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具應力奇異點之 Mindlin 板振動問題探討(I)

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具應力奇異點之 Mindlin 板振動問題探討(I)

Investigation in Vibrations of Mindlin Plates with Stress Singularities(I)

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中文摘要

板是工程設計上(土木工程、機械工程、航空工程…..等)之主要構件之一。Mindlin 板理論亦經常被使用於板相關問題分析上。由於外力點荷重、點彎矩及邊界之不連續性與尖角之存在，應力奇異點常發生於板相關問題。該奇異點須準確地處理，方能使得相關之數值分析解得到準確的答案。但依文獻回顧，目前對 Mindlin 板理論，由於邊界不連續或尖角之存在而引致之應力奇異階數，並未有一完整之探討。更不用論將其應用於含有應力奇異點且幾何較複雜問題之數值分析解。故本研究擬以三年之時間，深入探討此相關問題。

於第一年，本研究將以特徵函數展開法(eigenfunction expansion)，求解由於邊界不連續或尖角之存在所引致 Mindlin 板應力奇異之解析漸近解，以求得各種不同條件下之應力奇異階數及其對應漸近解函數。

關鍵詞：Mindlin 板理論；應力奇異；特徵函數展開法

1. Abstract

Plates are widely used components in engineering applications for civil engineering, mechanical engineering, and aerospace engineering. The Mindlin plate theory is often applied to describe the behaviors of plates. It is well known that stress singularities arise in the mathematical solutions of plate problems, which can be due to concentrated forces and moments, discontinuities in edge conditions or sharp corner. It has been pointed out and numerically shown that if singularities due to discontinuities in edge conditions or sharp corners are not properly considered in numerical solutions, significant errors will occur in the calculated global behavior of plates, such as static deflection, free vibration frequencies, forced dynamic response, and critical buckling load. However, there is no comprehensive study in the stress singularities for the Mindlin plate theory. Consequently, it is also short of accurate numerical solutions for the plates with stress singularities. It is the main purpose of the three-year proposal to investigate the stress singularity behaviors of Mindlin plates due to discontinuities in edge conditions or sharp corner and apply these results to some well known numerical solution techniques to solve some complicate vibration problems involving stress singularities.

In the first year, eigenfunction expansion approach will be applied to find the asymptotic solution for stress singularity behavior in the Mindlin plate theory. The singularity orders corresponding to various combinations of edge conditions will be determined and expressed in graphic form. The results will be compared with those for thin plate theory.

Keywords: Mindlin plate theory, stress singularities, eigenfunction expansion

2. Motive and Goal

Stress singularities in elastic plates frequently arise due to boundary conditions along the plate edges and the geometry of the plates. As well known, stress singularities exist at sharp corners in plates with V-notches or with irregular shapes of holes. Analytically determining the stress singularity behavior at a sharp corner is important not only for fracture mechanics [1] but also for numerical analysis of any complex problem involving such a sharp corner [2,3].

Some studies on stress singularities in plates have been undertaken according to classical plate theory

(CPT) or the plane stress assumption. Williams [4,5] pioneered the investigation of stress singularities of homogeneous, isotropic sector plates under bending and in-plane extension, due to various homogeneous boundary conditions. Williams and Chapkis [6] further considered the stress singularities for polarly orthotropic thin plates. Dempsey and Sinclair [7] proposed a new form of Airy stress function to reexamine the stress singularities in isotropic elastic plates under extension. Hein and Erdogan [8] and Bogy and Wang [9] used the Mellin transformation to study the stress singularities for bi-material wedges, while Dempsey and Sinclair [10] used an Airy stress function for the same purpose. Meanwhile, Ting and Chou [11] applied Stroh's approach [12] to examine the stress singularities at the vertex of anisotropic wedges under extension. Applying classical lamination theory, Ojikutu, Low, and Scott [13] considered stress singularities at the apex of a laminated composite wedge with simply supported radial edges.

