微積分實驗教材之研究

交大是一所以理工為主的大學，全校除了外文系及管理科學系社會組二班之外，所有的大一學生皆要修習微積分課程。但由於長期填鴨式教育及速食文化影響，學生的學習興趣及效果卻不盡理想。1999 年本人訪問美國數所參與微積分改革之學校後，深深體會，提供學生多元化的微積分學習環境並以電腦科技結合數學活動實為必然之趨勢。

自九十學年度開始，交大的微積分課程便採取不分系、由學生自由選班上課。並針對學生的不同需求，提供了三種課程：普通班課程、實驗班課程與榮譽班課程。其中，實驗班的教材與普通班相同，唯不同於一般只由老師課堂教授的模式，學生將透過電腦實驗主動學習。

這兩年本人均負責一班的實驗班，部分的教學則以本計劃所設計之 problem-based 實驗活動為教材。，讓學生兩人一組在電腦室中，透過網路取得教材，實地操作學習。

一. 活動教材之設計目標

以學習活動貫穿教材
從活動中引導實做,互相討論. --想法是"做出來"的,而非"獲得的".
強調學習過程而非僅止於學習結果.

二. 教學執行之情形

學生兩人一組在電腦室中，透過網路取得教材，實地操作學習，老師從旁指導。

三. 預期學生在電腦室中的活動

觀察(observation) → 認知(identification) → 探索(exploration) →

分析(analysis) → 解說(explanation)(口頭、文字)

四. 完成之教材

http://xserve.math.nctu.edu.tw/people/cpai/CalculusLab/index.htm

Lab 1 : Guessing limits Numerically ---Explore the concept of “ limit ” by graphs and numerical data.

Lab 2 : Mathematical Models --- Establish a mathematical model from given data with elementary functions such as polynomials, exponential functions.

Lab 3 : Implicit Functions and Implicit Differentiation --- Understand the concept of “ a function defined implicitly ” , visualize the idea of “ linearization “ and perform the procedure of implicit differentiation.

Lab 4 : Graphical Analysis --- What is a good representative plot of a function and how the derivatives of a function affect its graph.

Lab 5 : Area and Definite Integrals --- Start with the area problem and use the idea to formulate a definite integral.

Lab 6 : Approximation of Integrals --- Left endpoint approximation, right endpoint approximation, Midpoint rule and Simpson’s rule.

Lab 7 : Parametric Curves --- Understand the advantage of parametric descriptions of curves is that they are convenient for "combined motions." Realize that simple functions can do great graphic designs.

附註：此單元學生反應十分良好，學生作品請參考網頁：
http://xserve.math.nctu.edu.tw/people/cpai/demo/gallery/cal91.htm

Lab 8: Polar coordinates --- Be familiar with polar coordinates and explore some interesting curves defined by polar equations.

Lab 9 : Taylor Polynomials --- Explore the fact that a polynomial could be completely determined by its value and the values of its derivatives at $x = 0$. Find out that as terms of higher degree are added with the appropriate coefficients, approximation to the "target" polynomial improves in the sense that the two functions appear to match over a wider domain centered at 0. Further, extend this idea to approximations of a non-polynomial function.

Lab 10 : Cylinders and Quadratic Surfaces --- Explore the graphs of cylinders and quadratic surfaces by their traces. Also, discover the interesting shapes that members of family of surfaces $z = ax^2 + bxy + cy^2$ can take, by observing how the shape of the surface evolves as we vary the constants.

Lab 11 : Cylindrical and Spherical Coordinates --- Be familiar with cylindrical and spherical coordinates and explore some interesting surfaces parametrized by cylindrical or spherical coordinates.

Lab 12 : Limits of Multivariable Functions --- Understand the concept of “the limit of a two variable function” by level curves and graphs.

Lab 13 : Parametric Representations of Surfaces --- Represent a given surface with suitable parametric equations and identify the grid curves.

Lab 14 : Critical Points and Contour Plots --- Predict the location of the critical points of a two variable function f by its level curves and whether f has a saddle point or a local maximum or a local minimum at each of those points. Find the critical points of f by two-dimensional Newton’s method.

Lab 15 : Changes of Coordinates --- Investigate how a transformation can do to a region in R^2 and realize the “ Jacobian” of a transformation as “ change-in-area factor” for it.

四. 學生的反應

◎ Lab 的學習方式優於傳統教學?

A. **贊成**

(1) 可和同學討論，運用課堂上所學解決問題，定義及其運用可更了解。
(2) 可以了解圖形或者可以知道計算的思考路線。
(3) 可以快速的得到結果，而無須以繁瑣的步驟處理。
(4) 可以利用電腦軟體玩一些有趣的東西，提升學習興趣。
(5) 自己動手做，比較能學到東西，加深印象，且不會睡著。
(6) 可用電腦實做出比較難想像的東西。
(7) 可以主動參與學習，自己找尋問題與答案。不會和在一般課堂上一樣只單方面的教學，無互動。
(8) 我們可以利用電腦的繪圖能力，更佳的了解問題。
(9) 免除處理繁雜計算和作圖的時間，讓人可以更專注於解決問題的方向。
(10) 除了簡單的題目可以手算，大多數的題目都要使用電腦。
(11) 觀念會強，方法會學不少，有價值。
(12) 比一般死氣沉沉的上課方式好玩。
(14) 利用電腦軟體輔助教學可以提高學習效率。
(15) 可以加強空間的概念，由電腦軟體展示圖形，並提供較深刻之理解。
(16) 可以用動畫的方式使學生明白圖形如何隨著某些因素而改變, 更加了解圖形的含意。
17.透過實際操作，可以對函數圖形或是微積分的基本原理有更深的了解。

B.**不贊成**

(1) 習慣有老師在講解，自己看的話有些都看不出來。
(2) 因為 Lab 佔去部分上課時間，對於課本的內容就會比較不熟，希望有方法改進。
(3) 利用 Maple 做計算並不會提升自己的數學能力，考試時也不能使用 Maple。

C.**其他建議**

(1) Lab 內容可以再活潑一點。
(2) 第一次接觸 Lab，覺得滿難的，希望有多一點時間讓大家和教授討論。
(3) 一些較少用或較難的 Maple 指令教學可以放在網頁上, 以供忘記時查詢。
(4) 兩個人一組的好處是可以互相討論。但是往往最後會一人做一部分，學習到的東西就比較少。不分組自己做自己的話，雖然一開始會怕，但是會強迫自己去學。
(5) 建議不要強迫分組，自己找伙伴，增加學習興趣。
(6) 每次的工作量似乎有點多，希望採小組工作。
(7) 建議以後的學生要買一本 Maple 使用手冊，如此使用 Maple 會較順手。
因為題目不會做都是因為指令不會寫或看不懂，上課說的指令根本不

夠。

(8) 要先上課再上 Lab，否則英文看不懂根本無法了解題意。

(9) 因為我的電腦有問題，無法安裝 Maple，做 Lab 作業時都必須向別人借電腦，所以除了作業上的指令會用之外，沒有機會看它的設明檔或其他指令，覺得可惜！

◎ 印象最深刻的單元

(1) Lab 7： Parametric Curves

a. 圖形很有趣，尤其是最後一個圖。
b. 可看出參數函數圖形上的點隨著參數的改變而移動的情形。
c. 畫圖讓我們想了很久，可是很好玩，很有成就感。
d. 自己動手去設計自己喜愛的圖形，相當有新鮮感。
e. 讓我第一次體會到數學的應用，如畫圖。

(2) Lab 8： Polar Coordinates

a. 可以自由創作出令人意想不到的圖形，很好玩。
b. 起初對極座標的意義不甚了解，但經過 Maple 圖形的輔助，讓我有進一步的認識。
c. 做了很久，解決了全部的問題，才知道牛頓真厲害，可用極座標加上參數函數解釋行星運動的現象。

詳細活動內容如下:

Module 1

Guessing limits Numerically

Purpose:

Explore the concept of " limit "
by graphs and numerical data.

We write

$$\lim_{x \to a} f(x) = L$$

and say " the limit of f(x), as x approaches a , equals L. "

If we can make the values of f(x) arbitarily close to L (as close to L as we like) by taking x to be sufficiently close to a, but not equal to a.

In the module, we are going to have fun exploring some interesting limits, such as

$$\lim_{x \to 0} \frac{1-\cos(x)}{x^2}, \quad \lim_{x \to 0} \frac{1}{1-2^{\left(\frac{1}{x}\right)}},$$

$$\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right), \quad \lim_{x \to 0} x \sin\left(\frac{1}{x}\right).$$

We write

$$\lim_{x \to a} f(x) = L$$

and say "the limit of f(x), as x approaches a , equals L "

If we can make the values of $f(x)$ arbitarily close to L (as close to L as we like) by taking x to be sufficiently close to a, but not equal to a .

Part I

We are interested in estimating $\lim_{x \to 0} \frac{1-\cos(x)}{x^2}$.

1. Consider the function and plot the graph of

$$f(x) = \frac{1-\cos(x)}{x^2}$$

.

```
> f :=x ->(1-cos(x))/x ^2;
  plot(f(x),x=-1..1);
> f(0.0);
```

2. What is the domain of f ?

3. Let's look at the values of $f(x)$ for $x < 0$.

```
> for n from 1 to 6
  do # This is the beginning of a "do loop".
  x:=-1/2 ^n: # Let x = -1/2^n .
  print(evalf(x),evalf(f(x))); # Print the values of x and f(x).
  od: # This is the end of our "do loop".
```

4. Now let's look at some values of $f(x)$ for $x > 0$.

```
> for n from 1 to 6
  do
  x:=1/2 ^n:
  print(evalf(x),evalf(f(x)));
  od:
```

Remarks:

1. Different rates of convergence can be achieved by replacing by $\frac{1}{n^2}$ or $\frac{n}{10}$.

2. At the end of the do loops in the above code, Maple will think that $n = 6$ and $x = \pm \frac{1}{2^n}$. (You can check this by entering the commands **n;** and **x;** after each loop.) This is important to know since if, subsequent to the appropriate do loop, you wanted to reuse n or x as a variable then you would have to redefine it as a variable using the command **n := 'n'** or the command **x := 'x'**.

```
> n:='n'; x:='x';
```

5. On the basis of these data, do you think $\lim_{x \to 0} \frac{1 - \cos(x)}{x^2}$ exists? If so, what do you think it is (to 4 decimal places of accuracy)? Justify your answer.

Part II

> **restart;** # Clear Maple's memory.

1. Define the function $f(x) = \dfrac{1}{1 - 2^{\left(\frac{1}{x}\right)}}$, and plot the graph of f for x in [-1, 1].
2. Evaluate $f(x)$ for $x = 0.1, 0.09, 0.08, 0.07, ..., 0.01$.
3. Evaluate $f(x)$ for $x = -0.1, -0.09, -0.08, -0.07, ..., -0.01$.
4. On the basis of these data, do you think

$$\lim_{x \to 0} \frac{1}{1 - 2^{\left(\frac{1}{x}\right)}}$$ exists ? Justify your answer.

Part III

1. Define the function $g(x) = \sin\left(\frac{\pi}{x}\right)$, and plot the graph of g for x in [-2, 2].
2. Evaluate $g(x)$ for $x = 1, 1/2, 1/3, ..., 1/10$.

3. Evaluate $g(x)$ for $x = 2, 2/5, 2/9, 2/13, ..., 2/25$.

4. What can be said about the behavior of $g(x)$? Do you think $\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right)$ exists? Justify your answer.

Part IV

Explore the functions and $h(x) = x \sin\left(\frac{1}{x}\right)$. Do you think $\lim_{x \to 0} h(x)$ exists? If so, what do you think it is? Justify your answer.

Module 2

Mathematical Models

Contents

Purpose:

Establish a mathematical model with elementary functions such as polynomials, exponential functions.

Part I Linear Model

Table Shown below lists the average carbon dioxide level in the atmosphere, measured in parts per million at Mauna Lao Observatory from 1972 to 1990.

Year	CO_2 level (in ppm)
1972	327.3
1974	330.0
1976	332.0
1978	335.3
1980	338.5
1982	341.0
1984	344.3
1986	347.0
1988	351.3
1990	354.0

To enter this data in Maple, we define a list for each column and then "zip" the lists together to make the list of pairs, carbondata.

```
>Years:=[1972,1974,1976,1978,1980,1982,1984,1986,1988,1990];
 co2:=[327.3,330.0,332.0,335.3,338.5,341.0,344.3,347.0,351.3,354.0];
 co2data:=zip((x,y)->[x,y],Years,co2);
```

Edit and use the plot command below to generate a scatter plot of the data.

```
> plot(co2data, style=point, symbol=circle);
```

The ***style*** and ***symbol*** entries are called "options." You can use plot options to enhance your graphs in a variety of ways. The general format for plot options is

> **plot(data, option1, option2, option3, ...);**

You can specify the x and y ranges:

x = xmin..xmax
y = ymin..ymax

If you use either or both of these options, they must come before other options.

You can set the color of plotted points:

color = red (or **green** , **blue** , **yellow** , **violet** , etc.)

You can label your axes:

labels = [`Year`, `ppm`]

(Note backward quote, often found above the Tab key.)

See **?plot** , **?plot,options** for more details.

Complete and enter your enhanced plot command below.

> **plot(co2data, x=1970..1995, y=320..360, style=point, symbol=circle, labels = [`Year`, `ppm`]);**

1. Does the data points appear to lie close to a

   straight line ? If so, find the equation of the fitting line and explain your fitting procedure here.

   The ***display*** command in ***plots*** package used with several plots will plot them all on the same graph. Complete the following commands to plot your line and the data points together.

```
> with(plots):
  fitline:=plot(???, t=1970..1995, y=320..360,
  color=blue):
  dataplot:=plot(co2data, x=1970..1995, y=320..360,
  style=point, symbol=circle, labels = [`Year`,
  `ppm`]):
  display(fitline,dataplot);
```

2. Does the line look approximately like the data plot ?

   If not, rework last step.

   Maple has a built-in routine for fitting a line to a data set. In the ***stats*** package is a ***fit*** package that has a command called ***leastsquare*** . If the variables in this command are specified as ***[x,y]*** , then the output for the fitted line is of the form y = b + ax . Put your cursor in the line below, and press Enter to construct the "least squares" fitted line.

```
> with(stats):
  fit[leastsquare[[t, y], y=a*t+b]] ([Years, co2]);
```

Use copy and paste to define the equation above.

```
> y1 := t->???;
```

Next, we include the graph of this least squares line with the other two graphs.

```
> fitCurve := plot(y1(t), t=1970..1995, color=green):
  display(dataplot, fitline, fitCurve);
```

3. Which line fits better ?

4. Predict the CO_2 level in 1992.

5. According to your model, when will the CO_2 level exceed 400 parts per million ?

Part II Quadratic model

A ball is dropped from a tower, 450 meters above the ground, and its height h above the ground is recorded at 1-second intervals in the table below.

Time (seconds)	Height (meters)
0	450
1	445
2	431
3	408
4	375
5	332

6	279
7	216
8	143
9	61

1. Generate a scatter plot of the data.

2. Observe that a linear model is inappropriate. Does the data points may lie on a parabola? If so, try to find a parabola that fits the data.

3. Use your model to predict the time at which the ball hits the ground.

<u>Part III</u> Exponential model

<u>World Population in the 20th Century</u>

Year	Population (millions)
1900	1650
1910	1750
1920	1860
1930	2070
1940	2300
1950	2520
1960	3020
1970	3700
1980	4450
1990	5300
1996	5770

1. Generate a scatter plot of the data.

```
> Y:=[ ??? ]:
  P:=[ ??? ]:
  datP:=zip((a,b)->[a,b], Y, P):
  plot(datP, style=point, symbol=circle, color=blue);
```

There is a semilog command in the ***plots*** package called ***logplot*** , which works in much the same way as plot, but it does logarithmic (base 10) scaling of the vertical axis.

```
> with(plots):
  logdatP:=logplot(datP, style=point,
  symbol=circle):
  display(logdatP);
```

2. Does the data points in the semilog plot above look like a straight line ? Can you conclude from this that an exponential model should fit the population data ?

3. Find the equation of the line that fits the semilog plot. (Note that Maple's name for the base-10 logarithm is ***log10*** . Also recall ***evalf*** if you want to see numerical values.)

4. Find your population model here and compare it with the population data. Explain why or why not the model you get is a good one.

5. Predict the size of the world population in the year of 2001.

Module 3

Implicit Functions and Implicit Differentiation

Purpose:

Explore the concept of " a function defined implicitly ", visualize the idea of " linearization " and perform the procedure of implicit differentiation.

Part I

A function can be described either explicitly -- for example,

$$y = \sqrt{x^2 + 1}$$ or $$y = x \sin(x)$$

or, in general, $y = f(x)$. Some functions, however, are defined implicitly by a relation between x and y , such as

$$x^2 + y^2 = 25 \quad \text{or} \quad x^3 + y^3 = 3xy \,.$$

In some case, it is possible to solve such an equation for y as an explicit function (or several functions) of x. For instance, if we solve $x^2 + y^2 = 25$ for y,

```
> solve(x^2+y^2=25, y);
```

two functions determined by the equation $x^2 + y^2 = 25$ are $y_1(x) = \sqrt{25 - x^2}$ and $y_2(x) = -\sqrt{25 - x^2}$. The graphs of $y_1(x) = \sqrt{25 - x^2}$ and

$$y_2(x) = -\sqrt{25 - x^2}$$

are the upper and lower semicircles of the circle $x^2 + y^2 =$ 25.

The Maple command ***implicitplot*** is used for plotting equations.

```
> with(plots):
  implicitplot(x^2+y^2=25,x=-5..5,y=-5..5,scaling=co
  plot(sqrt(25-x^2),x=-5..5,scaling=constrained);
  plot(-sqrt(25-x^2),x=-5..5,scaling=constrained);
```

Another example is the **folium of Descartes** ("folium" means leaf), which is given by the equation $x^3 + y^3 = 3xy$. It is difficult to solve this equation for y explicitly as a function of x by hand. (A computer algebra system has no trouble, but the expressions it obtains are very complicated. If you are really curious, try it!) Here is its graph :

```
> eq := x^3+y^3=3*x*y;
  implicitplot(eq, x=-3..3, y=-3..3, grid=[50,50],
  scaling=constrained);
```

Note that this plot contains a loop, which cannot be described globally as the graph of one function $y = \mathrm{y}(x)$. However, the plot is the graph of some function near most points. For example, the lower piece of the loop over the interval [-1,1] is the graph of a function $\mathrm{y}(x)$. Finding formula for $\mathrm{y}(x)$, we need to solve the equation $x^3 + y^3 = 3xy$ for y in terms of x. This is difficult since this equation involves a cubic. It is possible to find numerical values of $\mathrm{y}(x)$ at specific values of x. For example, the values of y at $x = 1.5$ can be found by using the Maple command ***fsolve*** .

```
> x:=1.5;
  fsolve(eq, y, y=1..2);
```

A. Verify that $(1.5,1.5)$ is on the curve.

A plot over a small range that limits the range of x and y also reveals that the plot satisfies the vertical line test near $x = 1.5$. Hence, it is a graph of a function.

```
> x:='x':
  implicitplot(eq, x=1..1.75, y=1.25..1.75, scaling=co
```

Over a very small plot range, the graph looks like a straight line.

B. Do you think that the **folium of Descartes** has a tangent line at $(1.5,1.5)$?

If so, what is the equation of the tangent line? Justify your answer.

Implicit Differentiation is the procedure used to find the derivative of an implicitly defined function:

Step 1. Differentiate both sides of the equation with respect to x.

(by viewing y as a function $y(x)$ of x).

Step 2. Solve the resulting equation for y' (or $\frac{\partial}{\partial x} y(x)$).

The following sequence of commands used for implicit differentiation will be applied to the circle $x^2 + y^2 = 25$, but this sequence of commands also applies equally to other implicitly defined expressions.

```
> x:='x':
  eq1:= x^2+y^2=25;
  subs(y=y(x), %);

> diff(%,x);
  solve(%, diff(y(x), x));
```

The symbol $\frac{\partial}{\partial x} y(x)$ (***diff(y(x),x)***) stands for derivative of y with respect to x.

