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A priori estimate for non-uniform elliptic equations

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ABSTRACT

A priori estimate for non-uniform elliptic equations with periodic boundary conditions is concerned. The domain considered consists of two sub-regions, a connected high permeability region and a disconnected matrix block region with low permeability. Let ϵ denote the size ratio of one matrix block to the whole domain. It is shown that in the connected high permeability sub-region, the Hölder and the Lipschitz estimates of the non-uniform elliptic solutions are bounded uniformly in ϵ. But Hölder gradient estimate and L^p estimate of the second order derivatives of the solutions in general are not bounded uniformly in ϵ.

1. Introduction

A priori estimate for the solutions of non-uniform elliptic equations with periodic boundary conditions is presented. The problem may arise from the study of flows in fractured media or the study of stress in composite media, see [2,3,13,16] or references therein. Domain considered is $\Omega \equiv [0, \mathbf{L}]^3 \subset \mathbb{R}^3$ containing two sub-regions, a connected high permeability region and a disconnected matrix block region with low permeability. Assume ϵ is a positive number less than 1 and $Y \equiv [0,1]^3$ is a cell consisting of a sub-domain Y_m completely surrounded by another connected sub-domain Y_f $(\equiv Y \setminus Y_m)$. The disconnected matrix block sub-region of the domain Ω is $\Omega_m^\epsilon \equiv \{x\colon\ x \in \epsilon(Y_m + j) \subset \Omega \text{ for } j \in \mathbb{Z}^3\}$, the connected high permeability sub-region is $\Omega_f^\epsilon \equiv \Omega \setminus \Omega_m^\epsilon$,

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and the boundary of Ω (resp. Ω_m^ϵ) is $\partial\Omega$ (resp. $\partial\Omega_m^\epsilon$). The non-uniform elliptic equations in Ω are written as

$$\begin{cases} -\nabla\cdot(\mathbf{K}_\epsilon\nabla P_\epsilon + Q_\epsilon) = F_\epsilon & \text{in } \Omega_f^\epsilon, \\ -\epsilon\nabla\cdot(\epsilon\mathbf{k}_\epsilon\nabla p_\epsilon + q_\epsilon) = f_\epsilon & \text{in } \Omega_m^\epsilon, \\ (\mathbf{K}_\epsilon\nabla P_\epsilon + Q_\epsilon)\cdot\vec{\mathbf{n}}^\epsilon = \epsilon(\epsilon\mathbf{k}_\epsilon\nabla p_\epsilon + q_\epsilon)\cdot\vec{\mathbf{n}}^\epsilon & \text{on } \partial\Omega_m^\epsilon, \\ P_\epsilon = p_\epsilon & \text{on } \partial\Omega_m^\epsilon, \end{cases} \tag{1.1}$$

with periodic boundary conditions on $\partial\Omega$. Here $\vec{\mathbf{n}}^\epsilon$ is the unit outward normal vector on $\partial\Omega_m^\epsilon$. It is known that if $\mathbf{K}_\epsilon(>0)$, $\mathbf{k}_\epsilon(>0)$ satisfy periodic conditions and if $\mathbf{K}_\epsilon$, $\mathbf{k}_\epsilon$, Q_ϵ, q_ϵ, F_ϵ, f_ϵ are smooth, then the piecewise regular solutions of (1.1) exist [15]. By energy method, the L^2 gradient estimate of the H^1 solutions of (1.1) in the connected sub-region is bounded uniformly in ϵ. However, uniform estimate in ϵ for the solutions under Lipschitz or higher order norm is not clear [13,16,20], and in some cases, solutions under those norms may not be bounded uniformly (see [5,17] or one example in Section 2). Similar problems had been considered by other authors. Existence and uniform estimate in ϵ of the piecewise smooth solutions in Hilbert spaces for elliptic diffraction problems were studied in [13,15]. Uniform Lipschitz estimate in ϵ for the Laplace equation in perforated domains was given in [20], and uniform L^p estimate of the same problem was claimed in [18]. Lipschitz estimate for uniform elliptic equations could be found in [16]. Hölder and Lipschitz estimates uniform in ϵ for the solutions of uniform elliptic equations in periodic domains were given in [4]. This work gives estimates for non-uniform elliptic equations. It is shown that in the connected high permeability sub-region the Hölder and the Lipschitz estimates of the non-uniform elliptic solutions are bounded uniformly in ϵ.

The rest of the work is organized as follows: Notation and main results are stated in Section 2. Some auxiliary lemmas are given in Section 3. Uniform Hölder estimate of the solutions of (1.1) in connected sub-region is shown in Section 4. Uniform Lipschitz estimate of the solutions of (1.1) in connected sub-region is derived in Section 5. The last section is a proof of a trace theorem claimed in Section 5.