The stress singularities at the corners of moderately thick plates have seldom been addressed. Burton and Sinclair [14] considered the singularities due to six different combinations of homogeneous boundary conditions around a corner, for Reissner's theory. The authors reduced the three field equations of Reissner's theory to two Cauchy-Riemann equations by introducing a stress potential. Williams' procedure was then applied to find equations characterizing the stress singularity behaviors. However, moment singularities but no shear force singularities were found in their solution. Based on the Mindlin plate theory, Huang *et al.* [15] investigated the stress singularities at the vertex of a sector plate with simply supported radial edges by finding the exact solution for free vibrations of such a plate. That solution yielded both the moment singularity and the shear force singularity. The great similarity between Reissner's theory [16] and Mindlin's theory [17] leads one to expect very similar singular behaviors according to these two theories. Consequently, the singularity behaviors in thick plate theories require further study to resolve the conflicts between the conclusions of Burton and Sinclair [14] and those of Huang *et al.* [15].

This study thoroughly investigates the Williams type stress singularities in first-order shear deformation plate theory (FSDPT) due to ten different combinations of homogeneous boundary conditions. The three field equations in the first-order shear deformation plate theory are directly solved by adopting the eigenfunction expansion method recently proposed by Xie and Chaudhuri [18,19] for studying stress singularities in a three-dimensional problem. Notably, the method proposed by Xie and Chaudhuri [18,19] provides the same three-dimensional asymptotic stress fields in the vicinity of the front of crack as those obtained by Hartranft and Sih [20], even though the solution methodology used by Hartranft and Sih [20] is more complex than Xie and Chaudhuri's [18,19]. This study explicates not only the equations characterizing the moment and shear force singularities, but also the corresponding asymptotic displacement fields for stress singularities. The singularity orders of moments and shear force variations with the corner angles are graphically depicted for the various homogeneous boundary conditions. The obtained stress singularity orders are compared with those published in different theories or approaches, and especially in Williams' solution [4] for a thin plate.

3. Contents of the Research

3.1 Methodology

The equilibrium equations with no external loading, in terms of displacement components in polar coordinates in the first-order shear deformation plate theory are given

$$\frac{D}{2}\{(1-\nu)(\psi_{r,rr}+r^{-1}\psi_{r,r}+r^{-2}\psi_{r,\theta\theta}-r^{-2}\psi_r-2r^{-2}\psi_{\theta,\theta}) + (1+\nu)(\psi_{r,rr}-r^{-2}\psi_r+r^{-1}\psi_{r,r}-r^{-2}\psi_{\theta,\theta}+r^{-1}\psi_{\theta,\theta r})\} + \kappa^2 Gh(-\psi_r+W_{,r})=0 \quad (1)$$

$$\frac{D}{2}\{(1-\nu)(\psi_{\theta,rr}+r^{-1}\psi_{\theta,r}+r^{-2}\psi_{\theta,\theta\theta}-r^{-2}\psi_\theta+2r^{-2}\psi_{r,\theta}) + (1+\nu)(r^{-2}\psi_{\theta,\theta\theta}+r^{-2}\psi_{r,\theta}+r^{-1}\psi_{r,\theta r})\} + \kappa^2 Gh(-\psi_\theta+r^{-1}W_{,\theta})=0\,, \quad (2)$$

$$\kappa^2 Gh(W_{,rr}+r^{-1}W_{,r}+r^{-2}W_{,\theta\theta}-\Psi_{r,r}-r^{-1}\Psi_r-r^{-1}\Psi_{\theta,\theta})=0\,, \quad (3)$$

where W is the transverse displacement of the midplane; ψ_r and ψ_θ are the bending rotation of the midplane normal in the radial and circumferential directions; respectively, h is the thickness of the plate; $D=Eh^3/12(1-\nu^2)$ is the flexural rigidity; E is the modulus of elasticity; ν is Poisson's ratio; κ^2 is the shear correction factor, and G is the shear modulus.