C. Verify the formula for y' obtained above by differentiating the two functions $y_1(x) = \sqrt{25 - x^2}$ and $y_2(x) = -\sqrt{25 - x^2}$.

D. Find the equation of the tangent line to the circle $x^2 + y^2 = 25$ at ($\frac{5}{\sqrt{2}}, \frac{5}{\sqrt{2}}$).

E. Find all the points at which the formula for y' obtained above does not apply. Does the circle have tangent lines at those points?

F. Use the method of implicit differentiation to find the tangent line to the **folium of Descartes** at $(1.5,1.5)$.

G. Does the curve have a tangent line at ($2^{\left(\frac{2}{3}\right)}, 2^{\left(\frac{1}{3}\right)}$) ? Does the curve have a tangent line at $(0,0)$? Justify your answers.

Part II

Consider the curve with equation $2y^3 + y^2 - y^5 = x^4 - 2x^3 + x^2$.

A. Graph this curve and describe what the curve looks like.

B. At what point does this curve have horizontal tangent lines? Justify your answer.

C. Are there any points at which this curve have vertical tangent lines? Justify your answer.

Module 4

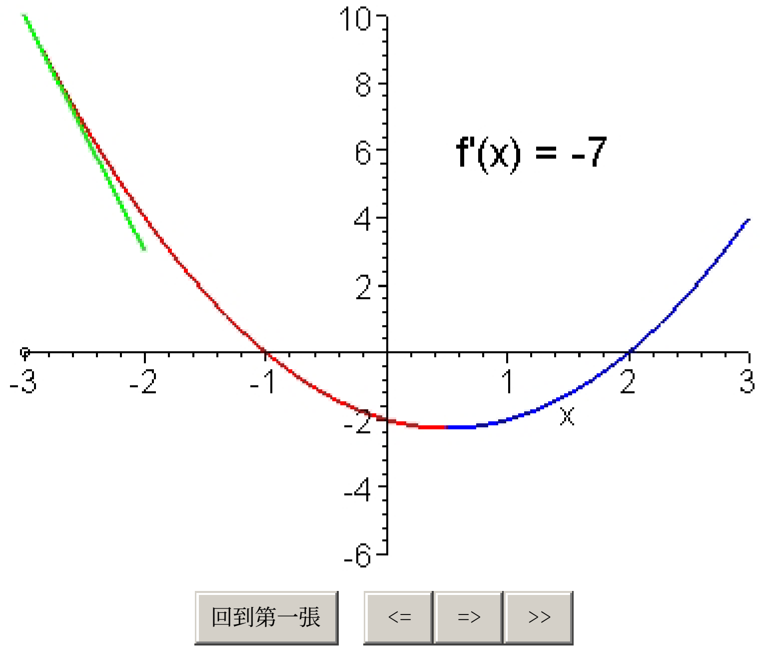
Graphical Analysis

Purpose:

Understand what is a good representative plot of a function and how the derivatives of a function affect its graph.

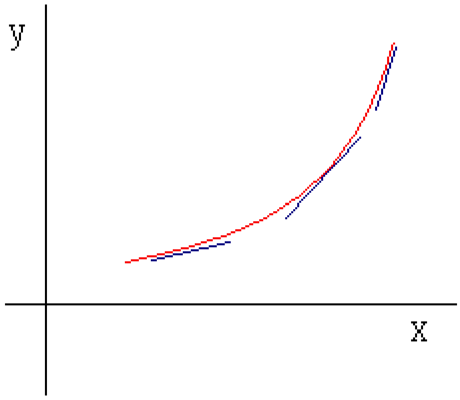
What Does f ' Say about f ?

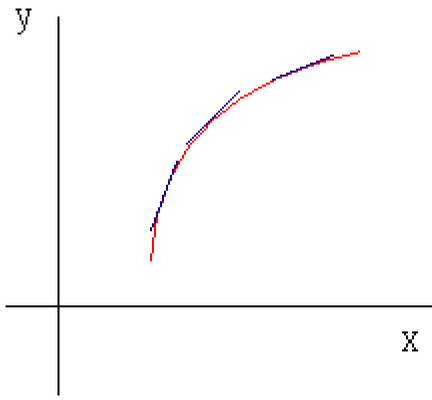
Play with the animation below and observe how the d
function affects the shape of its graph.



On the part of the graph of f which is colored red, the tangent lines have negative slope and so $f'(x) < 0$. While on the part of the graph of f which is colored blue, the tangent lines have positive slope and so $f'(x) > 0$. It appears that f decreases when $f'(x) < 0$ and increases when $f'(x) > 0$.

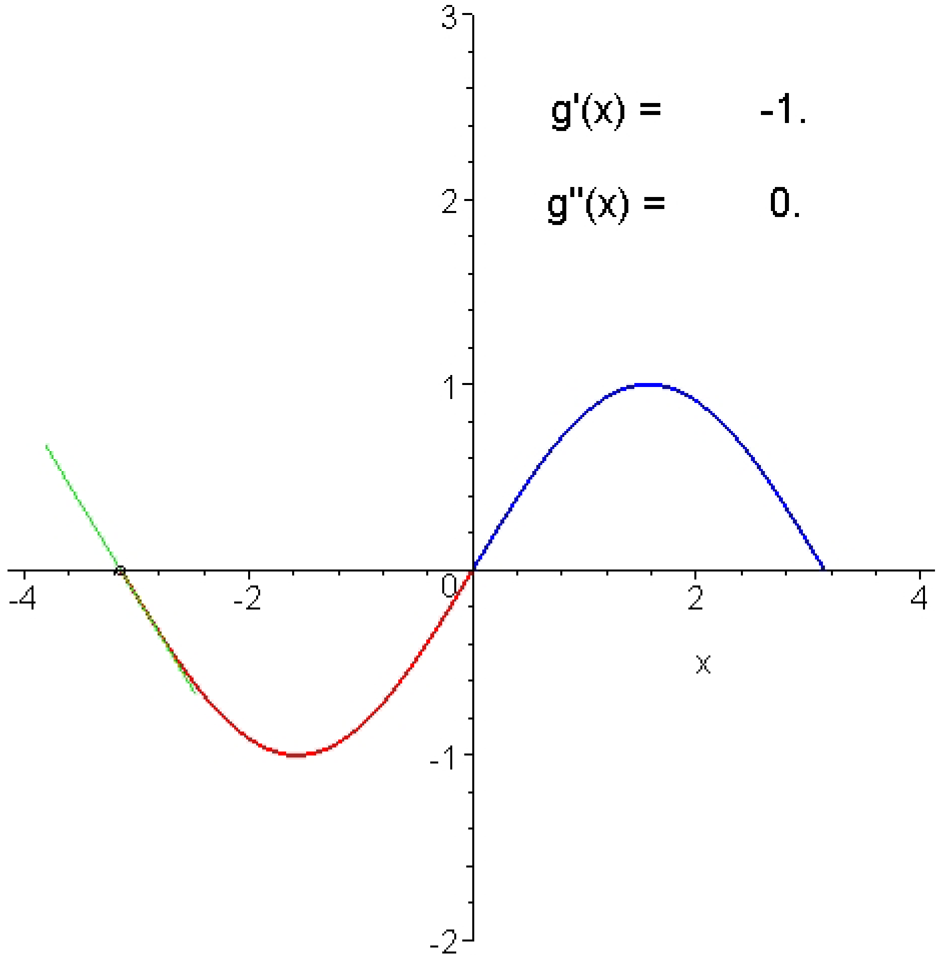
If the graph of f lies above all the tangent lines on an interval I, then it is called **concave upward** on I. If the graph of f lies below all the tangent lines on an interval I, then it is called **concave downward** on I. A point P on a curve is called an **inflection point** if the curve changes from concave upward to concave downward or from concave downward to concave upward at P.





The figure above shows the graphs of two increasing functions, in the graph on the left the curve lies above the tangents, so it is concave upward. In the graph on the right the curve lies below the tangents, so it is concave downward.

Play with the animation below.



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Notice that the interval on which the graph of f is colored red, f is concave upward; while the interval on which the graph of f is colored blue, f is concave downward. Do you see how the first and second derivatives help to determine the intervals of concavity and inflection points?

Good representative plots of functions try to exhibit all the changes in shape of the graph and give a strong flavor of the global scale behavior.

Part I

A. Plot $\frac{x^3}{e^x}$ for x in $[0, a]$, where a is chosen to be large enough to see the rising and falling of the curve.

```
> plot(x^3/exp(x),x=0..10);
```

B. Factor the derivative to find the exact turning point x at which the curve changes direction, and explain why the curve cannot change direction at any other point.

```
> diff(x^3/exp(x), x);
  factor(%);
```

C. Explore functions $\frac{x^6}{e^x}$ and $\frac{x^{12}}{e^x}$ as what you have done in A, B.

D. Given a positive number r , factor the derivative of $\frac{x^r}{e^x}$ to explain why the curve $y = \frac{x^r}{e^x}$ first goes up as x advances from 0 and grows until x reaches a point a after which the curve goes down. Find the exact value of the turning point x in terms of r .

E. How does the results above reflect the fact that in the global scale as x approaches ∞, the exponential growth of e^x dominates the power growth of x^r ?

Part II

Consider the function $$f(x) = \frac{x^7 - 58x^2 + 8}{2x^6 + 11}$$.

Plot the function and its derivative together.

```
> restart;
  with(plots):
  f:=x->(x^7-58*x^2+8)/(2*x^6+11):
  plot([f(x),D(f)(x)], x=-5..5, color=[red,blue], thickne
```

A. From the graph above, find the intervals of increase and the intervals of decrease of $f(x)$. Verify your answer by factoring the derivative of $f(x)$.

B. Determine the maximum and minimum values of $\frac{x^7 - 58x^2 + 8}{2x^6 + 11}$ for x in $[-1, 4]$.

C. Describe how the first derivative tells the concavity of the graph of $f(x)$.

D. Plot $f(x)$ and its second derivative together. Describe how the signs of the second derivative reflect the concavity of the graph of $f(x)$.

E. How does $f(x)$ behave as x approaches ∞ and as x approaches $-\infty$?

We say the line $y = mx + c$ is an asymptote of the graph of $f(x)$ if

$$\lim_{x \to \infty} [f(x) - (mx + c)] = 0$$

or

$$\lim_{x \to (-\infty)} [f(x) - (mx + c)] = 0 .$$

F. Find all the asymptotes of the graph of $f(x)$.

G. Does $f(x)$ have the maximum and minimum values for all x in $\boldsymbol{R}$? Justify your answer.

Part III

Plot the graph of the function $\dfrac{\sin(x)}{3 - 2x + x^2}$ over $[-6, 6]$ and discuss the important aspects of the function such as the intervals of increase or decrease, local maximum and minimum values, concavity and points of inflection, and asymptotes.

Part IV

Consider $f(x) = 2x^3 + cx^2 + 2x$.

A. Plot $f(x)$ for different values of c.

B. Use the command ***animate*** in the ***plots*** package to create an animation of $f(x)$.

```
> with(plots):
  animate(2*x^3+a*x^2+2*x, x=-10..10, a=-10..10, fra
  'view=[-10..10, - 40..100]');
```

To play an animation you must first select it by clicking on it. Then choose **Play** from the **Animation** menu.

C. Describe in words how the graph of $f(x)$ varies as c changes and confirm your answer with the help of calculus.

Summary

A. Why does a good representative plot of a function normally include all points at which its derivative is 0 ?

B. Comment on these statements:

1) If $f'(a) = 0$, then the plot of f is guaranteed to have a crest or a dip at $x = a$.
2) If the plot of f has a crest or dip at $(a, f(a))$, then it is automatic that $f'(a) = 0$.

C. Describe how the first derivative tells the concavity of a function?

D. What do you think the sign of f'' tells you about the concavity of the plot of f ?

E. If f has an inflection point at $(a, f(a))$, and $f''(a)$ exists. Is it always that $f''(a) = 0$?
On the other hand, does $f''(a) = 0$ guarantee that f has an inflection point at $(a, f(a))$?

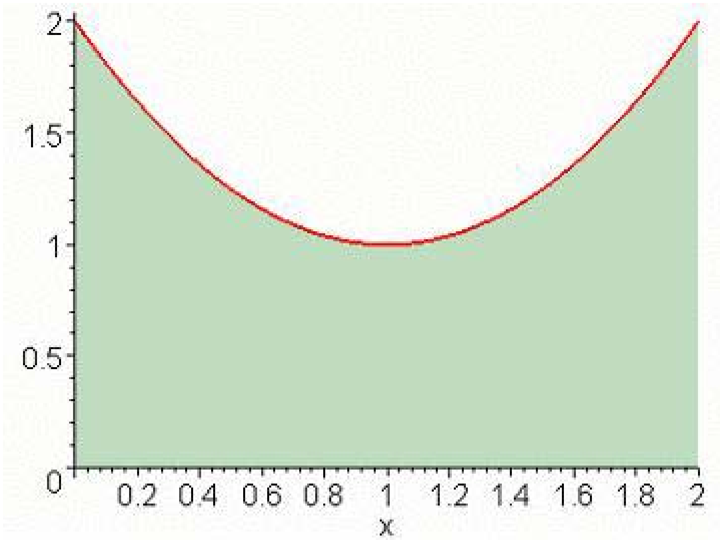
Module 5

Area and Definite Integrals

Purpose:

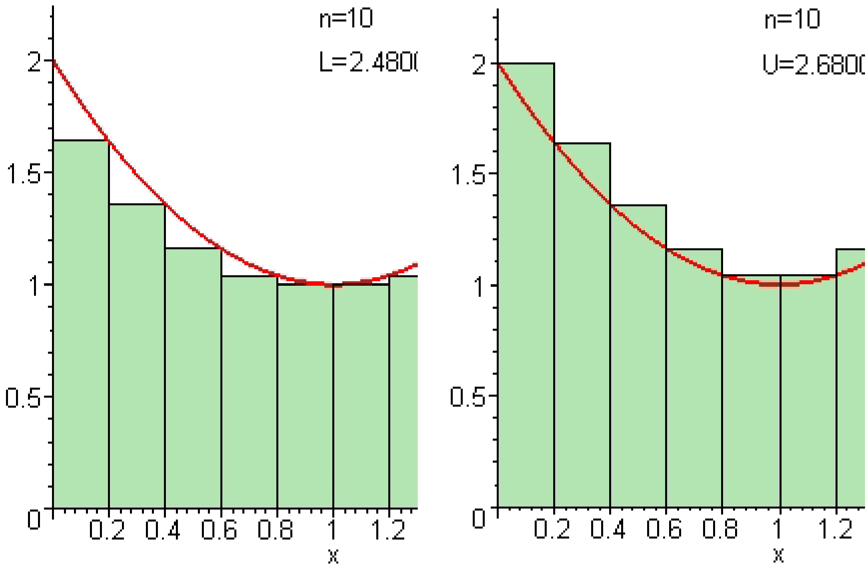
Start with the area problem and use the idea to formulate a definite integral.

積分問題的起源即是求面積的問題, 基本概念十分類似阿基米德的窮盡法(相關連結一, 相關連結二)。我們想要求出由紅色函數圖形、x 軸、$x = 0$ 與 $x = 2$ 所圍出區域的面積, 如下圖所示。



首先將 0 到 2 的區間分割成 n 個子區間, 左圖長方形的高度為該子區間函數的最小值, 所以長方形的面積總和必小於所求之面積。相反地, 右圖長方形的高度則為子區間內函數的最大值, 所以長方形面積的總和也會大於所求之面積。隨著 n 越來越大, 左圖長方形面積的總和 L 會越來越大, 而右圖長方形面積總和 U 會越來越小, 兩者之間的差 E 會趨近

零。也就是說，兩者會同時趨近所求的區域面積。

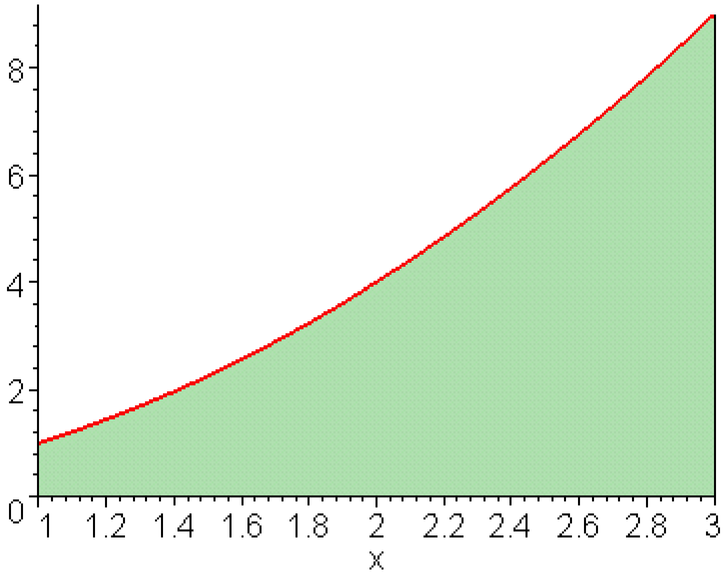


$E := .2000$

Restart <= => >>

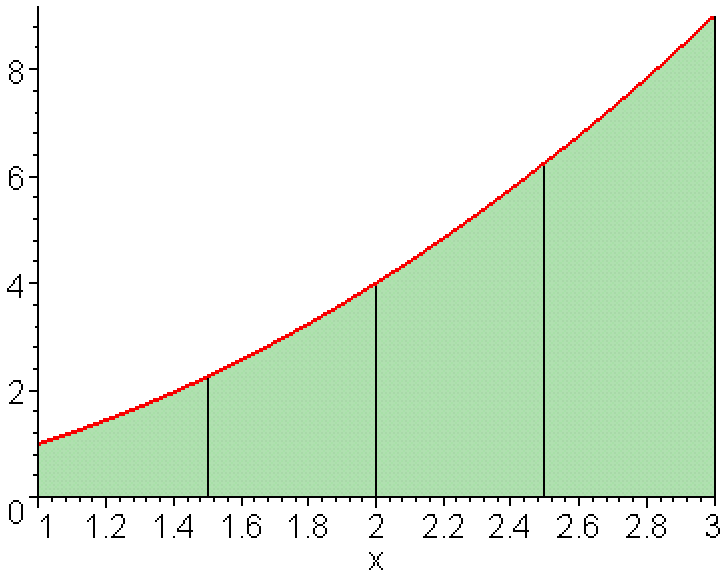
In this module we start with the area problem and use it to formulate the idea of a definite integral.

We begin by attempting to find the area of the region that lies under the curve $y = x^2$ from 1 and 3, illustrated below.



Suppose we divide the region into four strips by drawing the vertical lines

$x = \frac{3}{2}$, $x = 2$, and $x = \frac{5}{2}$.



We can approximate each strip by a rectangle whose base is the same as the strip and whose height is the same as the right edge of the strip.

Here we use Maple's ***rightbox*** command to visualize the process of approximating the area under a curve.

```
> f:=x->x^2:
```

In order to use Maple's ***leftbox*** (or ***rightbox***) command, one has to load the ***student*** package first.

```
> with(student):
  rightbox(f(x),x=1..3,4);
```

Moreover, we can compute the sum of these rectangles with the help of Maple's ***sum*** command.

```
> dx:=(3-1)/4;
  Sum(f(1+i*dx)*dx,i=1..4);
  evalf(%);
> sum(f(1+i*dx)*dx,i=1..4);
```

In general, we can divide the interval [$1, 3$] into n subintervals of eqaul length. The area under the graph of $f(x) = x^2$ is approximated by the sum of the areas of n rectangles where the base of a rectangle is one of the n subintervals and the height is the value of the function $f(x)$ at the right or left endpoint of the subinterval.

The ***leftbox*** command to illustrate rectangles that approximates area under the graph of $f(x)$ with the left endpoints of the intervals.

```
> eftbox(f(x),x=1..3,4);
```

1. What is the sum of those rectangles illustrated above?

Divide the interval into n subintervals of eqaul length, and let R_n and L_n be the sums of the rectangles with the heights of the right endpoints and left endpoints, respectively.