2. Notation and main result

Let $C^{k,\alpha}$ denote the Hölder space and $L^s, H^i, W^{i,s}$ denote the Sobolev spaces for $k\geqslant 0, \alpha\in(0,1)$, and $i,s>0$ [11]. Define $\|\zeta_1,\zeta_2,\ldots,\zeta_n\|_B \equiv \|\zeta_1\|_B + \|\zeta_2\|_B + \cdots + \|\zeta_n\|_B$ for any Banach space B. For any set D, $D/r\equiv\{x\colon\ rx\in D\}$ for $r>0$, $\overline{D}$ denotes the closure of D, $|D|$ is the volume of D, $\mathcal{X}_D$ is a characteristic function on D, and

$$\fint_D \zeta(y)\,dy \equiv \frac{1}{|D|}\int_D \zeta(y)\,dy \quad \text{if } \zeta\in L^1(D).$$

Let $B(x,r)$ denote a ball centered at x with radius $r>0$. For any $\zeta\in L^1(\Omega)$ and $B(x,r)\subset\Omega$,

$$(\zeta)_{x,r} \equiv \fint_{B(x,r)} \zeta(y)\,dy.$$

If $\zeta\in C^{k,\alpha}(D)$ for $k\geqslant 0, \alpha\in(0,1)$ (resp. $\zeta\in W^{i,s}(D)$ for $i,s>0$), define $|||\zeta|||_{C^{k,\alpha}(D)} \equiv \|\zeta(\epsilon x)\|_{C^{k,\alpha}(D/\epsilon)}$ (resp. $|||\zeta|||_{W^{i,s}(D)} \equiv \|\zeta(\epsilon x)\|_{W^{i,s}(D/\epsilon)}$).

We shall assume that there are constants $\delta, d_4, d_5 > 0$ such that

A1. Y_m is a smooth simply connected sub-domain of Y,

A2. $\mathbf{K}_\epsilon, \mathbf{k}_\epsilon \in [d_4, d_5]$, $|d_5 - d_4| + \|\nabla\mathbf{K}_\epsilon\|_{L^\infty(\Omega_f^\epsilon)} + \|\nabla\mathbf{k}_\epsilon\|_{L^\infty(\Omega_m^\epsilon)} \leqslant cd_4$ where c is a small number depending on Y_f, A3. $F_\epsilon \mathcal{X}_{\Omega_f^\epsilon} + f_\epsilon \mathcal{X}_{\Omega_m^\epsilon} \in L^{3+\delta}(\Omega)$, $Q_\epsilon \in W^{1,3+\delta}(\Omega_f^\epsilon)$, $q_\epsilon \in W^{1,3+\delta}(\Omega_m^\epsilon)$,

A4. $\sum_{j\in\mathbb{Z}^3} \mathbf{K}_\epsilon \mathcal{X}_{\epsilon(Y_f+j)} + \mathbf{k}_\epsilon \mathcal{X}_{\epsilon(Y_m+j)}$ (resp. $\sum_{j\in\mathbb{Z}^3} Q_\epsilon \mathcal{X}_{\epsilon(Y_f+j)} + q_\epsilon \mathcal{X}_{\epsilon(Y_m+j)}$) is a periodic function in $\mathbb{R}^3$ with period ϵY (resp. Ω).

Main results are:

Theorem 2.1. *Under* A1–A4, *the solutions of* (1.1) *satisfy*

$$\begin{aligned}
&[P_\epsilon]_{C^{0,\mu}(\Omega_f^\epsilon)} + \epsilon^{1-\mu} \|\!|\nabla P_\epsilon|\!\|_{C^{0,\mu}(\Omega_f^\epsilon)} + \epsilon^{3-\mu} \|\!|\nabla p_\epsilon|\!\|_{C^{0,\mu}(\Omega_m^\epsilon)} \\
&\quad \leqslant c\big(\|Q_\epsilon, F_\epsilon\|_{L^{3+\delta}(\Omega_f^\epsilon)} + \|q_\epsilon, f_\epsilon\|_{L^{3+\delta}(\Omega_m^\epsilon)} + \epsilon^{1-\mu}\|\!|Q_\epsilon|\!\|_{C^{0,\mu}(\Omega_f^\epsilon)} + \epsilon^{2-\mu}\|\!|q_\epsilon|\!\|_{C^{0,\mu}(\Omega_m^\epsilon)}\big), \quad (2.1)\\
&[P_\epsilon]_{C^{0,\mu}(\Omega_f^\epsilon)} + \epsilon^{1-\mu} \|\!|\nabla P_\epsilon|\!\|_{C^{0,\mu}(\Omega_f^\epsilon)} + \epsilon^{2-\mu} \|\!|\nabla p_\epsilon|\!\|_{C^{0,\mu}(\Omega_m^\epsilon)} \\
&\quad \leqslant c\big(\|Q_\epsilon, F_\epsilon\|_{L^{3+\delta}(\Omega_f^\epsilon)} + \|q_\epsilon, f_\epsilon\|_{L^{3+\delta}(\Omega_m^\epsilon)} + \epsilon^{1-\mu}\|\!|Q_\epsilon|\!\|_{C^{0,\mu}(\Omega_f^\epsilon)} + \epsilon^{1-\mu}\|\!|q_\epsilon|\!\|_{C^{0,\mu}(\Omega_m^\epsilon)}\big), \quad (2.2)
\end{aligned}$$