On the basis of separation of variables, the displacement components are assumed to take the following form:

$$\psi_r(r,\theta)=e^{p\theta}\Psi_r(r)\,,\quad \psi_\theta(r,\theta)=e^{p\theta}\Psi_\theta(r)\,, \text{ and}$$

$$W(r,\theta)=e^{p\theta}w(r)\,, \quad (4)$$

where p is commonly a complex number. Substituting Eq. (4) into Eqs.(1-3) with careful arrangement yields,

$$\frac{D}{2}\{(1-\nu)(\Psi_r''+r^{-1}\Psi_r'-(1+p^2)r^{-2}\Psi_r+2pr^{-2}\Psi_\theta) + (1+\nu)(\Psi_r''-r^{-2}\Psi_r+r^{-1}\Psi_r'-p\,r^{-2}\Psi_\theta+p\,r^{-1}\Psi_\theta')\} + \kappa^2 Gh(-\Psi_r+w')=0\,, \quad (5a)$$

$$\frac{D}{2}\{(1-\nu)(\psi_\theta'' + r^{-1}\psi_\theta' + (p^2-1)r^{-2}\psi_\theta + 2p\, r^{-2}\psi_r)$$
$$+ (1+\nu)(p^2 r^{-2}\psi_\theta + p\, r^{-2}\psi_r + p\, r^{-1}\psi_r')\}$$
$$+\kappa^2 Gh(-\psi_\theta + pr^{-1}w) = 0\,, \quad (5b)$$

$$\kappa^2 Gh(w'' + r^{-1}w' + p^2 r^{-2} w - \psi_r' - r^{-1}\psi_r - pr^{-1}\psi_\theta) = 0$$
$$, \quad (5c)$$

where the primes denote differentials with respect to r. The coupled ordinary differential equations (Eqs. (5)) are solved using the Frobenius method.

3.2 Singularity of bending moments

Let

$$\psi_r(r) = \sum_{m=0} a_{2m} r^{\lambda+2m}\,, \quad \psi_\theta(r) = \sum_{m=0} b_{2m} r^{\lambda+2m}\,,$$
and $$w(r) = \sum_{m=0} c_{2m} r^{\lambda+2m+1}\,, \quad (6)$$

where λ can be a complex number. Obviously, the real part of λ must be larger than zero to satisfy the regularity condition for the displacement components, as r approaches zero. The relations between stress resultants and displacement components reveal that the series given in Eq. (6) can lead to singular moments in the vicinity of r equal to zero, but no singularity for shear forces.

Substituting Eq. (6) into Eqs. (5) and satisfying the resulting equations corresponding to the smallest order in r (i.e., m=0) yield

$$p = \pm i(\lambda-1) \quad \text{and} \quad p = \pm i(\lambda+1)\,. \quad (7)$$

The general asymptotic form for the displacement components can be simply written as follows,

$$V_r(r,\theta) = (A_1 \cos(\lambda+1)\theta + A_2 \sin(\lambda+1)\theta +$$
$$A_3 \cos(\lambda-1)\theta + A_4 \sin(\lambda-1)\theta) r^{\lambda} + O(r^{\lambda+2}) \quad (8a)$$

$$V_\theta(r,\theta) = (A_2 \cos(\lambda+1)\theta - A_1 \sin(\lambda+1)\theta +$$
$$k_2 A_4 \cos(\lambda-1)\theta - k_2 A_3 \sin(\lambda-1)\theta) r^{\lambda} + O(r^{\lambda+2}) \quad (8b)$$

$$W(r,\theta) = (C_1 \cos(\lambda+1)\theta + C_2 \sin(\lambda+1)\theta +$$
$$\text{¥}_1 A_3 \cos(\lambda-1)\theta + \text{¥}_1 A_4 \sin(\lambda-1)\theta) r^{\lambda+1} + O(r^{\lambda+3}) \quad (8c)$$

where

$$k_1 = -\frac{[2(1-\nu) + (1+\nu)(\lambda+1)]}{[2(1-\nu) - (1+\nu)(\lambda-1)]}\,. \quad (9a)$$

Those coefficients in Eqs (8) and the values of λ are specified by the radial boundary conditions of a wedge. Table 1 lists the characteristic equations for λ corresponding to different combinations of boundary conditions. The characteristic equations based on thin plate theory [4] are also summarized in Table 1. Obviously, the characteristic equations for different plate theories are quite different, except for the case with simply supported radial edges. Figure 1 shows the variation of the smallest positive real part of λ with the vertex angle for different boundary conditions. The details of derivation of the characteristic equations and comparison of the values of λ for different plate theories are given in [22].