2. Find R_n and L_n with $n = 10$, 20, 40, 80. What do you find out?

The following commands compute the rightsum for general n, and the limit as n goes to infinity.

```
> dx:=(3-1)/n;
  right_area:=Sum(f(1+i*dx)*dx,i=1..n);
  right_limit:=Limit(right_area,n=infinity);
  value(%);
```

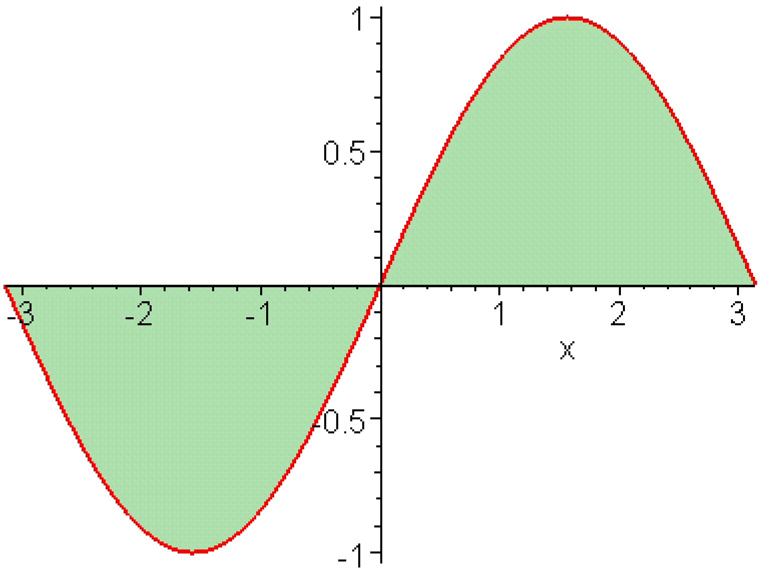
3. Find the limit of the leftsums as n goes to infinity. Does this limit agree with the one of the rightsums?

4. What do you think the area under the graph of $f(x)$ is ? Why ?

5. Approximating the area by the sum of the areas of n rectangles where the base of a rectangle is one of the n subintervals and the height is the value of the function $f(x)$ at any point x, instead of the right or left endpoint of the subinterval, d o you get the same answer as in (4) ?

6. Can you figure out the area under the graph of $f(x) = x^2$ over the interval [$-1, 2$] ? over the interval [a, b] for any $a < b$?

7. Use the same idea to find the area of the region bounded by the graph of $g(x) = \sin(x)$ on [$-\pi, \pi$] and the x -axis, illustrated below.



If f is a continuous function defined on [a, b], we divide the interval [a, b] into n subinterval of equal length $\Delta x = \frac{b-a}{n}$. We let $x_0 = a$, x_1, x_2 ,....., $x_n = b$ be the endpints of these subintervals, and let c_i be a point in the subinterval [x_{i-1}, x_i].

Then the **definite integral of** f **from** a **to** b is

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(c_i)\,\Delta x$$

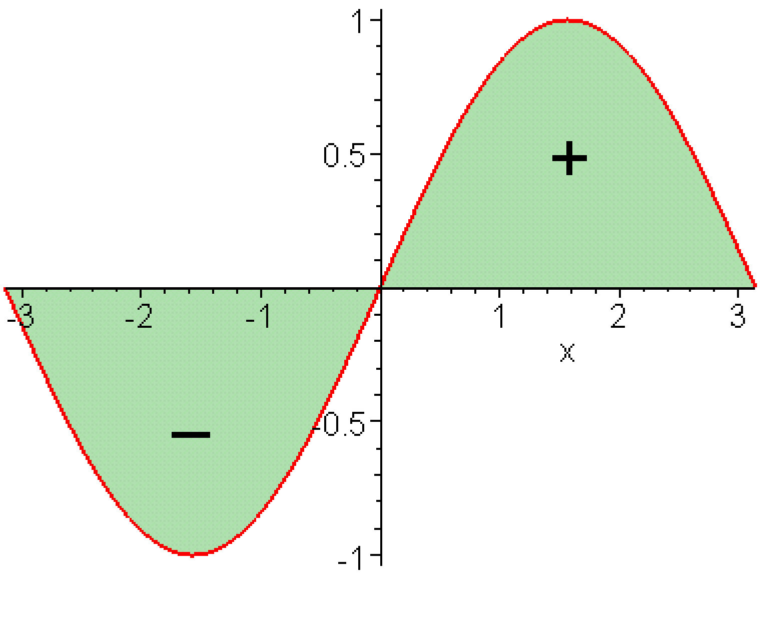
.

The sum $\sum_{i=1}^{n} f(c_i)\,\Delta x$ is called a **Riemann sum** .

Remarks :

A. In the definition above c_i can be chosen to be the right endpoint or the left endpoint of the subinterval [x_{i-1}, x_i].

B. We can view the definite integral $\int_a^b f(x)\,dx$ to be the " signed area" of f over the interval [a, b] as follows:



We can use Maple commands *int* or *Int* to Integrate functions or expressions.

For example, to integrate $f(x) = x^2$ over [a, b] one should enter

```
> Int(f(x),x=a..b);
  value(%);
```

or

```
> int(f(x),x=a..b);
```

The antideivative of f (or indefinite integral) can also be evaluated.

```
> Int(f(x),x);
> F(x):=int(f(x),x);
```

Note: Maple does not insert the constant of integration.

8. What is the relation between $\int_a^b f(x)\,dx$ and $F(x)$?

Module 6

Approximation of Integrals

Purpose:

Experiment with four different ways: Left endpoint approximation, right endpoint approximation, Midpoint rule and Simpson's rule, of approximating integrals, and find out which one is most efficient.

There are situations in which it is impossible to find the exact value of a definite integral. For examples :

$$\int_{-1}^{1} \sqrt{1+x^3}\, dx \quad \text{and} \quad \int_{0}^{1} e^{(x^2)}\, dx .$$

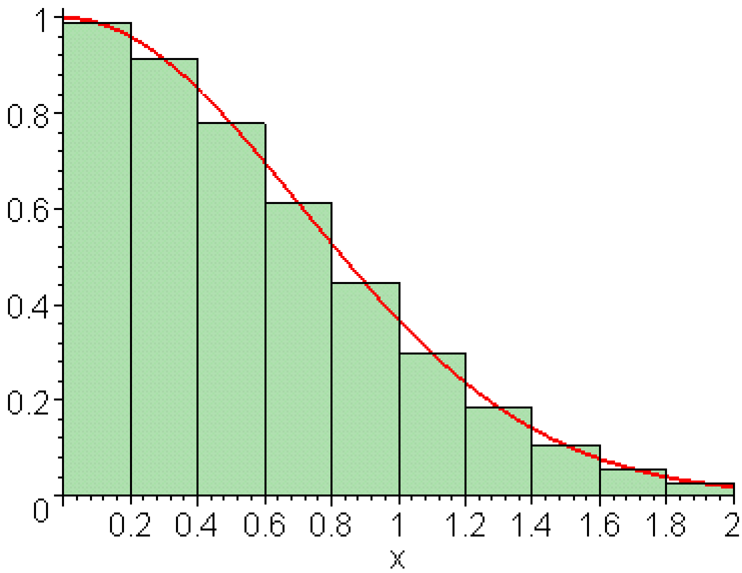
In these cases we need to approximate values of these definite integrals.

Recall that the definite integral $\int_a^b f(x)\, dx$ is defined as a limit of Riemann sums. If we divide $[a, b]$ into n subintervals $[x_{i-1}, x_i]$, $i = 1, \ldots, n,$ ($x_0 = a,\ x_n = b$), of equal length $\Delta x = \frac{(b - a)}{n}$, let c_i

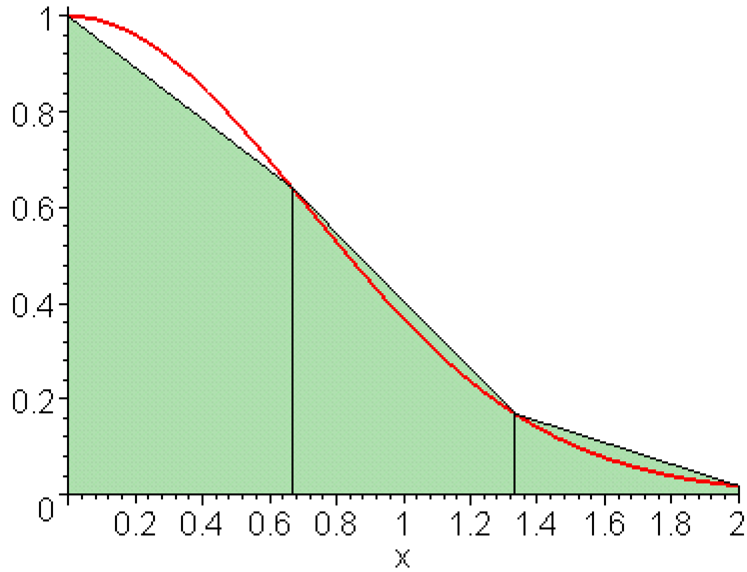
be any point in the i -th subinterval $[x_{i-1}, x_i]$, then $\sum_{i=1}^{n} f(c_i)\,\Delta x$ is a good approximation for $\int_a^b f(x)\,dx$ when n is sufficiently large.

By choosing c_i to be the left endpoint or the right endpoint of $[x_{i-1}, x_i]$, we have the **left endpoint approximation** or **right endpoint approximation,** respectively.

If we choose c_i to be the midpoint of $[x_{i-1}, x_i]$, then we have the **Midpoint Rule approximation** , as shown below.

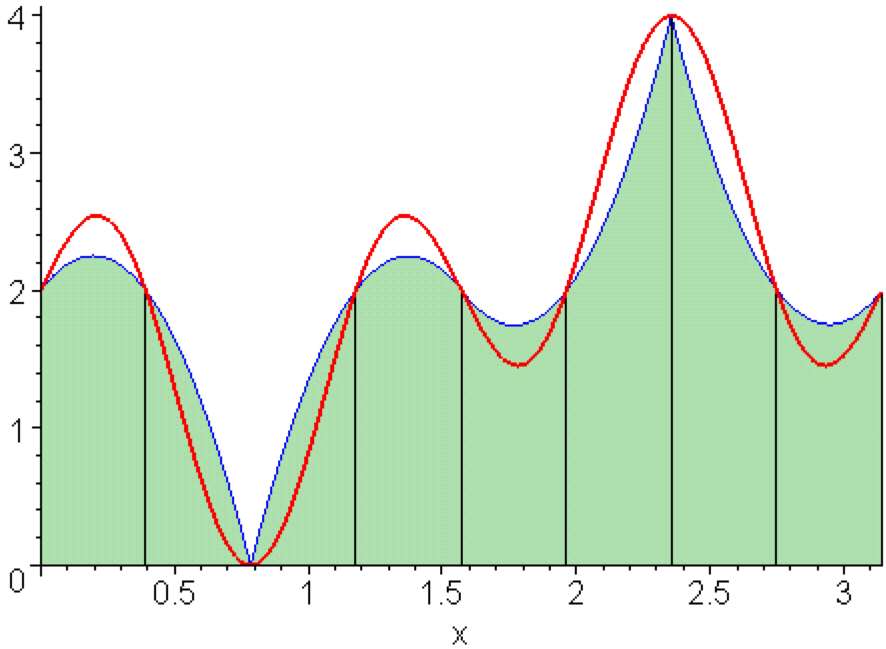


To compromize the difference between the values of left endpoints and right endpoints in each subintervals, we use the sum of areas of the trapezoids lies above the subintervals . This is called **Trapezoidal Rule**. The idea is shown below :



Simpson's Rule

Another rule for approximation integration results from using the parabolas. As before ,we divide $[a, b]$ into n subintervals $[x_{i-1}, x_i]$, $i = 1, \ldots, n$, ($x_0 = a,\ x_n = b$), of equal length $\Delta x = \frac{(b - a)}{n}$, but this time we assume that n is an **even** number. Then on each consecutive pair of intervals $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$, we approximate the curve $y = f(x)$ by a parabola passing through the points $(x_{i-1}, f(x_{i-1}))$, $(x_i, f(x_i))$, and $(x_{i+1}, f(x_{i+1}))$ as shown below. Let S_n denote the sum of the areas of these approximating parabolas.



In this module, we are going to explore these from methods and find out which one is most efficient.

Maple HTML Contents

Last Modified: 7.3.2003

Module 6

Approximation of Integrals

There are situations in which it is impossible to find the exact value of a definite integral. For examples :

$$\int_{-1}^{1} \sqrt{1+x^3}\, dx \quad \text{and} \quad \int_{0}^{1} e^{(x^2)}\, dx.$$

In these cases we need to approximate values of these definite integrals.

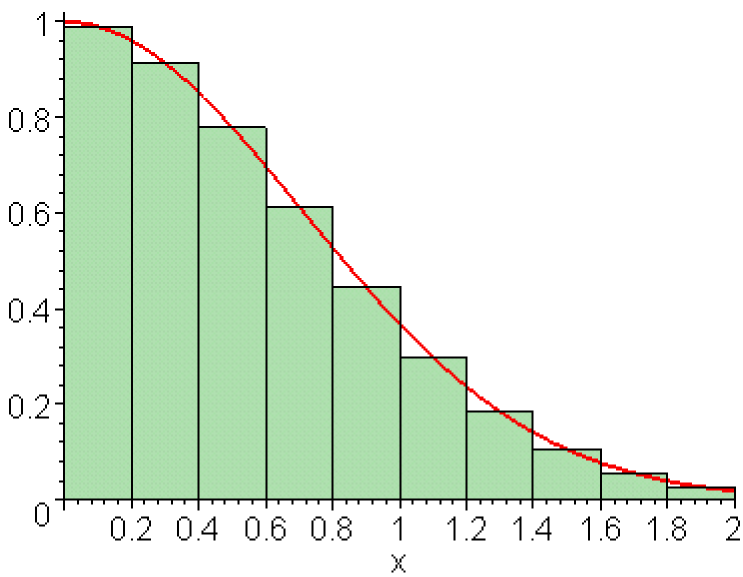
Recall that the definite integral $\int_a^b f(x)\, dx$ is defined as a limit of Riemann sums. If we divide $[a, b]$ into n subintervals $[x_{i-1}, x_i]$, $i = 1, \ldots, n$ ($x_0 = a$, $x_n = b$), of equal length $\Delta x = \frac{b-a}{n}$, let c_i be any point in the i -th subinterval $[x_{i-1}, x_i]$, then $\sum_{i=1}^{n} f(c_i)\, \Delta x$ is a good approximation for $\int_a^b f(x)\, dx$ when n is sufficiently large.

By choosing c_i to be the left endpoint or the right endpoint of $[x_{i-1}, x_i]$, we have the **left endpoint approximation** or **right endpoint approximation,** respectively.

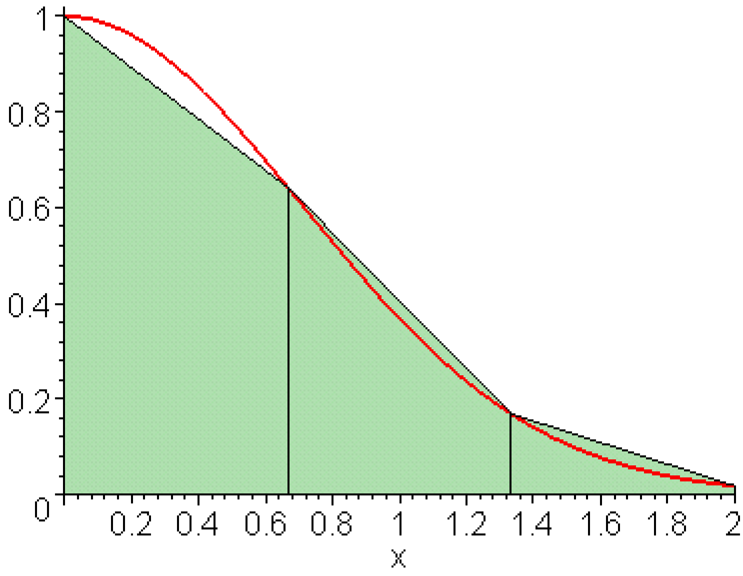
Divide $[a, b]$ into n subintervals **,** let L_n and R_n be the left endpoint approximation and the right endpoint approximation for $\int_a^b f(x)\,dx$, respectively.

1. Approximate $\int_0^2 e^{(-x^2)}\,dx$ by L_{10} and R_{10} and estimate the error. Justify your answer.

If we choose c_i to be the midpoint of $[x_{i-1}, x_i]$, then we have the **Midpoint Rule approximation** *,* as shown below.



Another approximation, called the **Trapezoidal Rule**. We use the sum of areas of the trapezoids lies above the subintervals . The idea is shown below :



Divide $[a, b]$ into n subintervals **,** let M_n and T_n be the Midpoint Rule approximation and the Trapezoidal Rule approximation, respectively.

$$T_n = \frac{L_n + R_n}{2}$$

2. Show that .

$$g(x) = \frac{1}{x^2}$$

3. Let on $[1, 2]$, for $n = 5, 10, 20$, compute L_n, R_n, M_n and T_n by defining L_n, R_n, M_n and T_n as functions of n.

As the command ***leftsum*** and ***rightsum***, the commands ***middlesum***, ***trapezoid*** can be found in Maple ***student*** package, you may have to apply the command ***evalf*** to get the numerical values.

Check your answers by using those commands.

```
> int(g(x),x=1..2);
```

Now we can make a table of the errors of the approximation above by the following commands.

```
> N:=3:
  A:=matrix(N+1,5,(Row,Col)->0):
  A[1,1]:='n': A[1,2]:='E[L]': A[1,3]:='E[R]': A[1,4]:='E|
  A[1,5]:='E[M]':
  for k from 1 to N
  do
  n:=2^(k-1)*5;
  A[k+1,1]:=n:
  A[k+1,2]:=0.5-evalf(L(n)):
  A[k+1,3]:=0.5-evalf(R(n)):
  A[k+1,4]:=0.5-evalf(T(n)):
  A[k+1,5]:=0.5-evalf(M(n)):
  od:
  eval(A);
```

4. What do you find out from the table above ?

Error Bounds

Suppose $|f''(x)| \leq K$ for $a \leq x \leq b$. If

$$E_{T_n} = \int_a^b f(x)\,dx - T_n$$

and

$$E_{M_n} = \int_a^b f(x)\,dx - M_n$$

are the errors involved in using the Trapezoidal and Midpoint Rules to approximate $\int_a^b f(x)\,dx$, then

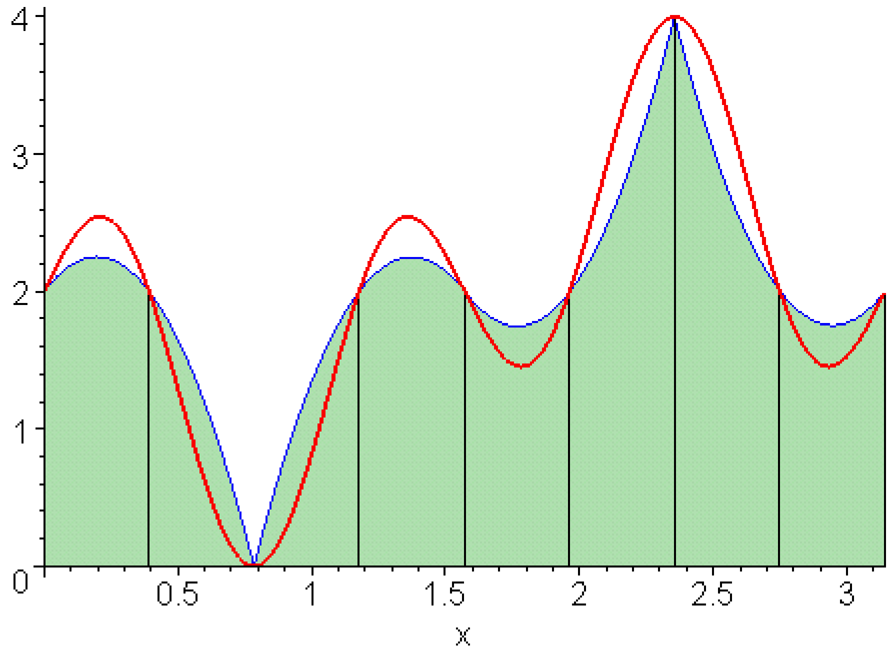
$$\left|E_{T_n}\right| \le \frac{K(b-a)^3}{12n^2} \qquad \text{and} \qquad \left|E_{M_n}\right| \le \frac{K(b-a)^3}{24n^2}$$

5. Approximate $\int_0^2 e^{(-x^2)}\,dx$ by the Trapezoidal and Midpoint Rule for $n = 10$ and estimate the errors of each approximation (i.e. $E_{M_{10}}$ and $E_{T_{10}}$) by the formula given above.

