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The Least Cost Super Replicating Portfolios in The Boyle-Vorst Model with Transaction Cost: The Two-Period Cases

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摘要

Boyle 和 Vorst 在 1992 年建立了二元樹財務定價模型，不同於一般的二元樹模型，在該模型中考慮了衍生性金融商品交易時的手續費。Boyle 和 Vorst 假設市場模型在買進和賣出衍生性金融商品時收取等比例的手續費，並且成功的推演出長部位和短部位選擇權的自我複製投資組合。

在 Boyle 和 Vorst 的市場模型裡，即使有「無風險套利」的假設，只有唯一複製投資組合的「或有權利」(contingent claim) 仍可能會有較便宜的「超複製投資組合」(super-replicating portfolio)。Bensaid, Lense, Pages 和 Scheinkman 在 1992 年證明了在某些條件下較便宜的「超複製投資組合」是不存在的。而 Stettner, Rutkowski 和 Palmer 則是把這些結果推廣到交易時非等比例手續費的市場模型。這些結果的條件包含了長部位的買權和賣權，但並不包含短部位的買權和賣權。

這篇研究報告是和彭柏堅教授（Ken Palmer）以及我的博士生陳冠宇共同研究的結果。在一個週期的定價模型裡，我們完全決定出「最低價超複製投資組合」。在兩個週期的定價模型裡，我們證明了「最低價超複製投資組合」的存在性，並且在某些條件下計算出「最低價超複製投資組合」。這些條件包含了短部位的買權和賣權。

關鍵字：二元樹模型、買權、賣權、複製投資組合、超複製投資組合。

Abstract

Boyle and Vorst (1992) work in the framework of the binomial model and derive self-financing strategies perfectly replicating the final payoffs to long and short positions in call and put options, assuming proportional transactions costs on trades in the stock and no transactions costs on trades in the bond.

Even when the market is arbitrage-free and a given contingent claim has a unique replicating portfolio, there may exist super-replicating portfolios of lower cost. However, Bensaid, Lesne, Pages and Scheinkman (1992) give conditions under which the cost of the replicating portfolio does not exceed the cost of any super-replicating portfolio. These results were generalized by Stettner, Rutkowski and Palmer to the case of asymmetric transactions costs. These results have the consequence that there is no super-replicating portfolio for long calls and puts of lower cost than the replicating portfolio. However, that is not true for short calls and puts.

In the joint work with Ken Palmer and Guan-Yu Chen, we first determine the least cost super-replicating portfolio for any contingent claim in a one-period binomial model. Then we prove the existence of a least cost super replicating portfolio for any contingent claim in a two-period binomial model and show that there are finitely many possibilities for such a portfolio. In particular, for short positions in calls and puts, we show that there are just five possibilities enabling the least cost super replicating portfolio to be easily determined. It turns out that such a portfolio is, in general, path-dependent.

Keywords: binomial model, call option, put option, replicating portfolio, super-replicating portfolio.

1. Introduction

Black and Scholes introduce their option pricing model in 1973, and show that in a complete, continuous-time financial market without transaction costs, every contingent claim can be replicated by starting with a certain initial capital at time $t=0$, and investing thereafter according to the Black-Scholes hedging portfolio. Under the assumption of no arbitrage opportunity, the initial value of the hedging portfolio is the price of the contingent claim at time $t=0$. They also give an explicit formula for the price of any contingent claims, but few of them, such as European put and call options, have closed forms. There are some numerical procedures that can be used to value derivatives, such as binomial trees, Monte Carlo simulation, and finite difference methods.

However, the Black-Scholes hedging portfolio requires trading at all time instant, and the total turnover of stock in a time interval is infinite. Therefore, in a model with transaction costs proportional to the amount of trading, the Black-Scholes hedging portfolio is prohibitively expensive. Many authors have attempted to develop the models with transaction costs since 1973. The groundwork of modeling the effects of transaction costs was done by Leland (1985). H. M. Soner, S. E. Shreve, and J. Cvitanic (1995) show that there is no nontrivial hedging portfolio for option pricing with transaction costs. Here we consider the binomial model given by Boyle and Vorst in 1992.

2. Terminologies

We consider a discrete-time model of a financial market with the set of dates $0,1,2,\cdots,n$, and with two securities: a risky asset, referred to as a stock, and a risk-free investment, called a bond. The stock price process S satisfies

$$\frac{S_{t+1}}{S_t} \in \{u, d\}$$

for $t = 0,1,2,\cdots,n-1$, where $0 < d < u$. The bond yields a constant rate of return r over each time period $[t, t+1]$, meaning that its price process B equals

$$B_t = (1+r)^t = R^t$$

where $R = 1 + r$.

We assume that proportional transaction costs are ${}^{a}S_t$ and «S_t while buying and selling one share of stock, where

$${}^{a} \geq 0, \qquad 0 \leq « < 1,$$

and no cost while trading in risk-free bonds.

