行政院國家科學委員會補助專題研究計畫成果報告

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計畫類別：□個別型計畫 □整合型計畫

計畫編號：NSC 89-2115-M-009-018

執行期間： 88 **年** 8 **月** 1 **日至** 89 **年** 7 **月** 31 **日**

計畫主持人：王夏聲

共同主持人：

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中 華 民 國 89 **年** 10 **月** 30 **日**

行政院國家科學委員會專題研究計畫成果報告

幾何問題奇點集的形成與結構

The Structure of Singular Sets in Some Geometric Problems

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† **八十六年度及以前的一般國科會專題計畫(不含產學合作研究計畫)亦可選擇適用，惟較特殊的計畫如國科會規劃案等，請先洽得國科會各學術處同意。**

一、中文摘要

我們考慮在四維緊緻里曼流型上楊-米爾斯熱流方程弱解在有限時間奇點產生的問題。我們證明了在有限時間內奇點產生所有的能量損失可由吹有限多個『泡泡』的能量所捕捉到。而這些『泡泡』是由在 S^4 上的正能量楊-米爾斯連絡所組成。相似的現象在定義域為二維緊緻里曼流型上的調和映照熱流方程亦有發生。

關鍵詞：四維緊緻里曼流型、楊-米爾斯熱流方程、弱解、S^4-泡。

Abstract

We show that there is no unaccounted energy loss for blow-ups of a solution of Yang-Mills flow equation in 4 dimensional manifolds. Any energy loss corresponds precisely to the product of bubbles, that is, nontrivial Yang-Mills connections on S^4. Our result is similar to the evolution of harmonic maps of Riemannian surfaces into spheres.

Keywords: compact Riemannian manifold, Yang-Mills heat equation, weak solution

1.Introduction

Let M be a closed, connected Riemannian 4-manifold and $\Pi : ¦ \to M$ a smooth vector bundle with fiber $\Pi^{-1} \cong R^n$, fiber metric $\langle \cdot,\cdot \rangle_x$ and compact structure group $G \subseteq SO(n)$. There is the group $\overline{G} = Aut(¦) = \mathrm{Y}_{x \in M} Aut(¦_x)$ of gauge transformations, where $Aut(¦_x) \subseteq G$ are automorphisms of the fiber $¦_x$. The group G acts on $\overline{G}$ by conjugation. We denote by $ad(¦) := \mathrm{Y}_{x \in M} ad(¦_x)$ the adjoint bundle whose sections $s \in \Omega^0(ad(¦))$ may locally represented as maps $s : U_{\sim} \to \overline{g}$, where $\overline{g}$ is the Lie algebra of G . G acts on $ad(¦)$ via adjoint action. Denote $\Omega^p(ad(¦))$ (or simply Ω^p), the spaces of $\overline{g}$-valued p-forms and $\overline{D}$, the affine space of connections, $D = D_{ref} + A, A \in \Omega^1$.

For $D \in \overline{D}$, $F(D) = F_D = D \circ D \in \Omega^2$ is a 0-order operator called the curvature of D.

The Yang-Mills action of a connection D with curvature F is defined by

$$YM(D) = \frac{1}{2}\int_M |F|^2 \, dx$$

where dx is the volume form of M, $|F|^2 = \langle F, F \rangle_{\overline{g}}$ and $\langle \cdot,\cdot \rangle_{\overline{g}}$ is the Cartan-Killing metric on $\overline{g}$. The Euler Lagrange equations is

$$D^{*}F = 0$$

where D^{*} is the adjoint operator of D with respect to the Riemannian metric of M.

M.Struwe, in [Stru1] showed the existence and uniqueness of the weak Yang-Mills flow in four dimensions:

$$(1.1)\ \frac{\partial}{\partial t}D = -D^{*},$$

$$(1.2)\ D(\cdot,0) = D_{0}.$$

. Concerning the blow-up analysis and the long-time behavior of the Yang-Mills flow, Struwe claimed and proved in detail by A.Schlatter in [Sch].