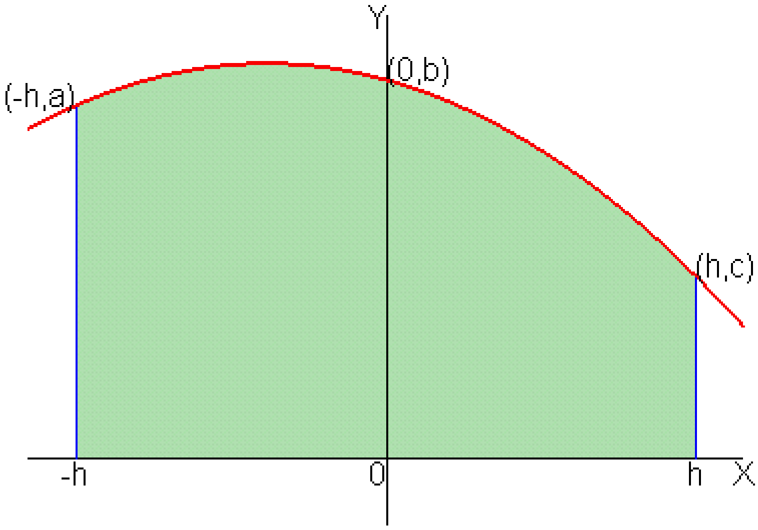
6. By the formula given above, how large should we take n in order to guarantee that the Midpoint Rule approximation and the Trapezoidal Rule approximation for $\int_0^2 e^{(-x^2)}\,dx$ are accurate to 10 decimal places. Which approximation is better ?

Simpson's Rule

Another rule for approximation integration results from using the parabolas. As before ,we divide $[a, b]$ into n subintervals $[x_{i-1}, x_i]$, $i = 1, \ldots, n$ ($x_0 = a, x_n = b$), of equal length $\Delta x = \frac{b-a}{n}$, but this time we assume that n is an **even** number. Then on each consecutive pair of intervals $[x_{i-1,} x_i]$ and $[x_{i,} x_{i+1}]$, we approximate the curve $y = \mathrm{f}(x)$ by a parabola passing through the points $(x_{i-1}, \mathrm{f}(x_{i-1}))$, $(x_i, \mathrm{f}(x_i))$, and $(x_{i+1}, \mathrm{f}(x_{i+1}))$ as shown below. Let S_n denote the sum of the areas of these approximating parabolas.



A typical parabola $y = Ax^2 + Bx + C$ passes through three consecutive points $(-h, a)$, $(0, b)$ and (h, c) as shown below.



7. Find the area of the region shown above.

8. Use the result in (7) to get a formula for S_6 and give a conjecture of the formula for S_n.

9. Use Simpson's Rule with $n = 6$ to approximate $\int_0^2 e^{(-x^2)}\,dx$. Check your answer with Maple command ***simpson*** . What can you say about this approximation?

<u>Error Bound for Simpson's Rule</u>

Suppose that $|f^{(4)}(x)| \le K$ for $a \le x \le b$. If $E_{S_n} = \int_a^b f(x)\,dx - S_n$ is the error involved in using Simpson's Rule to approximate $\int_a^b f(x)\,dx$, then

$$|E_{S_n}| \le \frac{K(b-a)^5}{180\,n^4}$$

10. By the formular given above, how large should we take n in order to guarantee that the approximation for $\int_0^2 e^{(-x^2)}\,dx$ using Simpson's Rule is accurate to 10 decimal places?

11. Compare the results from (6) and (10). What is your conclusion?

12. Suppose $f(x)$ is a cubic polynomial. Is the approximation of $\int_a^b f(x)\,dx$ exact by using Simpson's rule? Justify your answer.

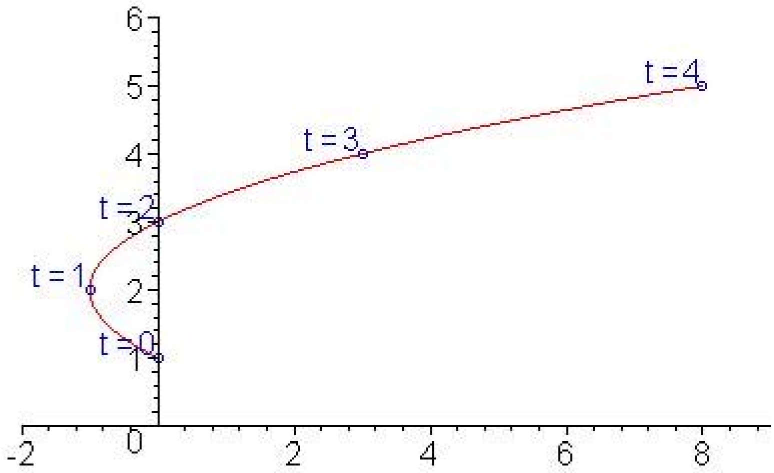
Module 7

Parametric Curves

Purpose:

Understand the advantages of parametric description of curves is that they are convenient for "combined motions." Realize that simple functions can do great graphic designs.

If a particle moves along the curve C shown below, then the x -coordinates and y -coordinates are functions of time. So we can write $x = f(t)$, $y = g(t)$.



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Notice that the consecutive points marked on the curve appear at equal time intervals but not at equal distances. That is because the particle slows down and speeds up as t increases.

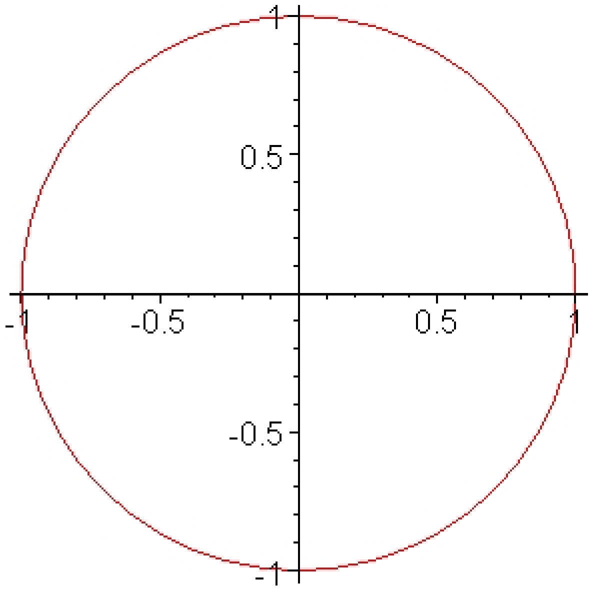
Suppose that x are y are both given as functions of a third variable t (called **parameter**) by the equations

$$x = \mathrm{f}(t), \quad y = \mathrm{g}(t)$$

(called **parametric equations**). Each value of t determines a point (x, y), which we can plot in a coordinate plane. As t varies, the point $(x, y) = (\mathrm{f}(t), \mathrm{g}(t))$ varies and traces out a curve $\boldsymbol{C}$, which is called a **parametric curve** If $\mathrm{f}(t)$ and $\mathrm{g}(t)$ are defined for all t in $[a, b]$, then $(a, \mathrm{f}(a))$ is called the **initial point** of $\boldsymbol{C}$ and $(b, \mathrm{f}(b))$ is called the **final point** of $\boldsymbol{C}$. Imagine that a particle moving along the curve $\boldsymbol{C}$, we can interpret t as time and $(x, y) = (\mathrm{f}(t), \mathrm{g}(t))$ as the position of a particle at time t. We say $\boldsymbol{C}$ is **closed** if initial point and final point of $\boldsymbol{C}$ are the same.

Take a close look at the following animations, you should be able to tell the difference of a curve, which is a set of points, and a parametric curve, in which the points are traced in a particular way.

C_1: $x = \cos(t)$, $y = \sin(t)$, where t in $[0, 2\pi]$.



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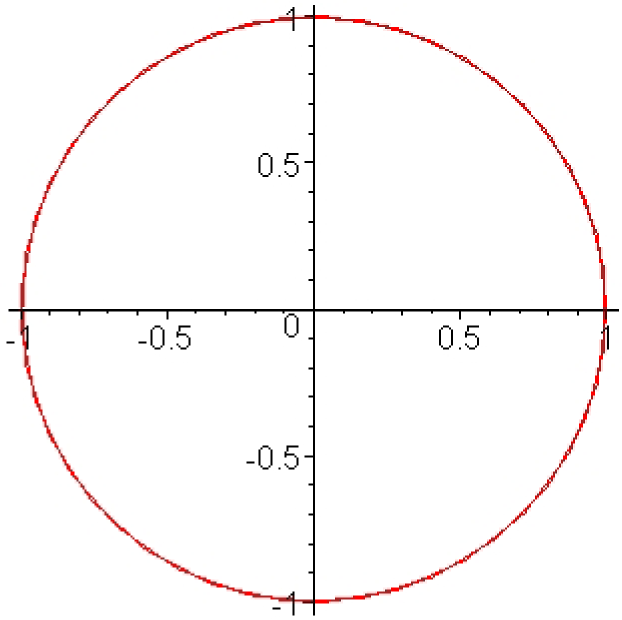
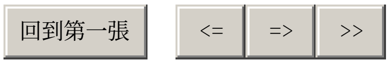
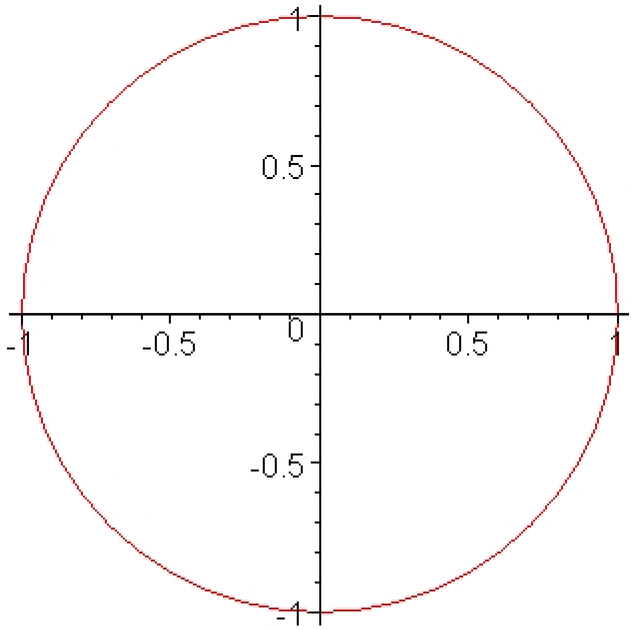
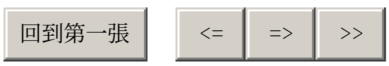
C_2: $x = \cos(2t)$, $y = \sin(2t)$, where t in $[0, 2\pi]$.



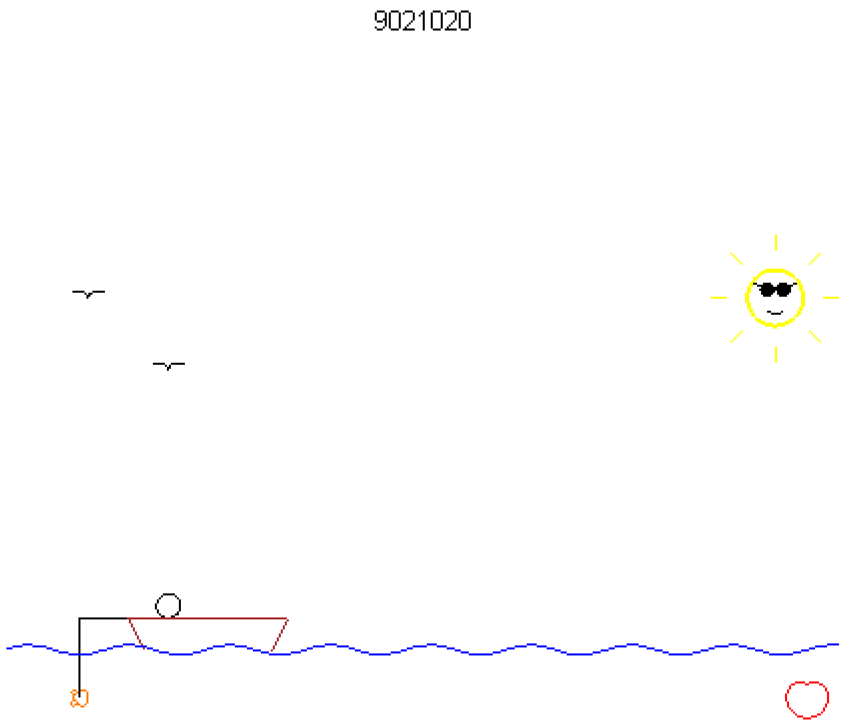


$\boldsymbol{C}_3$: $x = \cos(-t)$, $y = \sin(-t)$,   where t in $[0, 2\pi]$.





Parametric curves are used not only to represent letters and other symbols on the laser printer but also in graphic design. Here is an interesting graphic design using parametric curves by 電物 94 級的歐迪興同學. Try to make one yourself !



Take a close look at the following animations, you should be able to tell the difference of a curve, which is a set of points, and a parametric curve, in which the points are traced in a particular way.

```
> with(plots):
  animatecurve([cos(t), sin(t), t=0..2*Pi], scaling=co
> animatecurve([cos(2*t), sin(2*t), t=0..2*Pi] ,scaling
  numpoints=50);
> animatecurve([cos(-t), sin(-t), t=0..2*Pi], scaling=c
```

1. What are the differences between these three parametric curves

C_1 : $x = \cos(t)$ $y = \sin(t)$, for all t in [$0, 2\pi$] ,
C_2 : $x = \cos(2t)$ $y = \sin(2t)$, for all t in [$0, 2\pi$] ,
C_3 : $x = \cos(-t)$ $y = \sin(-t)$, for all t in [$0, 2\pi$] ?

To plot the parametric equations

$x = f(t)$ $y = g(t)$ where t is in [a, b]

first we define the functions f and g and type the command ***plot([f(t), g(t), t=a..b])*** .

2. Verify that an an ellipse centered at (x_0, y_0) with horizontal axes radius a and vertical axes radius b can be parametrized by $x = x_0 + a\cos(t)$ and $y = y_0 + b\sin(t)$. Plot an arc of an ellipse with the given parametric equations.

```
> a:=2:
  b:=5:
```

The following commands plots a small rectangle with viewing rectangle $[0, 2]$ by $[0, 2]$.

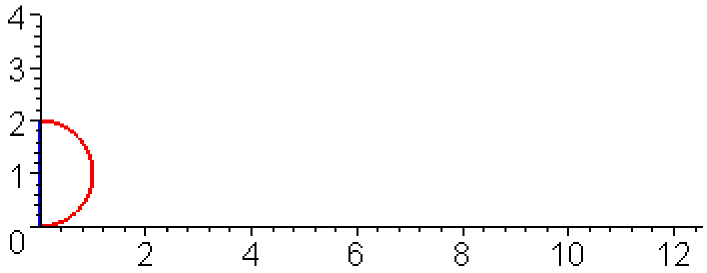
```
> u[1]:=1+.5*t:   v[1]:=1:
  line1:=plot([u[1], v[1], t=0..1], 0..2,0..2):
  u[2]:=1:   v[2]:=1-.5*t:
```

```
line2:=plot([u[2], v[2], t=0..1], 0..2,0..2):
u[3]:=1+.5*t:   v[3]:=.5:
line3:=plot([u[3],v[3],t=0..1], 0..2,0..2):
u[4]:=1.5:   v[4]:=1-.5*t:
line4:=plot([u[4], v[4], t=0..1], 0..2,0..2):
display([line1, line2, line3, line4], scaling=constrai
```

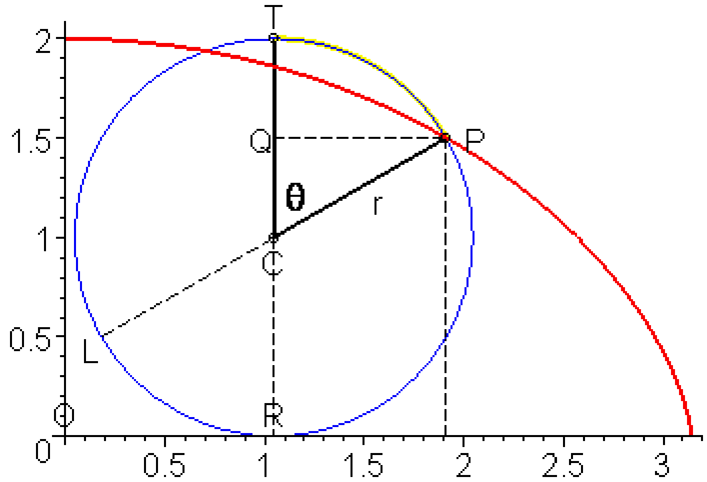
3. Plot a parallelogram with vertices ($1, 1$), ($2, 3$), ($5, 3$) and ($4, 1$) with viewing rectangle [$0, 6$] by [$0, 4$].
4. Plot the three parts of a capital letter B, using a straight line segment and either two semicircles, or two semi-ellipses.
5. Repeat problem (4), but with the letter moved .5 unit above the x -axis and .5 unit to the right of the y -axis, and with its size doubled.

One of the advantages of parametric description of curves is that they are convenient for "combined motions." This lets us plot curves obtained by adding parametric motions. Here is an example :

The curve traced out by a point P on the circumference of a circle of radius r as the circle rolls with a constant angular speed ω along a straight line is called **cycloid** . Play the animation below to get a better picture.



Click here to see how to derive the parametric equations for the cycloid and the commands for the animation above.



6. Using the graph above to show that t he cycloid is given by the parametric equations

$$x(\theta) = r\sin(\theta) + r\,\theta \quad \text{and} \quad y(\theta) = r + r\cos(\theta)$$

where $\theta = \omega\, t$ and t is the rolling time.

Parametric curves of the form $x = a\cos(m\,t)$, $y = b\sin(n\,t)$, with t in $[0, 2\pi]$, are known as *Lissajous curves.* Here, t is the parameter and a, b, m and n are constants which determine the particular curve in the family. Here are two examples:

```
> plot([2*cos(3*t), 7*sin(2*t), t=0..2*Pi]);
> plot([cos(5*t), 2*sin(3*t), t=0..2*Pi]);
```

Trace around these two curves until you understand how they are related to the equations which define them. Then ask *Maple* to plot one or two other Lissajous curves. See if you can guess what each one will look like before you plot it.

7. For fixed m and n, describe how the values of a and b affect the shape of the corresponding Lissajous curve.
8. For fixed a and b, consider the Lissajous curves with $m = 1$, $n = k$ for some integer k and t is in [$0, 2\pi$]. Are the curves closed ? Describe how the shape of the curve changes as k varies.
9. For fixed a and b, consider the Lissajous curves with $m = 1$, $n = r$ for some rational number r, where t is in [$0, 2k\pi$] for

some integer k. Are the curves still closed ? What if r is irrational ? Can you explain why ?