As usual, we make the following assumption throughout the report.

Assumption A. There are no transaction costs when a portfolio is established at time 0.

Assumption B. The no-arbitrage condition

$$0 < d < R < u.$$

3. Results for One-Period Cases

We suppose the current stock price to be $S > 0$ and at the next period is either Su or Sd. We also suppose that the contingent claim is settled by delivery and corresponds to stock and bond holdings at the end of the period of (Δ_u, B_u) and (Δ_d, B_d) in the up and down states respectively. Let (Δ, B) be the current holdings. In order that the claim is replicated, the *self-financing equations*

$$\Delta Su + BR = \Delta_u Su + B_u + [\text{«}(\Delta - \Delta_u)^+ + \text{ª}(\Delta - \Delta_d)^-]Su$$

$$\Delta Sd + BR = \Delta_d Sd + B_d + [\text{«}(\Delta - \Delta_d)^+ + \text{ª}(\Delta - \Delta_d)^-]Sd$$

must be satisfied, where

$$x^+ = \max\{x, 0\}, \quad x^- = \max\{-x, 0\}.$$

Theorem 1 *If*

$$« < 1$$

then the self-financing equations have a solution for Δ *and* B*, which is unique when*

$$\Delta_u \geq \Delta_d,$$

or

$$\Delta_u < \Delta_d, \;\; d(1+ ª) < u(1- «),$$

or

$$\Delta_u < \Delta_d, \;\; d(1+ ª) \geq u(1- «), \;\; a_u a_d > 0,$$

where

$$a_u = (\Delta_d - \Delta_u) Su(1- «) + B_d - B_u$$

and

$$a_d = (\Delta_d - \Delta_u) Sd(1+ ª) + B_d - B_u.$$

If $a_u a_d < 0$*, there are three solutions, if one of* a_u*,* a_d *is zero there are two solutions and if they are both zero there are infinitely many solutions.*

A self-financing trading strategy is called a super-replicating portfolio of a contingent claim if the terminal holdings of the portfolio hedge the contingent claim. In arbitrage-free markets, even when a contingent claim has a unique replicating portfolio, there may exist a lower-cost super-replicating portfolio.

Consider an arbitrary contingent claim $X = (g_n, h_n)$ and denote a self-financing trading strategy ϕ to be $(\sim_t, \ _t)$ for $t = 0,1,\ldots,n$. Let $V_0(¤)$ be the initial value of the trading strategy ϕ. Then under the consideration of no arbitrage opportunity, the reasonable price of the contingent claim is between $p_0^s(X)$ and $p_0^b(X)$, where

$$p_0^s(X) = \inf\{V_0(¤) \mid ¤ \text{ is self-financing}, \ \sim_n \geq g_n \text{ and } \ _n \geq h_n\}$$

$$p_0^b(X) = -\inf\{V_0(¤) \mid ¤ \text{ is self-financing}, \ \sim_n \geq -g_n \text{ and } \ _n \geq -h_n\}$$

In financial markets without transaction cost, $p_0^s(X) = p_0^b(X)$. But this is not necessary true in markets with transaction costs.

Theorem 2. *Consider a one-period model with contingent claim* (Δ_u, B_u) *in the up state and* (Δ_d, B_d) *in the down state. Then a least cost super-replicating portfolio exists and it is either a replicating portfolio or a super-replicating portfolio with holdings either* $(\Delta_u, B_u/R)$ *or* $(\Delta_d, B_d/R)$.

4. Results for Two-Period Cases

Consider a two-period model with parameters *S, u, d, R, ª,* and *«*. We assume that

$$0 < d < R < u\,,\ \ 0 \leq ª\,,\ \ 0 \leq « < 1$$

Theorem 3 *For any contingent claim in the two-period cases with holdings* (Δ_{uu}, B_{uu}), (Δ_{ud}, B_{ud}), *and* (Δ_{dd}, B_{dd}), *there is always a least cost super-replicating portfolio hedging the contingent claim.*

Theorem 4 *Consider a two period model with parameters S, u, d, R, «, and ª. For every short put option with exercise price K satisfying* $Sd^2 < K < Sud$, *there exists a least cost super-replicating portfolio which is one of the following five types:*

(I) the initial holdings are (0,0) and there are no other transactions additional to the terminal transactions;

(II) the initial holdings are (1,-K/R^2*) and the only additional transaction is selling the share in state u;*

(III) the initial holdings are (~,B/R), where (~,B) are the initial holdings in a replicating portfolio for the two-period portion {d,ud,dd}, and the only additional transaction is selling the shares in state u;

(IV) the initial holdings are (¢,B), where (¢,B) is such that BR is just enough to buy back the shares held short in state u and also such that the terminal holdings in state dd are (1,-K);

(V) a replicating portfolio for the whole two-period model.