In this paper, our purpose is to study what happens to the solutions near these singular points. The result we obtain as follows:

Main Theorem

Suppose $D(t)$ *is a solution of Yang-Mills flow equations and* $(\bar{x},\bar{t}) \in \Sigma$*, the set of singular set of the* $D(t)$ *at time* $\bar{t}$.
Then there exists finitely many nontrivial Yang-Mills connections $\{\Delta_j\}_{j=1}^{p}$ *over* S^4 *such that*

$$\lim_{t\to\bar{t}}\int_{B_R(\bar{x})} |F_D(\cdot,t)|^2\,dx = \int_{B_R(\bar{x})} |F_{D_1}|^2\,dx + \sum_{j=1}^{p} YM(\Delta_j)$$

for any disk $B_R(\bar{x}) = B_R$ *with* $\overline{B_{2R}(\bar{x})}\ \mathrm{I}\ \Sigma \setminus \{(\bar{x},\bar{t})\} = ¤$.

2. Solutions of the heat equations

In this section we study the local behavior of the solutions for the Yang-Mills equations near a singular point. In particular, we prove what is needed for the proof of our main theorem.

By using the local version of the monotonicity formula (see Lemma 2.2 of [Sch]) we have

Proposition 2.1

Suppose that $D(t)$ *is a smooth solution of Yang-Mills flow equation, and that* $(\bar{x},\bar{t}) \in \Sigma$ *and* B_R *are given as in Main Theorem. Then*

$$\lim_{t\to\bar{t}}\int_{B_R} |F_D|^2(\cdot,t)\,dx$$

exists.

Proof: A straightforward computation with the help of the local monotonicity formula (see Lemma 2.2 of [Sch]) .

With regarding the blowing up sequence of rescaled connections, we have the following more general result:

Proposition 2.2

For any sequence

$$\{D_j\}_{j=1}^{\infty} \subseteq H^{1,2}(B_j,\Omega^1)\ \textit{with}$$

$$YM(D_j) < \infty$$

and

$$D_j^{*}F_{D_j} \to 0\ \textit{in}\ L^2_{loc}(R^4).$$

Then there is a family of gauge transformations $\{\pm_j\}_{j=1}^{\infty} \subseteq H^{3,2-£}(R^4,G)$ *such that*

$$\hat{D}_j = \pm_j^{*}(D_j) \to D_\infty$$

weakly in $H^{2,2-£}_{loc}(R^4,\Omega^1)$. D_∞ is a nontrivial smooth Yang-Mills connection over $R^4\ \mathrm{Y}\ \{\infty\} \cong S^4$ so that

$$\int_{B_2(0)} |F_{D_\infty}|^2 \geq £_3,$$

a constant independent of j.

Proof:

Using exactly the same arguments of [Sch] (See pp. 17--20 of that paper) with necessary notational modification, we have the assertion.

3. Sketched proof of the main theorem

First of all, by the global monotonicity formula (see [Stru1]), Proposition 2.1 and 2.2

we know that $YM(D(t))$ is a nonincreasing function of t, the limit $\int_{B_R} |F_D(\cdot,t)|^2 dx$ exist as $t \to \bar{t}$ and it sufficies to show that we may blow the first bubble to compensate the loss of energy as time evolves, because we can keep blow bubbles within finite steps until the energy loss are captured by the bubble we blow near the singularity.

We write

$$\lim_{t\to\bar{t}} \int_{B_R} |F_D|^2(\cdot,t)dx = L + \int_{B_R} |F_{D_{\bar{t}}}|^2 dx,$$

where B_R as defined in the main theorem and $L > 0$ is a positive constant. For any $a_j \downarrow 0$, there are $x_j \to \bar{x}, t_j \uparrow \bar{t}$ such that for some $R_0 > 0$ we have

(3.1) $$\pounds_1 = \frac{\pounds_0}{N} = \int_{B_{a_j}} |F(t_j)|^2 dx = \sup_{x\in B_{2R_0}(\bar{x}),\, t_j - 2_j \le t \le t_j} \int_{B_{a_j}(x)} |F(t)|^2 dx$$

where N is a positive integer to be chosen later and $\pounds_0$ is the positive constant stated in Theorem 1.1(9) of [Sch],

$$2_j = \frac{\pounds_0 R_j^2}{4NYM(D_0)},$$

$$x_j \in B_{R_0}(\bar{x}),$$

and that the bundle ¦ is trivialized over $B_{2R_0}(\bar{x})$.