When m is an even number, the curve looks quite different.

```
> plot([cos(4*t),sin(5*t), t=0..2*Pi]);
```

10. Do you see what happened in the last curve? Explain it.

Here are some interesting parametric curves, explore how the shape of the curve varies for different values of m and n.

```
> m:=3:
  n:=2:
  plot([t^m,t^n,t=-2..2]);
> m:=2:
  n:=5:
  plot([t+2*sin(m*t),t+2*cos(n*t),t=0..2*Pi],scaling=c
> m:=2:
  n:=3:
  plot([t+sin(m*t),t+cos(n*t),t=0..2*Pi]);
```

11. Design an interesting picture with plots of parametric curves.

Module 8

Polar Coordinates

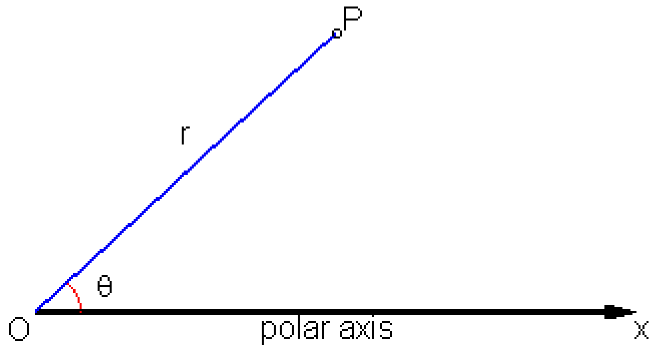
Purpose:

Be familiar with polar coordinates and explore some interesting curves defined by polar equations.

A coordinate system represents a point in the plane by an ordered pair of numbers called coordinates. So far we have being using Cartesian coordinates, which are directed distance from two perpendicular axes. Here we describe a coordinate system introduced by Newton, called **polar coordinate system**.

We choose a point in the plane called the **pole** (or the origin) and is labeled O. Then we draw a half-line starting from O called **polar axis**. This axis usually drawn horizontally to the right and corresponds to the positive x-axis in Cartesian coordinates.

If P is any other point in the plane, let r be the distance from P to O and let θ be the angle between the polar axis and the line OP as shown below. Then the point P is represented by the ordered pair (r, θ) and r, θ are called the **polar coordinates** of P.



We use the convention that an angle is positive if measured in the counterclockwise direction from the polar axis and negative in the clockwise direction. If $P = O$, then $r = 0$ and we agree that $(0, \theta)$ represents the pole for any value of θ.

We extend the meaning of polar coordinates (r, θ) to the case in which r is negative by agreeing that the points $(-r, \theta)$ and (r, θ) lie on the same line through O and the same distance $|r|$ from O, but on the opposite sides of O. Notice that $(-r, \theta)$ and $(r, \theta + \pi)$ represent the same point.

The connection between polar and Cartesian coordinates:

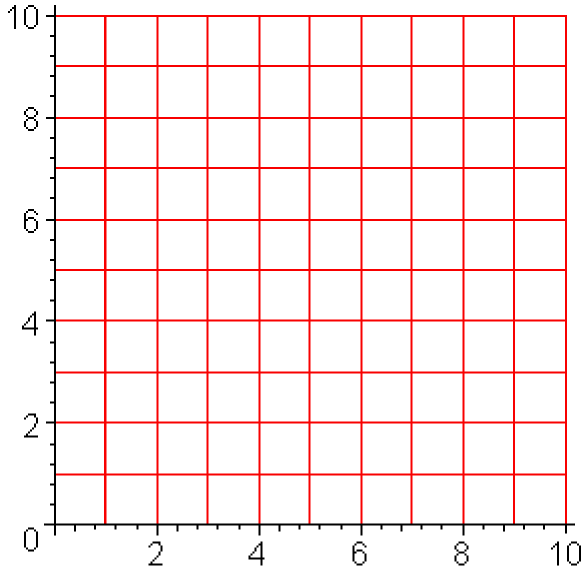
If the point P has Cartesian coordinates (x, y) and polar coordinates (r, θ), then

$$x = r\cos(\theta) \qquad y = r\sin(\theta)$$

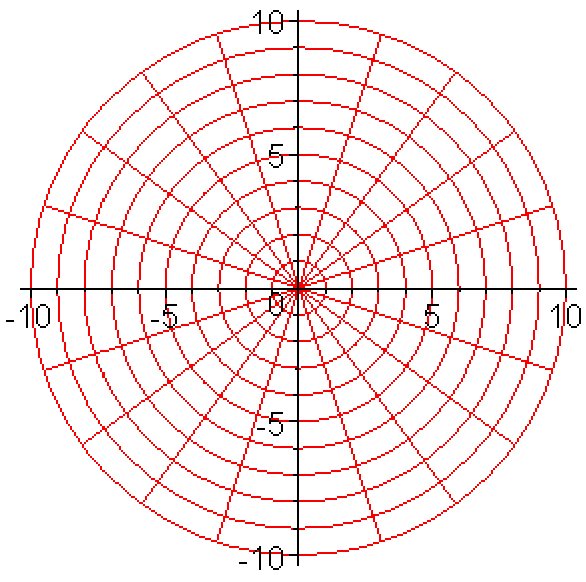
and

$$r^2 = x^2 + y^2 \qquad \tan(\theta) = y / x$$

Grids in Cartesian coordinates :



Grids in polar coordinates :



A. Convert the point ($2, \frac{\pi}{3}$), ($-3, \frac{3\pi}{4}$) from polar to Cartesian coordinates.

B. Find polar coordinates (r, θ), where $0 < r$ and $0 \leq \theta$ < 2π, of the points given by the Cartesian coordinates ($1, 1$) and ($2\sqrt{3}, -2$), and find polar coordinates (r, θ), where $r < 0$ and $0 \leq \theta$ < 2π, of the points given by the Cartesian coordinates ($1, -\sqrt{3}$) and ($-2\sqrt{3}, -2$).

The graph of a **polar equation** $F(r, \theta) = 0$, consists of all points P that have at least one polar representation (r, θ) whose coordinates satisfy the equation.

We can use the *plot* command the same way as for parametric equations by specifying the coordinates to be polar.

```
> plot([2, theta, theta=0..2*Pi], coords=polar,
  scaling=constrained);
> plot([r, Pi/3, r=0..1], coords=polar, scaling=constra
```

C. What curve is represented by the polar equation $r = 2$? What curve is represented by the polar equation $\theta = \frac{\pi}{3}$?

We can also use a command in Maple's *plots* package, called *polarplot* , to plot polar equations of the form $r = f(\theta)$. We start by loading the *plots* package.

D. Plot the curve $r = 2\cos(\theta)$ and find a Cartesian equation for this curve.

```
> with(plots):
f:=theta->2*cos(theta);
polarplot(f(theta),theta=0..2*Pi,scaling=constraine
```

The animation below will give you a better picture of how the curve goes.

```
> animatecurve([f(theta)*cos(theta), f(theta)*sin(thet
theta=0..2*Pi], scaling=constrained, numpoints=2(
```

E. Plot the curve $r = \cos(a\,\theta)$ for different integer a and describe how the curve varies with a .

F. Plot the curve $r = 1 + b\sin(\theta)$ for different value of b and observe how the curve varies with b . Find the transitional values of b where the basic shape of the curve changes.

The animation below probably will help.

```
> animate([1+b*sin(t), t, t=0..2*Pi], b=-2..2, coords=p
  scaling=constrained);
```

G. Graph the curve $(x^2+y^2)^3 = 4x^2y^2$ by finding a polar equation for the curve.

H. Graph the two ellipses $r = \frac{10}{3-2\cos(\theta)}$ and $r = \frac{10}{3-2\cos\left(\theta-\frac{\pi}{4}\right)}$, find the vertices and foci of each of them respectively. What is the relation between these two ellipses.

I. Graph the parabola given in polar form by $r = \frac{1}{1-\sin(\theta)}$ and find the Cartesian coordinate expression for this parabola.

Remark : Here you will find that the command *polarplot* will not give you a good picture. (Why?)

In order to get a good plot, you should get a proper parametric equation of the curve, then plot the parametic curve.

```
> polarplot(1/(1-sin(theta)), theta=-Pi/2..Pi/2, numpoi
> x:=???;
  y:=???;
  plot([x(t), y(t), t=0..2*Pi], -5..5, -5..5);
```

A polar equation of the form

$$r = \frac{a}{1 - e\cos(\theta)}$$ or $$r = \frac{a}{1 + e\cos(\theta)}$$ or $$r = \frac{a}{1 - e\sin(\theta)}$$ or

$$r = \frac{a}{1 + e\sin(\theta)}$$

represents a conic section with eccentricity e. The conic section is an ellipse if $e < 1$, a parabola if $e = 1$, or a hyperbola if $1 < e$.

```
> animate([cos(t)/(1+e*sin(t)), sin(t)/(1+e*sin(t)), t=0.
e=-1.5..1.5, view=[-10..10, -10..10], scaling=constra
numpoints=200, frames=50);
```

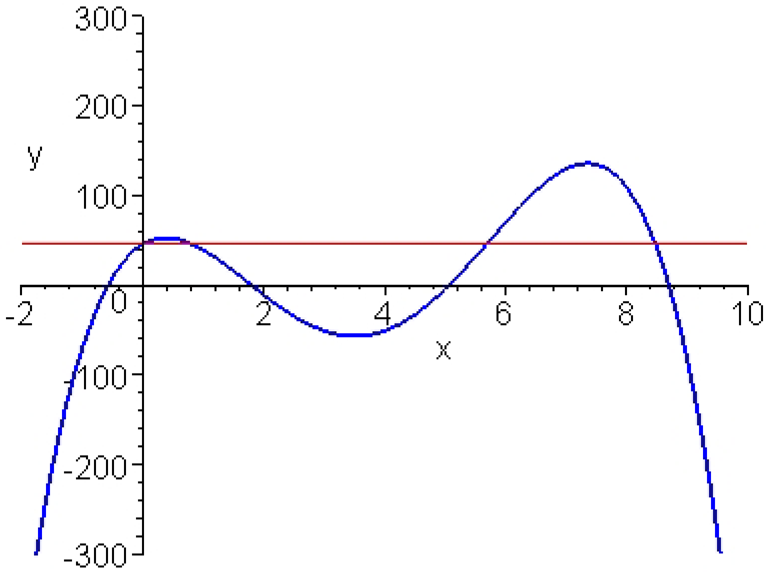
Module 9

Taylor Polynomials

Purpose:

Explore the fact that a polynomial could be completely determined by its value and the values of its derivatives at $x = 0$. Find out that as terms of higher degree are added with the appropriate coefficients, approximation to the "target" polynomial improves in the sense that the two functions appear to match over a wider domain centered at 0. Further, extend this idea to approximations of a non-polynomial function.

A polynomial can be completely determined by its value and the values of its derivatives at $x = 0$.



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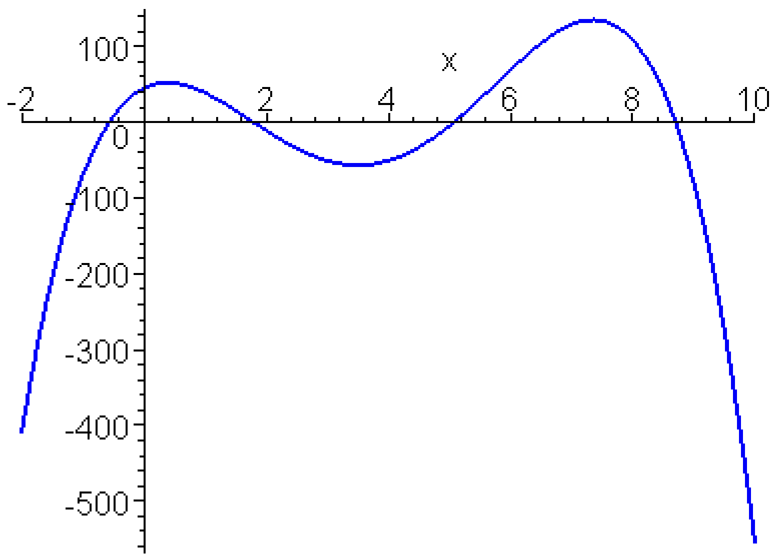
Can we extend this idea to approximations of a nonpolynomial function? Of course, we can't expect to get an exact fit in finite steps.

The idea of polynomial approximation is very powerful in later work, and we shall study it in the context of familiar functions like e^x and $\sin(x)$ in this module.

Part 1. Polynomial Coefficients

The following figure shows the graph of a fourth-degree polynomial, that is, a function $p(x)$ of the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$



We are given the following information about p and its derivatives at $x = 0$:

$$p(0) = 45$$
$$p'(0) = 40$$
$$p''(0) = -120$$
$$p^{(3)}(0) = 90$$
$$p^{(4)}(0) = -24$$

Our objective is to determine the coefficients a_0, a_1, a_2, a_3 , a_4 from this information.

1. How is a_n related to $p^{(n)}(0)$, for $n = 0$, 1 , 2 , 3 , 4 , respectively. Enter your answer here:

2. One at a time, determine appropriate values for the coefficients a_0 , ..., a_4 , and replace the 0 's in the following definitions. With each new definition of $\mathrm{p}(x)$, the plot will be automatic when you enter the following block of commands. Compare with the graph of $\mathrm{p}(x)$ in the web page.

```
> a[0]:=0; a[1]:=0; a[2]:=0; a[3]:=0; a[4]:=0;
  p:=x->a[0]+a[1]*x+a[2]*x^2+a[3]*x^3+a[4]*x^4;
  plot(p(x),x=-2..10,y=-600..200,thickness=2);
```

Part 2. Taylor Polynomials

In Part 1 we saw that a polynomial could be completely determined by its value and the values of its derivatives at $x = 0$. Further, we found that, as we added terms of higher degree (with the appropriate coefficients), our approximation to the "target" polynomial improved in the sense that the two functions appeared to match over a wider domain centered at 0 . In this part we extend this idea to approximations of a nonpolynomial function. Thus, we don't expect to get an exact fit in five steps -- or ever.

The idea of polynomial approximation is very powerful in later work, and it makes sense to study it first in the context of familiar functions.

1. How do we know that the exponential function $\mathbf{e}^{x}$ is not a polynomial ?

   State at least one property of this function that could not be a property of any polynomial.

2. Let $f(x) = \mathbf{e}^{x}$, find a polynomial

   $$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

   of degree 4 with the property that $f^{(n)}(0) = p^{(n)}(0)$, for $n = 0$, 1 , 2, 3, 4 .

   Enter functions and coefficients here, and plot f and p together.

```
> restart;
with(plots):
x:='x':f:=x->exp(x);
a[0]:=?; a[1]:=?; a[2]:=?; a[3]:=?; a[4]:=?;
p:=x->a[0]+a[1]*x+a[2]*x^2+a[3]*x^3+a[4]*x^4;
plot1:=plot(f(x),x=-3..3,y=-2..16,thickness=2, color
```

```
plot2:=plot(p(x),x=-3..3,y=-2..16,thickness=2, colo
display(plot1,plot2);
```

3. Plot the error function $f(x) - p(x)$ and describe the extent to which $p(x)$ does and does not approximate e^x.

Let's try to find better approximations of $f(x) = e^x$ with higher-degree polynomials. We look for an nth-degree polynomial $p_n(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$ + ... + $c_n x^n$ such that $f^{(k)}(0) = p^{(k)}(0)$, for all $k = 0$, $1, 2, .., n$. The resulting polynomial is called the **nth-degree Taylor polynomial of** e^x **centered at** 0.

4. How is c_k related to the *k* th-derivatives of e^x, for $k = 0$, 1, 2,.., *n*
Enter your answer here:

5. Enter the general formula for $p_n(x)$ and plot e^x and $p_{10}(x)$ together. Compare the approximation here and that in (2), which one looks better ? Try with larger n's, what do you find out ?
6. Find the general formula for the nth-degree Taylor polynomial centered at 0 for the function $g(x) = \sin(x)$. Graph g together with the Taylor polynomials of degree 2, 4, 6, 8 and comment on how well they approximate g.

In general, given a *n* -defferentiable function $f(x)$,the polynomial

$$T_n(x) = c_0 + c_1 (x-a) + c_2 (x-a)^2 + c_3 (x-a)^3 + \dots + c_n (x-a)^n$$

where $f^{(k)}(a) = T^{(k)}(a)$ for all $k = 0$, 1, 2,.., *n* , is called the **nth-degree Taylor polynomial of $f(x)$ centered at a**.

7. Find the general formula for $T_n(x)$.

8. Find the nth-degree Taylor polynomial centered at 1 for the function $\frac{1}{x}$.

9. Suppose that $f(x) = \sum_{n=1}^{\infty} c_n (x-a)^n$ for all $|x-a| < R$.

What is the nth-degree Taylor polynomial of f at a?

Module 10

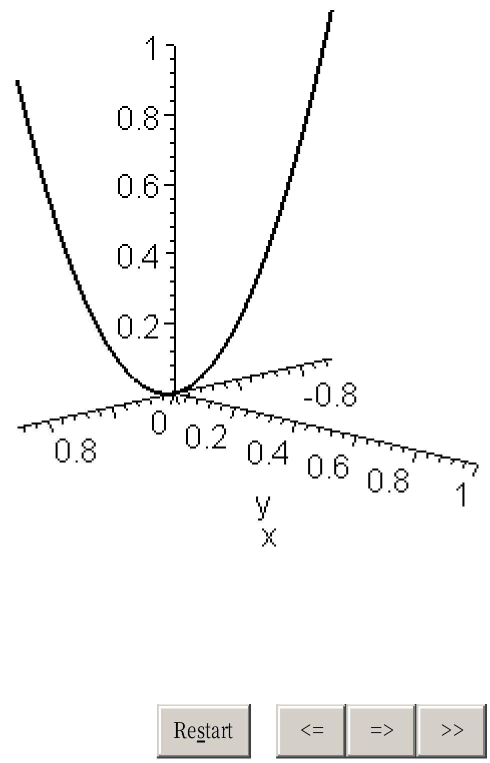
Cylinders and Quadratic Surfaces

Purpose:

Explore the graphs of cylinders and quadratic surfaces by their traces. Discover the interesting shapes that members of family of surfaces $z = a\,x^2 + b\,x\,y + c\,y^2$ can take, by observing how the shape of the surface evolves as we vary the constants.

In this project we investigate two types of surfaces --- cylinders and quadratic surfaces.

A **cylinder** is a surface that consists of all lines (called **rulings**) that are parallel to a given line and pass through a given plane curve. The animation below shows how the surface is formed by taking the parabola $z = x^2$ in the xz -plane and moving in the direction of the y -axis.