3.3 Singularity of shear forces

Starting with assuming

$$\psi_r = \sum_{n=0} \bar{a}_{2n} r^{\bar{\lambda}+2n+1}\,, \quad \psi_\theta = \sum_{n=0} \bar{b}_{2n} r^{\bar{\lambda}+2n+1}\,, \quad \text{and}$$
$$w = \sum_{2n=0} \bar{c}_{2n} r^{\bar{\lambda}+2n} \quad (10)$$

for Eqs. (5), one is able to find the general asymptotic form for the displacement components simply written as follows,

$$V_r(r,\theta) = [\bar{A}_1 \cos\bar{\lambda}\theta + \bar{A}_2 \sin\bar{\lambda}\theta +$$
$$\bar{A}_3 \cos(2+\bar{\lambda})\theta + \bar{A}_4 \sin(2+\bar{\lambda})\theta] r^{\bar{\lambda}+1} + O(r^{\bar{\lambda}+3})$$
$$V_\theta(r,\theta) = [\bar{B}_1 \cos\bar{\lambda}\theta + \bar{B}_2 \sin\bar{\lambda}\theta +$$
$$\bar{A}_4 \cos(2+\bar{\lambda})\theta - \bar{A}_3 \sin(2+\bar{\lambda})\theta] r^{\bar{\lambda}+1} + O(r^{\bar{\lambda}+3})$$
$$W(r,\theta) = [\bar{l}_1(\bar{A}_1 \cos\bar{\lambda}\theta + \bar{A}_2 \sin\bar{\lambda}\theta) +$$
$$\bar{l}_2(\bar{B}_2 \cos\bar{\lambda}\theta - \bar{B}_1 \sin\bar{\lambda}\theta)] r^{\bar{\lambda}} + O(r^{\bar{\lambda}+2})$$
(11a-11c)

where $\bar{l}_1 = \frac{-D}{2\kappa^2 Gh}(3-\nu+(1+\nu)(1+\bar{\lambda}))$ and (4i)

$$\bar{l}_2 = \frac{D}{2\kappa^2 Gh}(2(1-\nu)-(1+\nu)\bar{\lambda})\,.$$

The coefficients $\bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{A}_4, \bar{B}_1$, and $\bar{B}_2$ are also determined from the radial boundary conditions. Table 2 lists the characteristic equations for $\bar{\lambda}$ corresponding to different combinations of boundary conditions. Figure 2 shows the variation of the smallest positive real part of $\bar{\lambda}$ with the vertex angle for different boundary conditions.

4. Discussion

This investigation has presented the Williams type asymptotic solution at a corner of a thick plate with various boundary conditions using an eigenfuction expansion technique to solve the three partial differential equations for displacement components in first-order shear deformation plate theory. The characteristic equations for determining the singularity orders for moments and shear forces at the corner, with corresponding corner functions for various boundary conditions, were also fully developed. Notably, under identical boundary conditions, the equations characterizing the singularity behaviors of moments are totally different from those characterizing the singularity behaviors of shear forces.

The validity of the solution was confirmed by comparing the moment and shear force singularity behaviors for type I simply supported conditions with those from the exact solution of free vibrations of a sector plate with the same boundary conditions along radial edges. Furthermore, the obtained characteristic equations for a free-free boundary condition were consistent with those for a completely free wedge from three-dimensional elasticity solution.

The characteristic equations for first-order shear deformation plate theory are completely different from those for the classical thin plate theory, except in the case of simply supported (S(I)) radial edges. The boundary conditions and the vertex angle determine which theory, FSDPT or CPT, produces a stronger moment singularity. Nevertheless, the classical theory always leads to a stronger shear force singularity than does the first-order shear deformation plate theory because the former does not consider the shear deformation.

The corner functions corresponding to various boundary conditions presented here can be applied to numerical analysis for the complex problems of moderately thick plates with corner singularities. McGee et al. [23] and Leissa et al. [3] applied the Ritz method using the corner functions for classical plate theory as admissible functions to examine the free vibrations of skewed plates and sectorial plates.

