)$

$if\quad h_0(\phi) = h_0(\pi-\phi), v_0(\phi) = -v_0(\pi-\phi)$.

In the case of surface of revolution, the Willmore flow involves two unknown functions h and v which satisfy a system of fourth order equations.

4. Solutions in the Generalized Sense.

As a special case, if the initial surface M_0 is an ellipsoid which is a surface of revolution of the form $\frac{x^2}{a^2}+\frac{y^2}{a^2}+\frac{z^2}{b^2}=1$ in the 3-dimenensional Euclidean space, there exists a solution of a flow in the following form

$X_t = -{}^{\circ}\tau N + \text{tangential component}$, where ρ is a positive function defined on $[0,\infty)\times M$ (see [Lee]). Moreover this solution converges to a round sphere as the time tends to infinity. In fact the same argument is also work when the initial surface is a circular torus in R^3. In this case the generalized flow converges to the Clifford torus. In both cases the behavior of the solutions look just like the numerical scheme given by Mayer and Simonett (see [MS]). These solutions are constructed by solving a system of ordinary differential equations without using the maximum principle. It should be noted that the main tool for the second order parabolic equation is the maximum principle which has no known counterpart for fourth order equation. The problem of constructing a solution of fourth order equation will be simplify if we can observe the solution is of certain type. By studying the numerical example given by Mayer and Simonett, we may expect the solution is in the form of

$X=(h(t,\psi)\cos\beta, h(t,\psi)\sin\beta, v(t,\psi))$.

with $h(t,\psi)=\rho(t,\theta(t))\cos\theta(t)-\rho(t,\psi)\cos\psi$

and $v(t,\psi)=\rho(t,\psi)\sin\psi$. Furthermore, after rescaling in horizontal and vertical directions, these functions h and v are deformed by the sine function and cosine function. Under this observation, ρ must given implicitly by

$a(t)\sin F_1(t,\psi)+F_2(t,b(t)-c(t)\rho(t,\psi))=\phi$, where F_1 and F_2 are increasing functions,

$F_i(t,\phi)=\frac{3}{2}\phi$ for all t and $F_i(t,\phi)$ converges to ϕ as t tends to the maximal existence time T uniformly on any closed intervals contained in $[\frac{\phi}{2},\phi)$ for I = 1,2.

To show this solution is blow up in finite time, it suffices to show that there exists a constant R and functions a(t), b(t) and c(t) such that a(t) - c(t)R converges to zero as the time t tends to T and

$$\rho(t,\theta(t))c(t)F_2'(t,b(t)-c(t)\rho(t,\theta(t)))$$
$$=-a(t)\cos F_1(t,\theta(t))F_1'(t,\theta(t))\tan\theta(t),$$

for all $0\le t<T$. To construct the solution X precisely, we need to find suitable deformations F_1 and F_2 , and then show that the corresponding X satisfies a generalized flow. If it is done, the problem of constructing example will be reduced to a scalar equation involved only one unknown function ρ. There are more work to be done in this direction.

5. The Flux Formulas.

In this section we establish by specializing the flux formulas to a Willmore flow with a symmetric initial surface. Let X be a solution of the Willmore flow, and let

$$y^1(t,\cdot) = \frac{H}{2}x_3 + n_3,\ y^2(t,\cdot) = \frac{|X|^2 - 1}{4}H + X\cdot N$$

$$and\ y^3(t,\cdot) = \frac{|X|^2 + 1}{4}H + X\cdot N,$$

where x_3 and n_3 are the z-component of the position vector and unit normal, respectively. In fact, y's are component functions of the conformal Gauss map(see [P]). Then we have

$\Delta y^{\sim} + \Phi y^{\sim} = {}^{2}z^{\sim}$, for all α =1,2,3, where

$$z^1 = \frac{x_3}{2},\ z^2 = \frac{|X|^2 - 1}{4},\ z^3 = \frac{|X|^2 + 1}{4}.$$

Denote by $\mu_{\sim} = y^{\sim} * dy \ - y \ * dy^{\sim}$

for α, β =1,2,3. Then we have

$$d\mu_{\sim} = (y^{\sim}\Delta y \ - y \ \Delta y^{\sim})\mu_1 \Lambda \mu_2.$$

In particular, if X is a Willmore surface then $\mu_{\sim}$ are closed 1-forms. Now let Σ be a part of truncated surface in the z-axis of the solution. Taking integration over Σ, the stokes' theorem implies

$$\oint_{\partial\Sigma} \mu_{\sim} = \iint_{\Sigma} (y^{\sim}\Delta y \ - y \ \Delta y^{\sim}),$$

for all α, β =1,2,3. We consider the following three cases: α =1, β =2; α =1, β =3 and α =2, β =3. Then we have the following flux formulas

$$\oint_{\partial\Sigma}(\frac{H}{2}-h_{22})H\cos ¥+\frac{\partial H}{\partial n}\sin ¥=\iint_{\Sigma}{}^{2}n_3,$$

$$\oint_{\partial\Sigma}(\frac{H}{2}-h_{22})H(-v\cos ¥+h\sin ¥)$$

$$-\frac{\partial H}{\partial n}(v\sin ¥+h\cos ¥)$$

$$=\iint_{\Sigma}{}^{2}X\cdot N$$

and

$$\oint_{\partial\Sigma}(\frac{H}{2}-h_{22})[H\frac{v}{2}(-v\cos ¥+h\sin ¥)$$

$$+\mathrm{H}\frac{h^2+v^2}{4}\cos ¥+(-v\cos ¥+h\sin ¥)\sin ¥$$

$$+(v\sin ¥+h\cos ¥)\cos ¥]$$

$$+\frac{\partial H}{\partial n}[\frac{h^2+v^2}{4}\sin ¥-(v\sin ¥+h\cos ¥)\frac{v}{2}]$$

$$=\iint_{\Sigma}{}^{2}[\frac{h^2+v^2}{4}n_3+\frac{v}{2}X\cdot N],$$

where $\frac{\partial H}{\partial n}$ is the outer ward normal derivative of H, $n_3 = \sin\gamma$, $X\cdot N=v\sin ¥+h\cos ¥$ on the boundary of Σ. Since the boundary of Σ is a circle which intersects the plane of the circle in an angle depending only on time t, by the Joachimsthal theorem, the boundary is a line of curvature and h_{11} is the normal curvature. These formulas will be more simple for special choice of Σ. In particular, if Σ cuts from $\cdot=\frac{®}{2}$, then $\gamma=\pi$. In this case the flux formulas will be written as

$$-(\frac{1}{h}-\frac{h''}{(v')^2})(\frac{1}{h}+\frac{h''}{(v')^2})®h\Big|_{\cdot=\frac{®}{2}}=\iint_{\Sigma}{}^{2}n_3,$$

$$0=\iint_{\Sigma}{}^{2}X\cdot N$$

and

$$(\frac{1}{h}+\frac{h''}{(v')^2})(\frac{3}{4}h+\frac{h^2}{4}\frac{h''}{(v')^2})®h\Big|_{\cdot=\frac{®}{2}}$$

$$=\iint_{\Sigma}{}^{2}[\frac{h^2+v^2}{4}n_3+\frac{v}{2}X\cdot N].$$

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