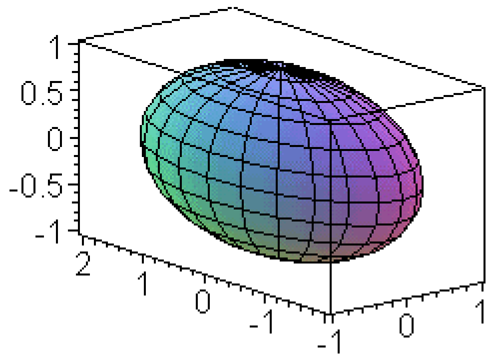


A **quadratic surface** is the graph of a second-degree equation in three variables x, y and z. The most general such equation is

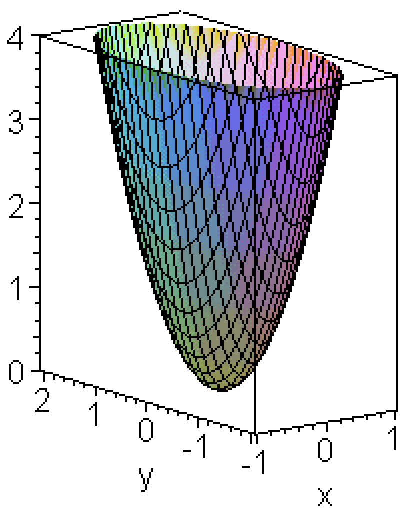
$$A\,x^2 + b\,y^2 + C\,z^2 + D\,x\,y + E\,y\,z + F\,x\,z + G\,x + H\,y + I\,z + J = 0$$

where $A, B, C, \ldots, J$ are constants. There are six basic shapes :

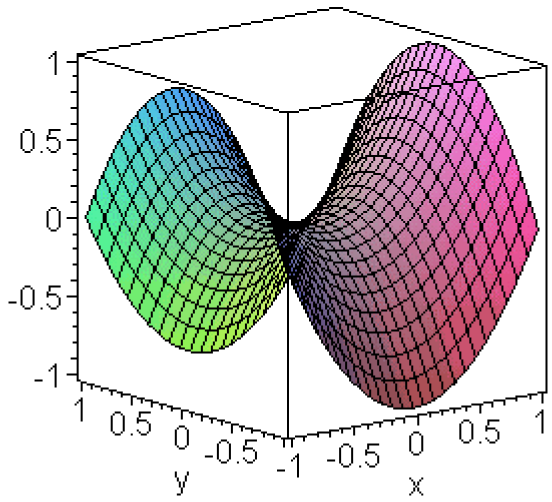
(1) ellipsoid



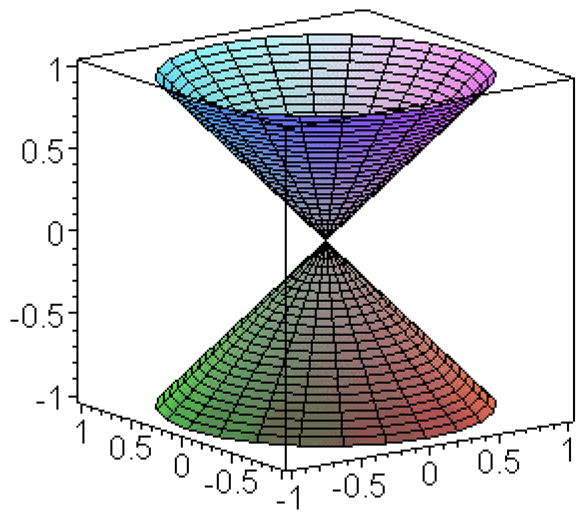
(2) Elliptic Paraboloid



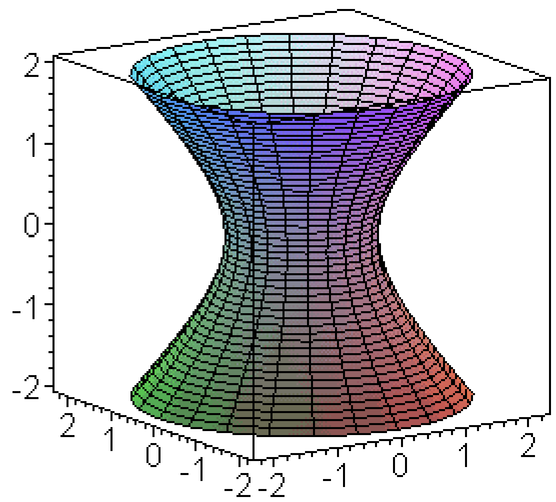
(3) Hyperbolic Paraboloid



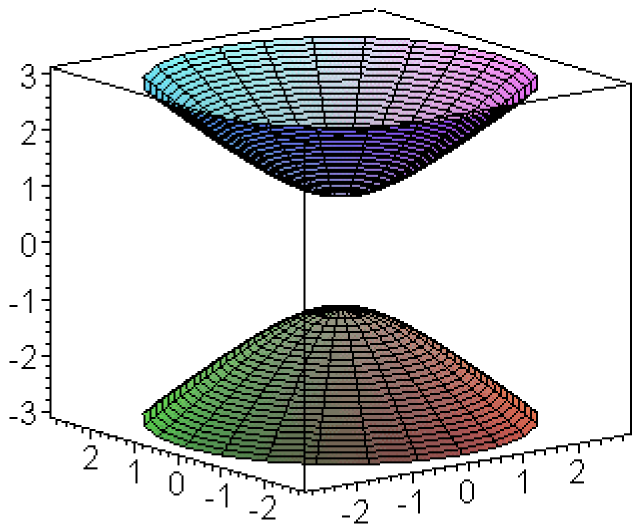
(4) Cone



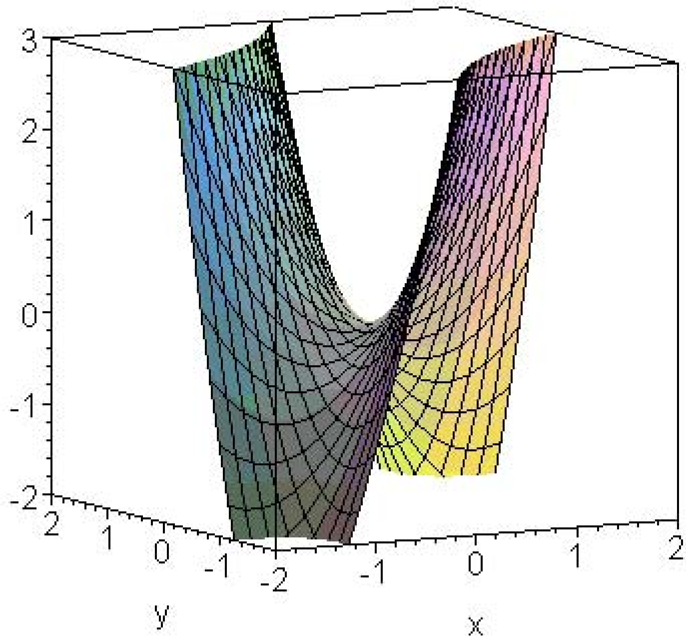
(5) Hyperboloid of one sheet

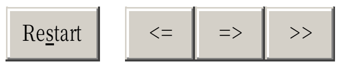


(6) Hyerboloid of Two sheets



We will also discover the interesting shapes that members of family of surfaces $z = a\,x^2 + b\,y^2 + c\,x\,y$ can take, by observing how the shape of the surface evolves as we vary the constants.





PART I Cylinders

A **cylinder** is a surface that consists of all lines (called **rulings**) that are parallel to a given line and pass through a given plane curve.

You may use the Maple command *plot3d* to plot an explicit function $z = f(x, y)$ or use the command *implicitplot3d* in the *plots* package to plot a surface defined by the equation $F(x, y, z) = 0$.

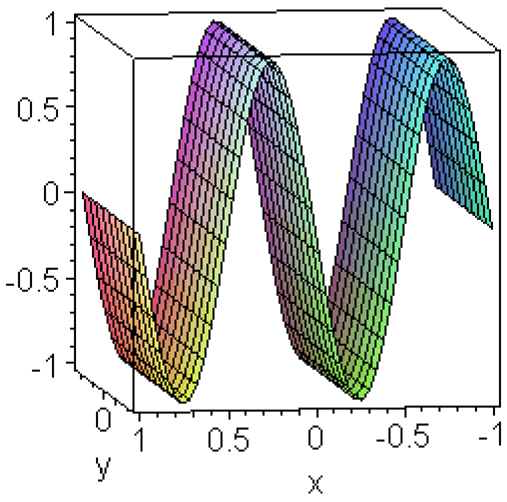
1. Graph the **parabolic cylinder** $z = y^2$.

> plot3d(y^2, x=-4..4, y=-4..4, view=0..16, axes=norm
> implicitplot3d(z=y^2, x=-4..4, y=-4..4, z=0..16, grid=
> axes=normal);

2. Graph the **parabolic cylinder** $y = x^2$ and compare with the one in 1.

3. Graph and compare the circular cylinders $x^2 + y^2 = 1$ and $y^2 + z^2 = 1$.

4. Here is a graph of a cylinder. Observe the graph, make a good guess of its equation and justify your answer.



PART II Quadratic Surfaces

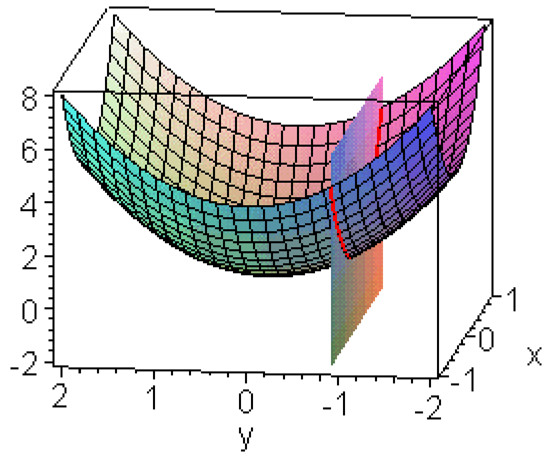
A **quadratic surface** is the graph of a second-degree equation in three variables x, y and z. The most general such equation is

$$A\,x^2 + b\,y^2 + C\,z^2 + D\,x\,y + E\,y\,z + F\,x\,z + G\,x + H\,y + I\,z + J = 0$$

where A, B, C, ... , J are constants. There are six basic shapes :

In order to sketch the graph of a surface, it is useful to determine the curves of intersection of the surface with planes parallel to the coordinate planes. These curves are called **traces** (or cross-sections) of the surface. The following animation shows the vertical traces in $y = k$ of the surface $z = 4\,x^2 + y^2$

.



$$x^2+\frac{y^2}{9}+\frac{z^2}{4}=1$$

1. Graph the **ellipsoid** and identify the horizontal traces and vertical traces.

$$z=4\,x^2+y^2$$

2. Graph the **elliptic paraboloid** and identify the horizontal traces and vertical traces.

$$z=x^2-y^2$$

3. Graph the **hyperbolic paraboloid** and identify the horizontal traces and vertical traces.

$$\frac{x^2}{4}+y^2-\frac{z^2}{4}=1$$

4. Graph the **hyperboloid of one sheet** and identify the horizontal traces and vertical traces.

$$4\,x^2-y^2+2\,z^2+4=0$$

5. Graph the **hyperboloid of two sheet** and identify the horizontal traces and vertical traces.

PART III Families of Surfaces

1. Investigate the family of surfaces $z = x^2 + y^2 + c\,x\,y$. In particular, you should determine the transitional values of c for which the surface changes from one type of quadratic surface to another. Justify your answer.

2. Investigate the family of surfaces $z = a\,x^2 + b\,y^2 + c\,x\,y$. In particular, you should determine the transitional values of a, b and c for which the surface changes from one type of quadric surface to another. Justify your answer.

Module 11

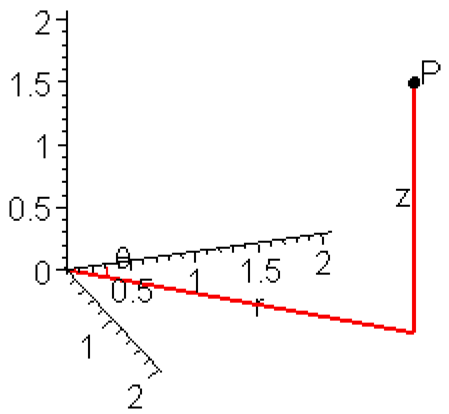
Cylindrical and Spherical Coordinates

Purpose:

Be familiar with cylindrical and spherical coordinates and explore some interesting surfaces parametrized by cylindrical or spherical coordinates.

<u>PART I</u>   Cylindrical coordinates

In cylindrical coordinate system, a point P in three-dimentional space is represented by the ordered triple (r, θ, z), where r and θ are polar coordinates of the projection of P (as shown below) onto the *xy* -plane and z is the directed distance from the *xy* -plane to P .



To convert from cylindrical to rectanglular coordinates we use the equations

$$x = r\cos(\theta) \quad y = r\sin(\theta) \quad z = z$$

whereas to convert from rectanglular to cylindrical coordinates we use the equations

$$r^2 = x^2 + y^2 \quad \tan(\theta) = \frac{y}{x} \quad z = z$$

1. Find the rectangular coordinates of the point with cylindrical coordinates
   ($4, -\frac{\pi}{3}, 4$).

2. Find the cylindrical coordinates of the point with rectangular coordinates
   ($1, -\sqrt{3}, 2$).

We can plot a surface with equation $r = f(\theta)$ in cylindrical coordinates using the Maple command ***plot3d*** with the option specifying cylindrical coordinates :

3. What is the surface with equation $r = 1$ in cylindrical coordinates ?

```
> plot3d(1,theta=0..2*Pi,z=0..1,coords=cylindrical);
```

We can also plot a surface given by parametric equations in cylindrical coordinates using the Maple command ***plot3d*** with the option specifying cylindrical coordinates :

```
> plot3d([r,Pi/4,z],r=0..6,z=0..4,axes=normal,scaling=
coords=cylindrical);
```

4. What is the surface with equation $\theta = \frac{\pi}{4}$ in cylindrical coordinates ?

5. What is the surface with equation $z = 1$ in cylindrical coordinates ?

```
> plot3d([r, theta, 1], r=0..6, theta=0..2*Pi, view=0..6,
  scaling=constrained, coords=cylindrical);
```

6. Plot the surface with equation $r = \theta$ in cylindrical coordinates.

7. Plot the surface with equation $z = r$ in cylindrical coordinates and find the equation of the surface in rectangular coordinates .

8. Plot the surface with equation $z = \theta$ in cylindrical coordinates.

<u>PART II</u> Spherical coordinates

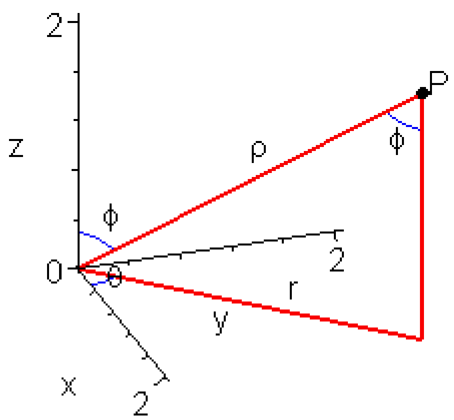
The spherical coordinates (ρ, θ, ϕ) of a point P in space are shown below, where ρ is the distance from the origin O to P, θ is the same angle as in cylindrical coordinates, and ϕ is the angle between the positive *z*-axis and the line segment joining the origin and P. Note that $0 \le \rho$ and ϕ is in [$0, \pi$].

The relationship between rectangular and spherical coordinates is given by the equations:

$$x = \rho \sin(\phi) \cos(\theta) \quad y = \rho \sin(\phi) \sin(\theta) \quad z = \rho \cos(\phi)$$

and

$$\rho^2 = x^2 + y^2 + z^2$$



1. Find the rectangular coordinates of the point ($2, \frac{\pi}{4}, \frac{\pi}{3}$) given in spherical coordinates.

2. Find the spherical coordinates of the point ($0, 2\sqrt{3}, -2$) given in rectangular coordinates.

We can plot a surface with equation $\rho = f(\theta, \phi)$ in cylindrical coordinates using the Maple command ***plot3d*** with the option specifying cylindrical coordinates; we can also plot a surface given by parametic equations in cylindrical coordinates using the Maple command ***plot3d*** with the option specifying cylindrical coordinates.

```
> plot3d(1, theta=0..2*Pi, phi=0..Pi, coords=spherica
  scaling=constrained);
> plot3d([1, theta, phi], theta=0..2*Pi, phi=0..Pi, coor
  scaling=constrained);
```

3. Find the equation in rectangular coordinates of the surface given by the equation $\rho = 1$ in spherical coordinates.

4. Plot the surface with equation $\theta = \frac{\pi}{4}$ in spherical coordinates. What is the surface?

5. Plot the surface with equation $\phi = \frac{\pi}{4}$ in spherical coordinates. What is the surface?

6. Plot the surface with equation $\rho = \phi$ in spherical coordinates.

7. Plot the surface with equation $\rho = \theta$ in spherical coordinates.

8. Plot the surface with equation $\rho = \sin(\theta)\sin(\phi)$ in spherical coordinates. Find the equation of the surface in rectangular coordinates and identify the surface.

PART III

1. Draw a picture of the solid that remains when a whole of radius 2 is drilled through the center of a sphere of radius 3.

2. Members of the family of surfaces given in spherical coordinates by the equation

$$\rho = 1 + .2 \sin(m\,\theta)\sin(n\,\phi)$$

have been suggested as models for tumors and have been called *bumpy spheres* and *wrinkled spheres* .

Investigate this family of surfaces, assuming that m and n are positive integers. What roles do the values of m and n play in the shape of the surfaces?

Module 12

Limits of multivariable Functions

Purpose:

Understand the concept of " the limit of a two variable function " by level curves and graphs of the function.

Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then we say

$$\lim_{x, y \to (a, b)} f(x, y) = L$$

if for every $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$|f(x, y) - L| < \varepsilon$$

whenever (x, y) in D and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$.

If $\lim_{x, y \to (a, b)} f(x, y) = L$, then $f(x, y)$ approaches L as (x, y) approaches (a, b) along any path C in D. In other words, if $f(x, y)$ approaches L_1 as (x, y) approaches (a, b) along a path C_1 in D and $f(x, y)$ approaches L_2 as (x, y) approaches (a, b) along a path C_2 in D, where ,

where $L_1 \neq L_2$, then $\lim_{x, y \to (a, b)} f(x, y)$ does not exist.

Let *f* be a function of two variables whose domain *D* includes points arbitrarily close to (*a* , *b*). Then we say

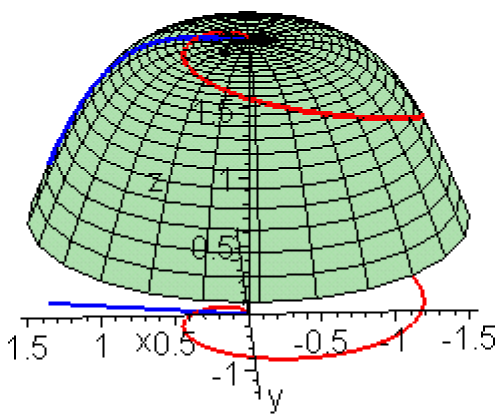
$$\lim_{x, y \to (a, b)} f(x, y) = L$$

if for every ε > 0 there is a corresponding number δ > 0 such that

$$|f(x, y) - L| < \varepsilon$$

whenever (*x* , *y*) in *D* and 0 < $\sqrt{(x-a)^2 + (y-b)^2}$ < δ .

If $\lim_{x, y \to (a, b)} f(x, y) = L$, then *f* (*x* , *y*) approaches *L* as (*x* , *y*) approaches (*a* , *b*) along any path *C* in *D* . In other words, if *f* (*x* , *y*) approaches L_1 as (*x* , *y*) approaches (a, b) along a path C_1 in *D* and *f* (*x* , *y*) approaches L_2 as (x, y) approaches (*a* , *b*) along a path C_2 in *D,* where , where $L_1 \neq L_2$, then $\lim_{x, y \to (a, b)} f(x, y)$ does not exist.



1. Let $f(x, y) = \frac{x \sin(y)}{x^2 + y^2}$, plot $f(x, y)$ together with the paths you picked as the graph above and determine whether

$$\lim_{x, y \to (0, 0)} \frac{x \sin(y)}{x^2 + y^2}$$ exists. Explain your answer.

2. Let $g(x, y) = \frac{x y^3}{x^2 + y^6}$, plot $g(x, y)$ together with the paths you picked as the graph above and determine whether

$$\lim_{x, y \to (0, 0)} \frac{x y^3}{x^2 + y^6}$$ exists. Explain your answer.

The fact that $\lim_{x, y \to (0, 0)} \frac{x^2 y}{x^2 + y^2} = 0$ and $\lim_{x, y \to (0, 0)} \frac{x^2}{x^2 + y^2}$ does not exist can be detected using contour plots.

3. Execute the following command and execute this commands again with 0.01 replaced by 0.001; does the pattern seem to change ?

```
with(plots):
contourplot(x^2*y/(x^2+y^2),x=-0.01..0.01,y=-0.01..0.01
```

grid=[40,40]);

4. How does those graphs support the conclusion that

$$\lim_{x,y\to(0,0)} \frac{x^2 y}{x^2+y^2}$$

exists ?

5. How do the contour plots support the conclusion that

$$\lim_{x,y\to(0,0)} \frac{x^2}{x^2+y^2}$$

does not exist ?

6. Based on the contour plots, do you think that $\lim_{x,y\to(0,0)} \frac{x\sin(y)}{x^2+y^2}$ exists ? Explain your answer.