5. Comment and Conclusion

We have achieved the goals of the project given in the proposal. Based on the results in this work, one paper has been published in *International Journal of Mechanical Science* [22]. This work has also been extended to investigate the stress singularities in bi-material plates, which has been published in *Composite Structure* [24].

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Table 1 Comparison of characteristic equations for first-order shear deformation plate theory and classical plate theory

Case No.	Boundary conditions	Characteristic equations	
		FSDPT	CPT
1	Simply supported (I) -Simply supported (I)	$(\cos((\lambda-1)\alpha/2))(\cos((\lambda+1)\alpha/2))=0$ (S*) $(\sin((\lambda-1)\alpha/2))(\sin((\lambda+1)\alpha/2))=0$ (A*)	$\cos\alpha\lambda=-\cos\alpha$ (S*) $\cos\alpha\lambda=+\cos\alpha$ (A*)
2	Clamped-Free	$\sin^2\lambda\alpha=\frac{4-\lambda^2(1+\nu)^2\sin^2\alpha}{(3-\nu)(1+\nu)}$	$\sin^2\lambda\alpha=\frac{4-\lambda^2(1-\nu)^2\sin^2\alpha}{(3+\nu)(1-\nu)}$
3	Simply supported (I) - Free	$\sin 2\lambda\alpha=\lambda\sin 2\alpha$	$\sin 2\lambda\alpha=\frac{\lambda(1-\nu)}{-3-\nu}\sin 2\alpha$
4	Simply supported (I) - clamped	$\sin 2\lambda\alpha=\frac{\lambda(1+\nu)}{-3+\nu}\sin 2\alpha$	$\sin 2\lambda\alpha=\lambda\sin 2\alpha$
5	Free-Free	$\sin\lambda\alpha=-\lambda\sin\alpha$ (S*) $\sin\lambda\alpha=\lambda\sin\alpha$ (A*)	$\sin\lambda\alpha=-\frac{\lambda(1-\nu)}{-3-\nu}\sin\alpha$ (S*)

			$\sin {}^{a}\sim = \frac{{}^{a}(1-{}^{3})}{-3-{}^{3}}\sin\sim$ (A*)
6	Clamped-Clamped	$\sin {}^{a}\sim = -\frac{{}^{a}(1+{}^{3})}{-3+{}^{3}}\sin\sim$ (S*) $\sin {}^{a}\sim = \frac{{}^{a}(1+{}^{3})}{-3+{}^{3}}\sin\sim$ (A*)	$\sin {}^{a}\sim = -{}^{a}\sin\sim$ (S*) $\sin {}^{a}\sim = {}^{a}\sin\sim$ (A*)
7	Simply supported (II) -Simply supported (II)	$\sin\lambda\alpha = -\lambda\sin\alpha$ (S*) $\sin\lambda\alpha = \lambda\sin\alpha$ (A*)	
8	Clamped- Simply supported (II)	$\sin^{2} {}^{a}\sim = \frac{4-{}^{a2}(1+{}^{3})^{2}}{(3-{}^{3})(1+{}^{3})}\sin^{2}\sim$	
9	Simply supported (I) -Simply supported (II)	$\sin 2\lambda\alpha = \lambda\sin 2\alpha$	
10	Simply supported (II) -Free	$\sin {}^{a}\sim$ **Z ï** ${}^{a}\sin\sim$	

Note: * S: symmetric case, A: anti-symmetric case

Table 2 Characteristic equations for the singularities of shear forces

Boundary conditions	S (I)-S(I), F-F, C-C, S(II)-S(II)	C-F, S(I)-F, S(II)-F	S(I)-C, C-S(II), S(I)-S(II)
Characteristic equation	cos $\overline{a}$ ~ / 2 Z 0	cos $\overline{a}$ ~ Z 0	sin $\overline{a}$ ~ Z 0