Now we do the scaling procedure by setting

$D_j(x,t) = d + a_j A(x_j + a_j x, - a_j^2 t + t_j)$

We note that $D_j(t)$ is a classical solution of (1.1) on $B_{R_0/a_j}(\bar{x}) \times [t_0, 0] = B_j \times I_0$, where the $*$-operator is taken with respect to the rescaled metric and $t_0 = \frac{-\pounds_0}{4NYM(D_0)}$.

By the conformal invariance of the L^2-norm of the curvature F_j,

$$\sup_{|x|<R_0/a_j,\, t\in I_0} \int_{B_1(x)} |F_j(t)|^2 dx \le \int_{B_1(0)} |F_j(0)|^2 = \pounds_1,$$

Fix $x \in R^4$. Since $a_j \downarrow 0$, $B_1(x) \subset B_j$ for $j > j_x$, and $D_j(t)$ is defined on $B_1(x)$ for all $j > j_x$. By choosing sufficiently large N, by (3.1), we may apply Theorem 2.1 of [U1] to $D_j(t)$, $j > j_x$, $t \in I_0$ to get a sequence of gauge transformation $\{\pm_{j,t}\} \subset C^\infty(B_R, G)$ such that the transformed connection

$\overline{D_j}(t) = \pm^*_{j,t}(D_j(t)) = d + \overline{A_j}(t)$

satisfy

(i) $d^* \overline{A_j}(t) = 0$;

(ii) $\| \overline{A_j} \|_{H^{1,2}} \le CYM(\overline{D_j}) \le C\pounds_1$.

We notice that

$$\int_{t_0}^0 \int_{B_j} |\partial_t A_j|^2 dxdt = \int_{t_j - 2_j}^{t_j} \int_M |\partial_t A|^2 dxdt \to 0$$

there is a sequence $\{s_j\}_j$ $s_j \in I_0$,, such that

(3.2) $$\| \partial_t A_j(s_j) \|_{L^2(K)} \to 0$$

as $j \to \infty$ for any compact $K \subset R^4$.

We also notice that $\pm_{j,s_j} \in C^\infty(B_1(x), G)$.

Define

$$\tilde{D}_j(t) := \pm^*_{j,s_j}(D_j(t)) = d + \tilde{A}_j(t)$$

which satisfies

$$\frac{d}{dt}\tilde{D}_j = -\tilde{D}_j^* \tilde{F}_j$$

with $\tilde{F}_j = F_{\tilde{D}_j}$. By (3.2)

$$\|\widetilde{D}_j^* \widetilde{F}_j\|_{L^2(K)} = \|\frac{d}{dt}\widetilde{D}_j\|_{L^2(K)}$$
$$= \|\partial_t \widetilde{A}_j(s_j)\|_{L^2(K)} \to 0$$

as $j \to \infty$ for any compact $K \subset R^4$.

Now we may apply Proposition 2.2 to $\widetilde{D}_j$ to obtain the first bubble $\Delta_1 = D_\infty$ with

$$\int_{B_2(0)} |F_\infty|^2 dx \geq \frac{\pounds_0}{2N}.$$

4 Some remarks

In the case of the heat flow of harmonic maps from surfaces into spheres, much more is known about the structure of the bubbles (see [Q]. In our case, the space of connections is an affine space and they are invariant under gauge transformations, so we have to construct various time-independent gauge transformations to blow our S^4-bubbles. This difficulty make it more challenging to find similar convergence results as in the harmonic maps case.

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