If (r, θ) are polar coordinates of the point (x, y) with $0 \le r$, note that $r \to 0^+$ as $(x,y) \to (0,0)$. Hence

$\lim_{x,y\to(a,b)} f(x,y) = L$ exists if and only if $\lim_{r\to 0+} f(r\cos(\theta), r\sin(\theta))$ exists.

7. Let $f(x,y) = \dfrac{x^3 + y^3}{x^2 + y^2}$, graph $f(x,y)$ and use polar coordinates to determine whether $\displaystyle\lim_{x,y \to (0,0)} \frac{x^3 + y^3}{x^2 + y^2}$ exists .

8. Let m and n be positive integers. Find all the values of m and n such that $\displaystyle\lim_{x,y \to (0,0)} \frac{x^m y^n}{x^2 + y^2}$ exists.

Module 13

Parametric representations of Surfaces

Purpose:

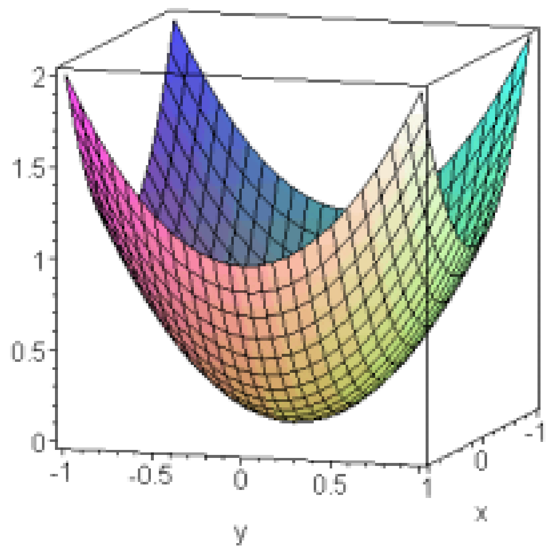
Represent a given surface with suitable parametric equations and identify the grid curves.

So far we have learned to describe surfaces in three dimensional space as :

graphs of functions of two variables,

level sets for functions of three variables,
graphs of equations in three variables.

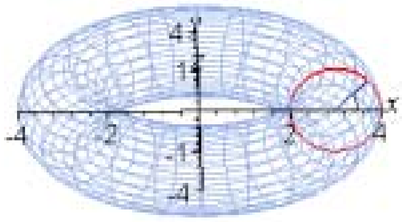
For example, the surface below



can be described as

the graph of the function $f(x, y) = x^2 + y^2$,
the graph of the equation $z = x^2 + y^2$, or
a level set of the function $f(x, y, z) = x^2 + y^2 - z$

Unfortunately, some surfaces are hard to be represented in any of those ways, for example, the torus shown below.



Recall that we described a space curve by a vector function of a single parameter t

$$r(t) = (x(t), y(t), z(t))$$

In much of the same way, we can describe a surface by a vector function of two parameters u and v

$$r(u, v) = (x(u, v), y(u, v), z(u, v)).$$

So far we have learned to describe surfaces in three dimensional space as :

- graphs of functions of two variables,
- level sets for functions of three variables,
- graphs of equations in three variables.

For example, the surface below

> **plot3d(x^2+y^2, x=-1..1, y=-1..1, axes=boxed);**

can be described as

- the graph of the function $f(x, y) = x^2 + y^2$,
- the graph of the equation $z = x^2 + y^2$, or
- a level set of the function $f(x, y, z) = x^2 + y^2 - z$

> **with(plots):**
> **implicitplot3d(z=x^2+y^2,x=-1..1,y=-1..1,z=0..2,axe**

Some surfaces are hard to be represented in any of those ways.

Recall that we described a space curve by a vector function of a single parameter t

$$\mathbf{r}(t) = (x(t), y(t), z(t))$$

In much of the same way, we can describe a surface by a vector function of two parameters u and v

$$\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))$$

Take the surface above, for instance, we can parameterize the surface in a natural way :

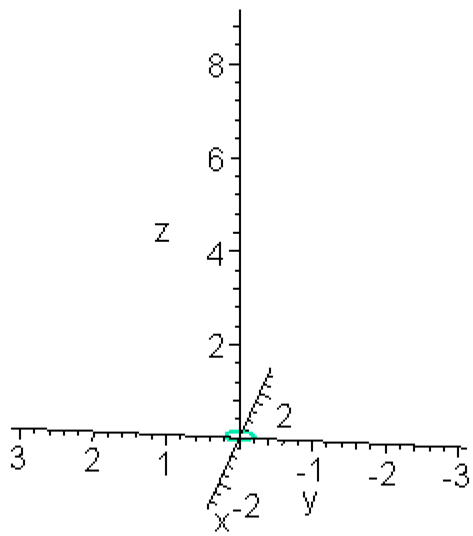
$$\mathbf{r}(u, v) = (u, v, u^2 + v^2)$$

However, there is another parameterization that is better for plotting the surface. Notice that the level curves of $f(x, y) = x^2 + y^2$ are circles centered at $(0, 0)$, so we can parameterize them with

$$x = r\cos(\theta), \quad y = r\sin(\theta)$$

We can get the surface by putting together these circles with various sizes of radius. Hence we get the following parameterization

$$\mathbf{r}(u, v) = (r\cos(\theta), r\sin(\theta), r^2) \text{ ----- (1)}$$



1. Use the same idea to find a parametric representation of the elliptic paraboloid given by the equation $z = \frac{(x-2)^2}{4} + \frac{(y-1)^2}{9}$ and check your answer by plotting the parameterized surfaces.

If a parametric surface *S* is given by a vector function **r** (u, v) , then there are two useful families of curves that lies on *S* , one family with u constant and the other with v constant.

If we keep *u* constant by putting $u = u_0$, then **r** (u_0, v) defines a curve C_1 lying on *S* . Similarly, if we keep v constant by putting $v = v_0$, then **r** (u, v_0) defines a curve C_2 lying on *S* . We call these curves **grid curves** . Notice that when Maple graphs a parametric surface, it usually depicts the surface by plotting these grid curves, as we see in the following example.

```
> plot3d([x, y, x^2+y^2], x=-3..3, y=-3..3, axes=boxec
  plot3d([r*cos(theta), r*sin(theta),r^2], r=0..4, theta=
  axes=boxed);
```

2. Find the grid curves of the surface

$$\mathbf{r}(u, v) = (2\cos(u), v, 2\sin(u))$$

Which grid curves have u constant ? Which grid curves have v constant ?

Another way to look at the parametric equation in (1) is by converting from Cartesian coordinates to Cylindrical coordinates.

We routinely use parameterized surfaces when we are converting a surface from a coordinate system to another coordinate system. We may, for example, want to consider a sphere of radius 1 centered about the origin, which is easy in spherical coordinates, in Cartesian coordinates for some reason. Then we can parameterize the sphere by

$$x = \cos(\theta)\sin(\phi), \quad y = \sin(\theta)\sin(\phi), \quad z = \cos(\phi)$$

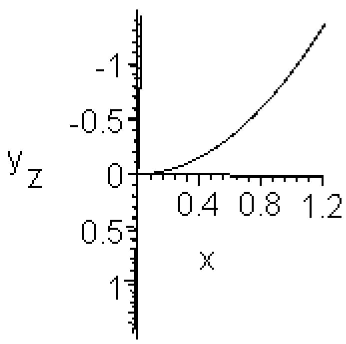
where θ is from 0 to 2π and ϕ is from 0 to π.

```
> plot3d([cos(theta)*sin(phi), sin(theta)*sin(phi), cos
  theta=0..2*Pi, phi=0..Pi, axes=boxed, scaling=cons
```

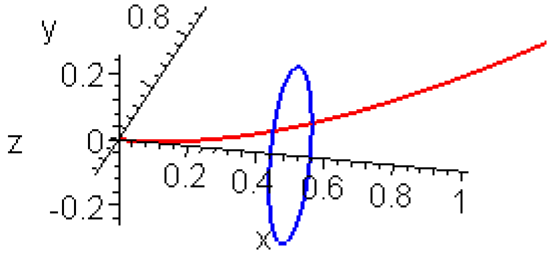
3. Find a parametric representation of the ellipsoid $\dfrac{x^2}{4} + \dfrac{(y-1)^2}{9} + \dfrac{z^2}{16} = 1$ and check your answer by plotting the parameterized surfaces.
4. Find a parametric representation of the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the cone $z = \sqrt{x^2 + y^2}$.
5. Graph the surface with parametric equations $x = u\,v$, $y = u + v$, $z = u - v$, where $u^2 + v^2 \le 1$

   Do you recognize this surface ?

We are of course, interested in using parameterization to describe surfaces that can easily be parameterized, but are hard to describe as graphs of functions. A class of examples is **surfaces of revolution** . For example, the curve $y = f(x)$, with *x* in $[a, b]$, revolved around the x axis.



Notice that the vertical trace corresponding to $x = c$, where c is a constant, is a circles of radius $f(c)$ centered at ($c, 0, 0$).



6. Find a parametric representation of the surface obtained by revolving the curve

$$y = 2x^2 - x^3$$

, with x in [$0, 2$]

around the x -axis and check your answer by plotting the parameterized surfaces.

7. Find a parametric representation of the surface obtained by revolving the curve

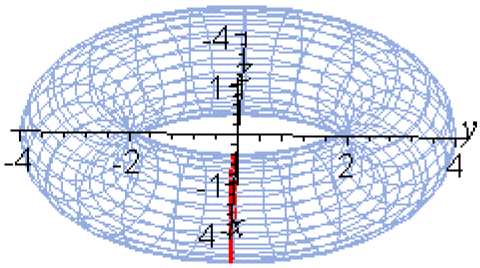
$$y = 2x^2 - x^3$$

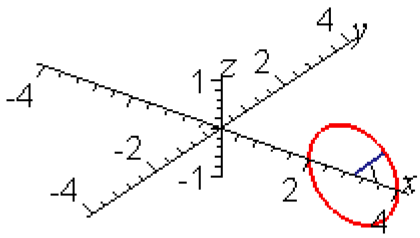
, with x in [$0, 2$]

around the *y* -axis and check your answer by plotting the parameterized surfaces.

The same construction works when the original curve is a parameterized curve rather than the graph of a function.

8. Find a parametric representation for the **torus** obtained by rotating about the z-axis the circle in xz-plane with center ($3, 0, 0$) and radius 1 .





9. Graph the surface

r $(u, v) = ((3+\cos(v))\cos(u), (3+\cos(v))\sin(u), u+\sin(v))$,

where u, v are in [$0, 2\pi$] .

Find the grid curves of the surface. Which grid curves have u constant ? Which grid curves have v constant ?

Compare this surface with the torus above and state your comment.

10. (a) What happens to the spiral tube in problem 7 if we replace $\cos(u)$ by $\sin(u)$ and $\sin(u)$ by $\cos(u)$?

(b) What happens if we replace $\cos(u)$ by $\cos(2u)$ and $\sin(u)$ by $\sin(2u)$?

11. The surface with parametric equations

$$x = 2\cos(\theta) + r\cos\left(\frac{\theta}{2}\right)$$

$$y = 2\sin(\theta) + r\cos\left(\frac{\theta}{2}\right)$$

$$z = r\sin\left(\frac{\theta}{2}\right)$$

with r in [$-\frac{1}{2}, \frac{1}{2}$] and θ in [$0, 2\pi$] , is called a **Mobius strip** . Graph this surface. What is unusual about it ?

Module 14

Critical Points and Contour Plots

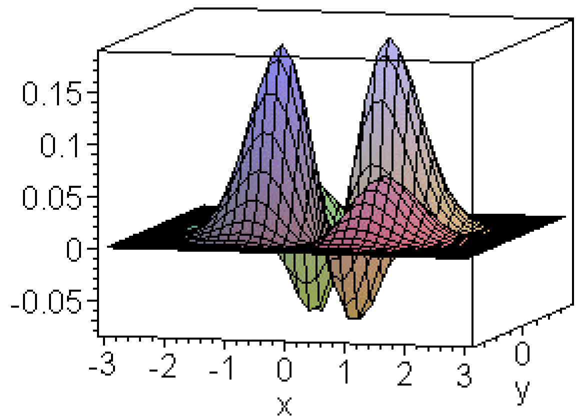
Purpose:

Predict the location of the critical points of a two variable function f by its level curves and determine whether f has a saddle point or a local maximum or a local minimum at each of those points. Find the critical points of f by two-dimensional Newton's method.

Here we will investigate the critical points of the function

$$f(x, y) = \cos(x - y)\, x\, y\, e^{(-x^2 - y^2)}$$

by its level curves.



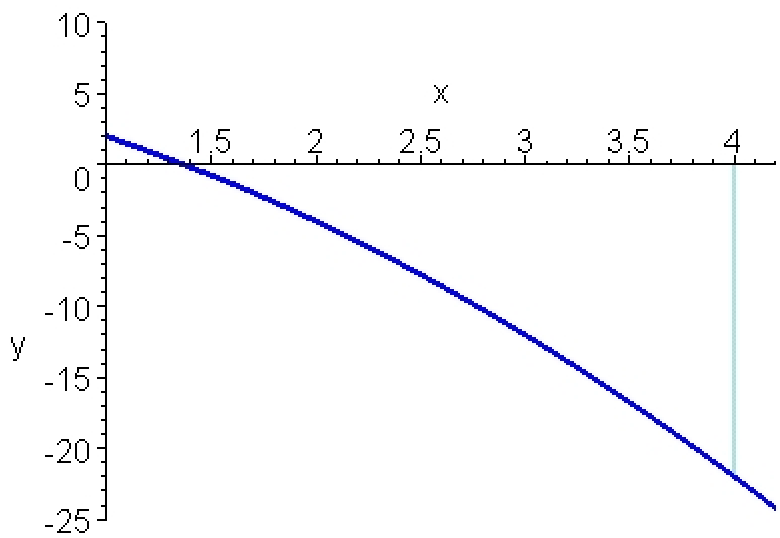
Theoretically, given a two variable function *f*, we can find all the critical points of *f* by solving

$$\frac{\partial f}{\partial x}(x, y) = 0$$
$$\frac{\partial f}{\partial y}(x, y) = 0$$

However, we often encounter the cases that it is alomost impossible to solve the equations above.

The one dimensional Newton's Method uses the linear approximation to find an approximate solution to an equation of the form $f(x) = 0$. If x_0 is an initial approximation to the solution, then the tangent line to $y = f(x)$ at $x = x_0$ intersects the x-axis at a point $(x_1, 0)$ and x_1 is usually a better approximation to the solution than x_0 . So the process can be iterated using x_1 as the new initial approximation. A short derivation shows that at each stage

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}.$$



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The same as the one dimensional Newton's Method, the 2 dimensional Newton's Method uses the linear approximation to find an approximate solution to a pair of equations of the form $f(x, y) = 0$ and $g(x, y) = 0$. Basically, if (x_0, y_0) is an initial approximation to the solution, then the tangent plane to $z = f(x, y)$ at (x_0, y_0) and the tangent plane to $z = g(x, y)$ at (x_0, y_0) intersect the xy-plane at a common point ($x_1, y_1, 0$) and (x_1, y_1) is usually a better approximation to the solution than (x_0, y_0). So the process can be iterated using (x_1, y_1) as the new initial approximation.

PART I Exploration of a Surface

Here we will investigate the critical points of the function

$$f(x, y) = \cos(x - y)\, x\, y\, e^{(-x^2 - y^2)}$$

The graph of this function over the domain [-3,3] x [-3,3] is shown in the following figure.

```
> f :=(x,y)->cos(x-y)*x*y*exp(-x^2-y^2);
  plot3d(f(x,y),x=-3..3,y=-3..3,grid=[35,35],axes=box
```

1. Make a contour plot of f over the domain [-3,3] x [-3,3], and identify a part of the domain that you think contains a local maximum or minimum. Explain what features of the contour plot indicate a local maximum or minimum.

```
> with(plots):
  contourplot(f(x,y), x=-3..3, y=-3..3, contours=15, gı
  coloring=[yellow,red], filled=true);
```

2. Zoom in on your selected part of the contour plot until you can find a two-significant-digit (2SD) approximation to the coordinates of the critical point.

3. Return to the original domain, and identify another region that you think contains a saddle point. Explain what features of the contour plot indicate a saddle point.

4. Zoom in on this new region until you can find a 2SD approximation to the coordinates of this critical point.

5. Calculate and display the partial derivatives for f_x and f_y. Explain why it is likely to be difficult to solve $f_x = 0$ and $f_y = 0$ for critical points.

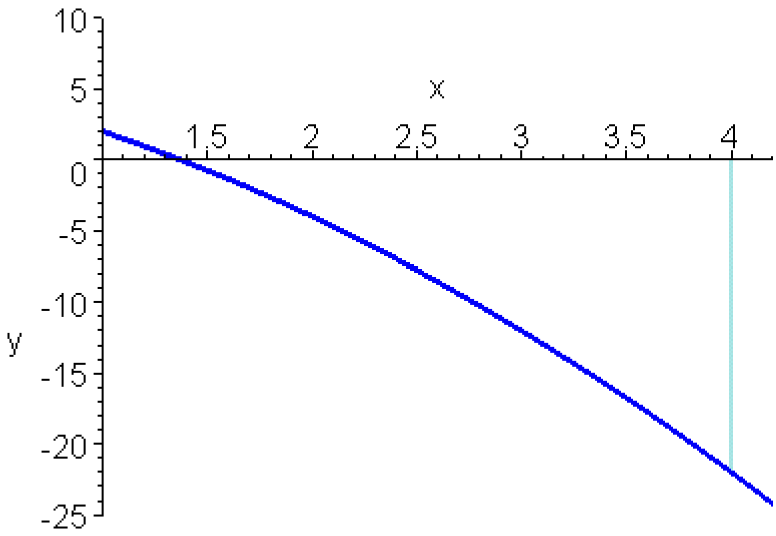
   The following commands define the partial derivatives f_x and f_y as functions.

```
> fx:= D[1](f);
  fy:= D[2](f);
```

PART II Newton's method in 2 dimensions

The one dimensional Newton's Method uses the linear approximation to find an approximate solution to an equation of the form $f(x) = 0$. If x_0 is an initial approximation to the solution, then the tangent line to $y = f(x)$ at $x = x_0$ intersects the x-axis at a point ($x_1, 0$) and x_1 is usually a better approximation to the solution than x_0. So the process can be iterated using x_1 as the new initial approximation. A short derivation shows that at each stage

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$



The same as the one dimensional Newton's Method, the 2-dimensional Newton's Method uses the linear approximation to find an approximate solution to a pair of equations of the form $\mathrm{f}(x, y) = 0$ and $\mathrm{g}(x, y) = 0$. Basically, if (x_0, y_0) is an initial approximation to the solution, then the tangent plane to $z = \mathrm{f}(x, y)$ at (x_0, y_0) and the tangent plane to $z = \mathrm{g}(x, y)$ at (x_0, y_0) intersect the xy -plane at a common point ($x_1, y_1, 0$) and (x_1, y_1) is usually a better approximation to the solution than (x_0, y_0). So the process can be iterated using (x_1, y_1) as the new initial approximation.

1. Derive that at each stage

$$x_{i+1} = x_i - \frac{f g_y - f_y g}{f_x g_y - f_y g_x} \quad \text{and} \quad y_{i+1} = y_i - \frac{f_x g - f g_x}{f_x g_y - f_y g_x}$$

where the functions f and g and their partial derivatives f_x , f_y , g_x and g_y are all evaluated at (x_i, y_i).