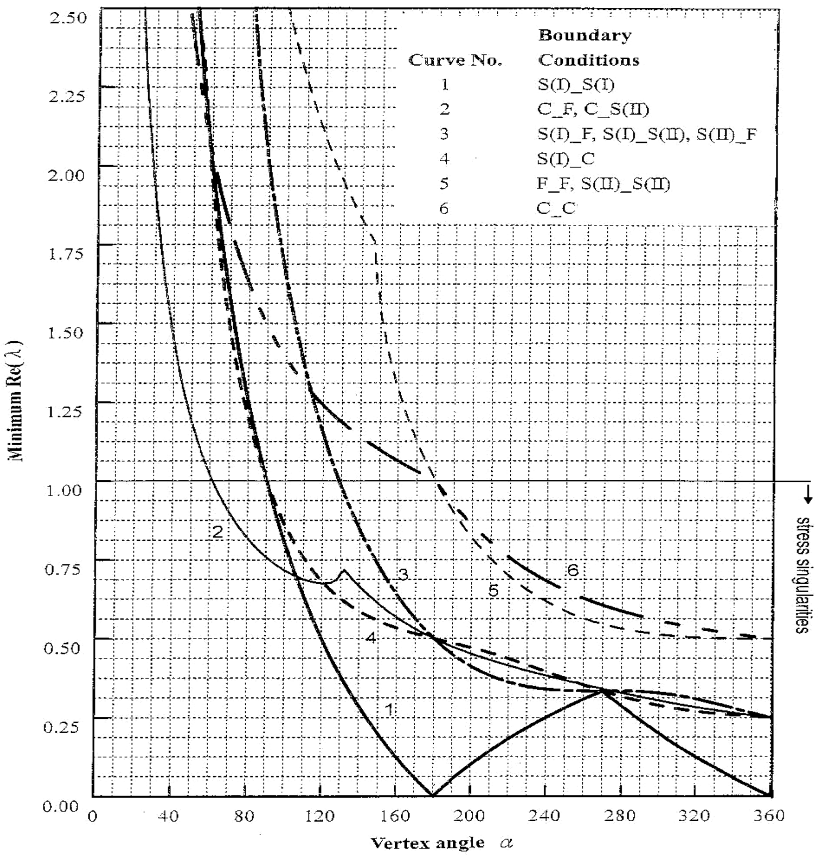

Figure 1 Variation of minimum positive Re(λ) with vertex angle α for FSDPT.

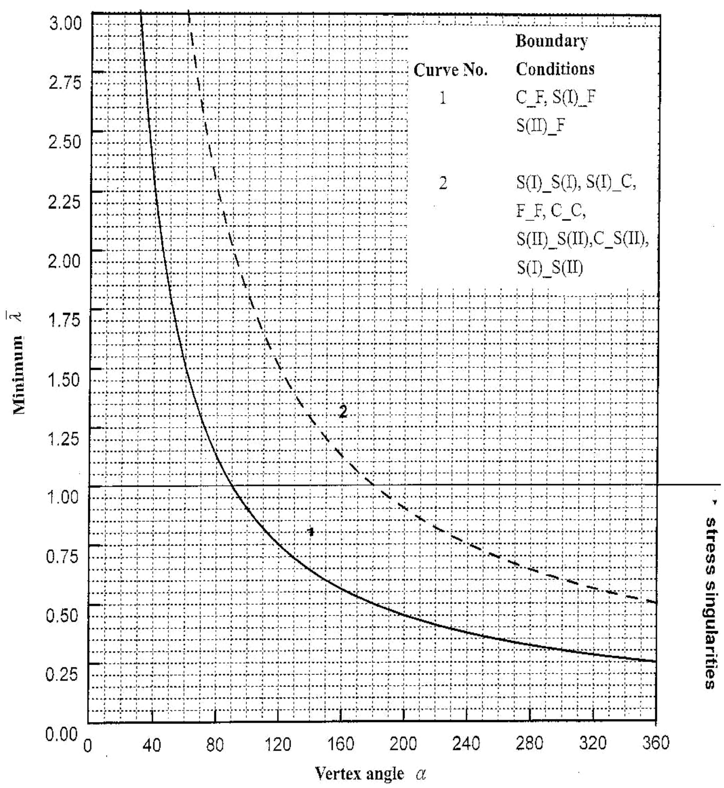

Fig. 2 Variation of minimum positive $\overline{a}$ with vertex angle ~ for FSDPT