2. Construct a *Maple* function called newt2d which acts on an initial approximation (x_0, y_0) and produces the next approximation. (Or write a *Maple* procedure which will automatically control the iterations of newt2d. The procedure should take as arguments, the functions f and g , the number of digits of accuracy desired and the maximum number of iterations to allow to prevent an infinite loop.)
3. Enter the first of your approximate critical points in PART I -- the one that should lead to a maximum or minimum value of f -- as (x_0, y_0) . Use your *Maple* function or your *Maple* procedure to find the solution to the equations $f_x = 0$ and $f_y = 0$ in the region that you have chosen for this critical point. Give your answer to 10 digits of accuracy (See ?Digits). you can use **fsolve** to check your solution.

```
> evalf(sqrt(2),10);
  x0:=???;
  y0:=???;
  fsolve({fx(x,y)=0,fy(x,y)=0},{x,y},x=?..?,y=?..?);
```

4. Have you found a local maximum point or a local minimum point? How can you tell? (Hint: Use the second derivative test.)

```
> fxx:=D[1,1](f):
  fxy:=D[1,2](f):
  fyy:=D[2,2](f):
```

5. Repeat the process in problem 3 for your second estimated critical point, the one that should lead to a saddle point.
6. Have you in fact located a saddle point? How can you tell?
7. Go back to your contour plot of f in PART I, and approximate a third critical point. If you have already found a local maximum, find a local minimum. If you have found a local minimum, find a local maximum. Then use your *Maple* function or your *Maple* procedure to find the coordinates of this point to 10 digits of accuracy.
8. Go back to your contour plot of f in PART I, and approximate a fourth critical point that should lead to another saddle point. Then use your *Maple* function or your *Maple* procedure to find the coordinates of this point to 10 digits of accuracy.

<u>PART III</u> Different behavior of functions of two variables

For functions of one variable it is impossible for a continuous function to have two local maxima and no minimum. But for functions of two variables such functions exist.

1. Show that the function

$$f(x, y) = -(x^2 - 1)^2 - (x^2 y - x - 1)^2$$

has only two critical points, but has local maxima at both of them. Then use *Maple* to produce a graph with carefully chosen domain and viewpoint to see how this is possible.

If a function of one variable is continuous on an interval and has only one critical point, then a local maximum has to be an absolute maximum. But this is not true for functions of two variables.

2. Show that the function $g(x, y) = 3\,x\,e^{y} - x^{3} - e^{(3\,y)}$ has exactly one critical point, and that g has a local maximum there that is not an absolute maximum. Then use *Maple* to produce a graph with carefully chosen domain and viewpoint to see how this is possible.

Module 15

Changes of Coordinates

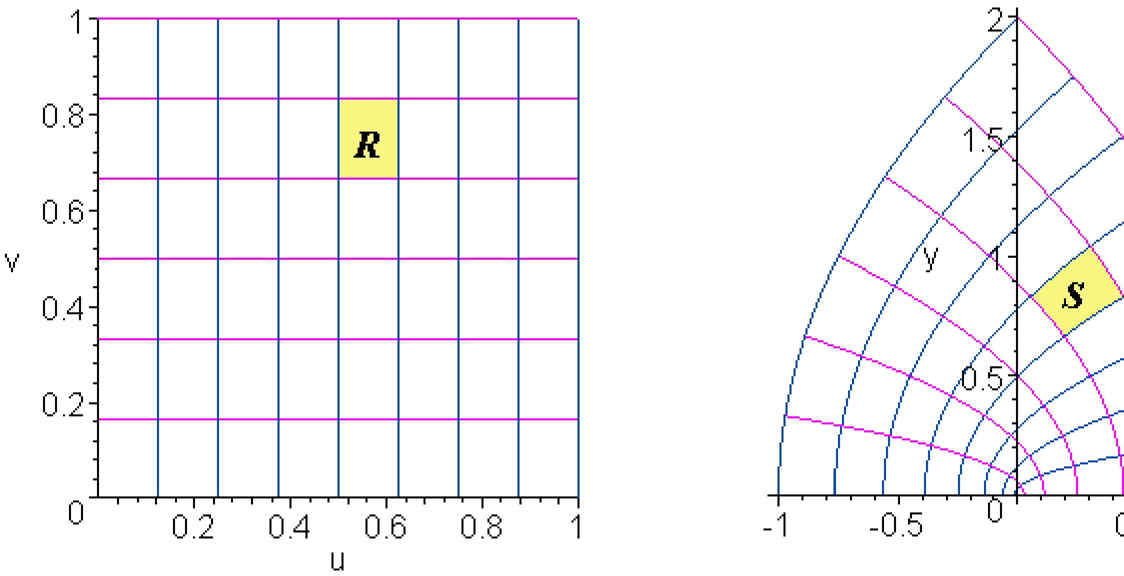
It is often useful to convert one set of parameters to another. This is called a change of coordinates (or changes of variables) and can be expressed as a set of functions (a pair of functions in two-dimensional case) from one set of parameters, or coordinates, to the other set.

Given a transformation

$$T(u, v) = (x, y)$$

where x and y are related to u and v by the equations

$$x = \mathrm{x}(u, v), \quad y = \mathrm{y}(u, v)$$



The left figure shows a coordinate grid in the uv -plane, with the curves u = constant in blue and the curves v = constant in red. Then a blue curve, say $u = c$, is transformed in the right figure to a blue curve parameterized by $(\mathrm{x}(c, v), \mathrm{y}(c, v))$. Similarly, a red curve, say $v = k$, is transformed in the right figure to a red curve parameterized by $(\mathrm{x}(u, k), \mathrm{y}(u, k))$. In the process, a typical coordinate rectangle R in the uv -plane is transformed into a "curvilinear rectangle" S in the xy -plane. The boundaries of S are formed by the parameterized curves.

The local change-in-area factor is the ratio of the area of S to the area of R -- that is, the factor by which the area grows or shrinks under the transformation.

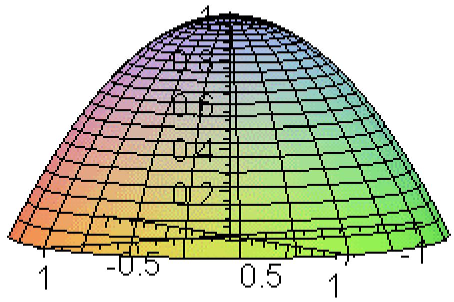
Given a transformation

$$T(u, v) = (x(u, v), y(u, v))$$

What is the local change-in-area factor for this transformation ?

It is often useful to convert one set of parameters to another. This is called a change of coordinates (or changes of variables) and can be expressed as a set of functions (a pair of functions in two-dimensioal case) from one set of parameters, or coordinates, to the other set. For example, as we experienced in the Module "Parametric surfaces", to better describe the surface $z = 1 - x^2 - y^2$ above the xy -plane, we use the polar coordinates instead of the Cartesian coordinates. The new variables r and θ are related to the old variables x and y by the equations

$$x(r, \theta) = r\cos(\theta), \qquad y(r, \theta) = r\sin(\theta)$$



More generally, in two-dimentioal case, we consider a change of variables that is given by a **transformation** T from uv -plane to xy -plane :

$$T(u, v) = (x, y)$$

where x and y are related to u and v by the equations

$$x = \mathrm{g},(u, v) \qquad y = \mathrm{h}(u, v)$$

or, as more often we write

$$x = \mathrm{x},(u, v) \qquad y = \mathrm{y}(u, v).$$

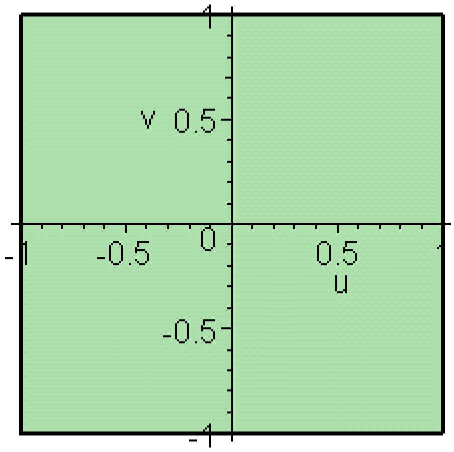
If $\mathrm{T}(u_1, v_1) = (x_1, y_1)$, then the point (x_1, y_1) is called the **image** of the point (u_1, v_1). If *T* transform a region S in the uv -plane onto a region R in the xy -plane, then we say R is the image of S .

Let $S = [-1, 1]$ x $[-1, 1]$ in the uv -plane (See the figure below.) and let

$$T(u, v) = (x, y)$$

where x and y are related to u and v by the equations

$$\mathrm{x}(u, v) = u + v \quad , \quad \mathrm{y}(u, v) = u - v$$



1. Let R be the image of S under the transformation T , graph the image of S in the $x\,y$-plane.

   (Hint: Consider separately the boundary $u = -1$, $u = 1$, $v = -1$, $v = 1$. In each case you get a parametric representation for one of the boundaries of R .)

2. How is the area of R related to the area of S ?
3. Let $f(x, y) = x\,y$. Set $g(u, v) = f(x(u, v), y(u, v))$, where the change of variables is the one in (1). Graph $g(u, v)$ over S = [$-1, 1$] x [$-1, 1$] , compare it with the graphs of $f(x, y)$ by plotting them together. Are they the same ? What are the ranges for x and y ?
4. How do your answers from (1) and (2) change if you change the coordinate functions to

$x(u, v) = u + 2v$ and $y(u, v) = u - 2v$?

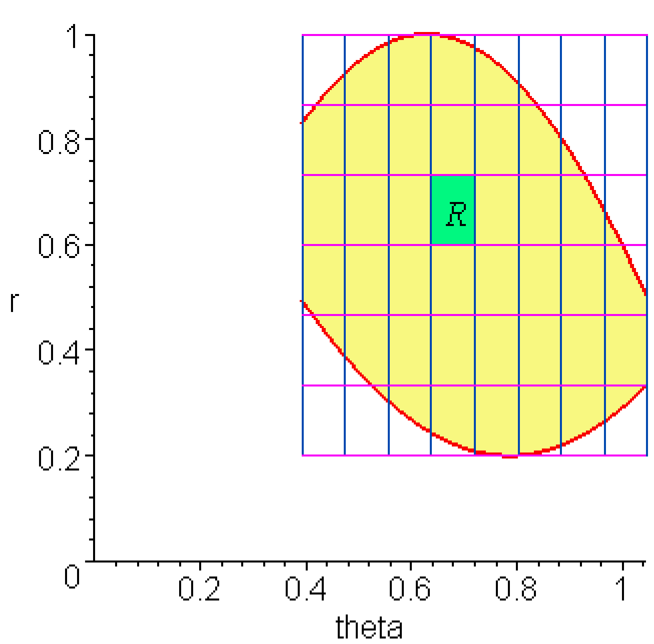
We have already seen one useful changes of variables for simplifying certain integrals, the change from Cartesian to polar coordinates in the plane.

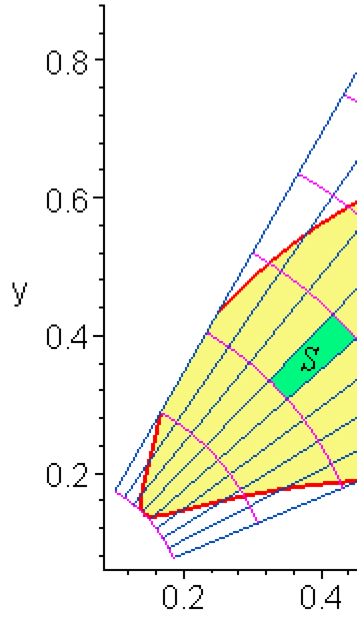
5. Compute the iterated integral

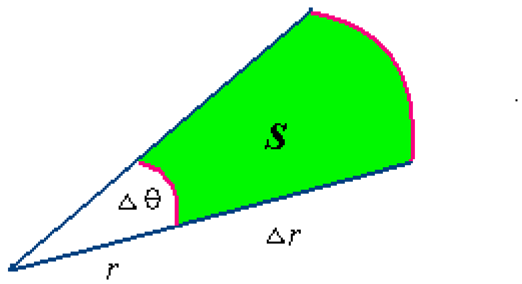
$$\int_1^2 \int_0^{\sqrt{2x - x^2}} \frac{1}{\sqrt{x^2 + y^2}} \, dy \, dx$$

Plot the domain over which the integration is being carried out.

In the shift from Cartesian to polar coordinates in double integrals, we see that $dxdy$ becomes $rdrd\theta$, but where does the "r" come from? In deriving this integral formula, as we subdivide the domain with a polar grid, as shown in the left graph below, we have to calculate the area of each little region.



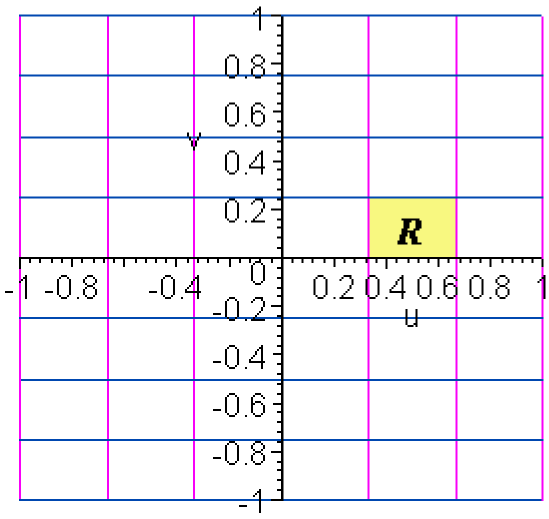


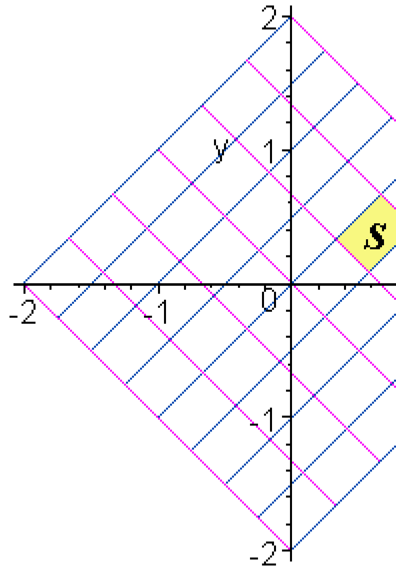


Elementary geometry shows that the area of the region ***S*** shown above is approximately equal to $r \, \Delta r \, \Delta \theta$, that means we have to "scale" each little region by its distance from the origin. So r is the local change-in-area factor for the Cartesian-to-polar transformation.

Now let's reconsider the transformation given by the equations

$$\mathrm{x}(u, v) = u + v, \quad \mathrm{y}(u, v) = u - v$$





The figure on the left shows a coordinate grid in the $u\,v$-plane, with the curves u = constant in red and the curves v = constant in blue. A red curve, say $u = c$, is transformed in the right figure to a red curve parameterized by ($v + c, c - v$), which is a segment on the line $x + y = 2\,c$. Similarly, a blue curve, say $v = k$, is transformed in the right figure to a blue curve parameterized by ($u + k, u - k$) , which is a segment on the line $x - y = 2\,k$. In this way, a typical coordinate rectangle R in the $u\,v$-plane is transformed into a parallelogram S in the $x\,y$-plane as shown above.

6. What is the local change-in-area factor for this transformation ?

7. Use the given transformation to evaluate the double integral $\iint_R xy\,dA$, where R is the region bounded by $x+y=2$, $x+y=-2$, $x-y=2$, $x-y=-2$. Check your answer by evaluating the integral using Cartesian coordinates.

In general case, given a transformation

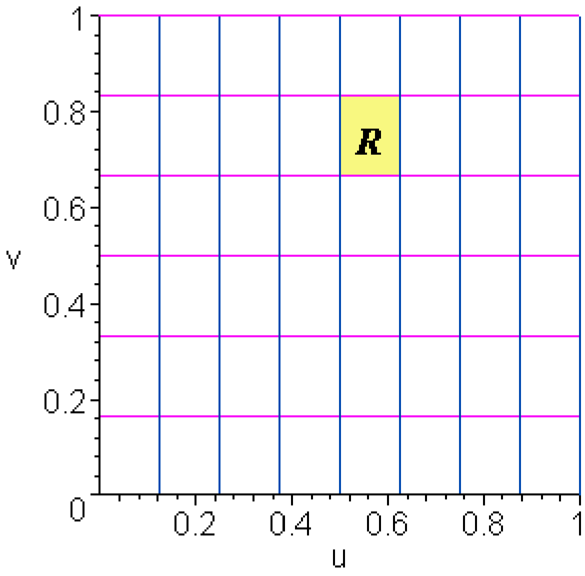
$$T(u, v) = (x, y)$$

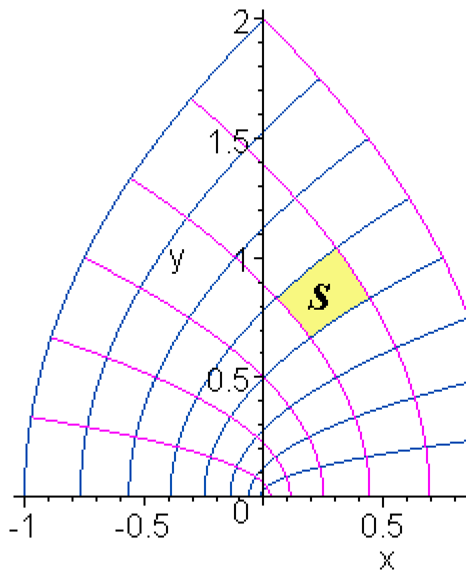
where x and y are related to u and v by the equations

$$x = \mathrm{x},(u, v) \quad y = \mathrm{y}(u, v)$$

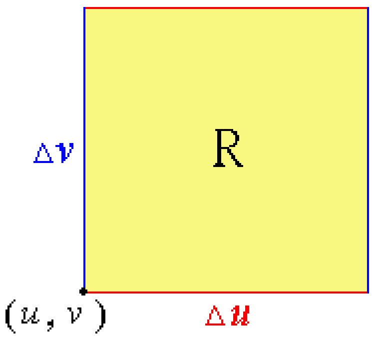
What is the local change-in-area factor for this transformation?

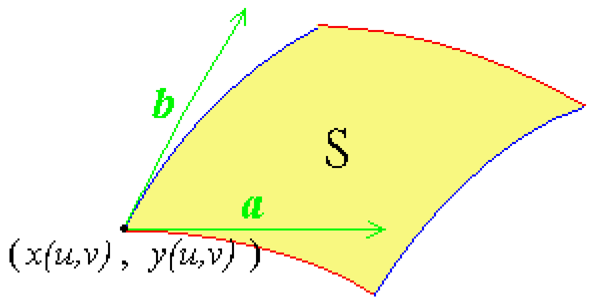
The left figure shows a coordinate grid in the $u\,v$-plane, with the curves u = constant in blue and the curves v = constant in red. Then a blue curve, say $u=c$, is transformed in the right figure to a blue curve parameterized by ($\mathrm{x}(c, v), \mathrm{y}(c, v)$) . Similarly, a red curve, say $v=k$, is transformed in the right figure to a red curve parameterized by ($\mathrm{x}(u, k), \mathrm{y}(u, k)$) . In the process, a typical coordinate rectangle R in the $u\,v$-plane is transformed into a "curvilinear rectangle" S in the $x\,y$-plane. The boundaries of S are formed by the parameterized curves.





Here is a closer look at *R* and *S* :





The local change-in-area factor is the ratio of the area of *S* to the area of *R* -- that is, the factor by which the area grows or shrinks under the transformation. We can calculate the area of *S* approximately as the area of the parallelogram determined by the two tangent vectors shown on the right.

8. Show that

$a = (x_u(u, v)\,\Delta u, y_u(u, v)\,\Delta u)$ and $b = (x_v(u, v)\,\Delta v, y_v(u, v)\,\Delta v)$

So the approximate area of *S* is | a x b |, where now we are thinking of the planar vectors as being in space.

9. Show that, in general, the local change-in-area factor is

$$\left|x_u(u, v)\,y_v(u, v) - x_v(u, v)\,y_u(u, v)\right|.$$

This expression is called the **Jacobian** of the coordinate transformation.

10. Show that the polar coordinate change-in-area factor is r. What is the image in the $x\,y$-plane of the coordinate rectangle [$0, 1$] x [$0, 2$] in the $r\,\theta$-plane ? How is the area of this image related to the area of the coordinate rectangle ?

11. Calculate the local change-in-area factor for the transformation

$\mathrm{x}(u, v) = u + v$, $\mathrm{y}(u, v) = u - v$.

How is this related to your answer in (6) ?