where $\delta > 0$, $\mu \equiv 1 - \frac{3}{3+\delta}$, *and the constant* c *is independent of* ϵ.

Let $\vec{e}_i$, $i = 1,2,3$, be the unit vector in coordinate direction x_i in $\mathbb{R}^3$. Define $P_\epsilon^{\dagger_i}(x) \equiv \frac{P_\epsilon(x+5\epsilon\vec{e}_i)-P_\epsilon(x)}{5\epsilon}$ for $i = 1,2,3$. Symbols $\mathbf{K}_\epsilon^{\dagger_i}$, $Q_\epsilon^{\dagger_i}$, $F_\epsilon^{\dagger_i}$, $\mathbf{k}_\epsilon^{\dagger_i}$, $p_\epsilon^{\dagger_i}$, $q_\epsilon^{\dagger_i}$, $f_\epsilon^{\dagger_i}$ are defined in a similar way.

Theorem 2.2. *Under* A1–A4, *the solutions of* (1.1) *satisfy*

$$\begin{aligned}
&\sup_{\substack{j\in\mathbb{Z}^3\\ \epsilon(Y_f+j)\subset\Omega_f^\epsilon}} \|\!|\nabla P_\epsilon|\!\|_{W^{1,3+\delta}(\epsilon(Y_f+j))} + \sup_{\substack{j\in\mathbb{Z}^3\\ \epsilon(Y_m+j)\subset\Omega_m^\epsilon}} \epsilon\|\!|\nabla p_\epsilon|\!\|_{W^{1,3+\delta}(\epsilon(Y_m+j))} \\
&\quad + \sum_{i=1}^{3}\big(\big\|P_\epsilon^{\dagger_i}\big\|_{C^{0,\mu}(\Omega_f^\epsilon)} + \epsilon^{1-\mu}\big\|\!\big|\nabla P_\epsilon^{\dagger_i}\big|\!\big\|_{C^{0,\mu}(\Omega_f^\epsilon)} + \epsilon^{2-\mu}\big\|\!\big|\nabla p_\epsilon^{\dagger_i}\big|\!\big\|_{C^{0,\mu}(\Omega_m^\epsilon)}\big) \\
&\leqslant c\Bigg(\sum_{i=1}^{3}\big(\big\|Q_\epsilon^{\dagger_i}, F_\epsilon^{\dagger_i}\big\|_{L^{3+\delta}(\Omega_f^\epsilon)} + \big\|q_\epsilon^{\dagger_i}, f_\epsilon^{\dagger_i}\big\|_{L^{3+\delta}(\Omega_m^\epsilon)} + \big\|\!\big|\epsilon^{1-\mu}Q_\epsilon^{\dagger_i}\big|\!\big\|_{C^{0,\mu}(\Omega_f^\epsilon)} \\
&\quad + \big\|\!\big|\epsilon^{1-\mu}q_\epsilon^{\dagger_i}\big|\!\big\|_{C^{0,\mu}(\Omega_m^\epsilon)}\big) + \sup_{\substack{j\in\mathbb{Z}^3\\ \epsilon(Y_f+j)\subset\Omega_f^\epsilon}}\big(\|\!|Q_\epsilon|\!\|_{W^{1,3+\delta}(\epsilon(Y_f+j))} + \epsilon\|\!|F_\epsilon|\!\|_{L^{3+\delta}(\epsilon(Y_f+j))}\big) \\
&\quad + \sup_{\substack{j\in\mathbb{Z}^3\\ \epsilon(Y_m+j)\subset\Omega_m^\epsilon}}\big(\|\!|q_\epsilon|\!\|_{W^{1,3+\delta}(\epsilon(Y_m+j))} + \|\!|f_\epsilon|\!\|_{L^{3+\delta}(\epsilon(Y_m+j))}\big)\Bigg), \quad (2.3)
\end{aligned}$$

where $\delta > 0$, $\mu \equiv 1 - \frac{3}{3+\delta}$, *and the constant* c *is independent of* ϵ.

Clearly if the right-hand side of (2.1) is bounded independently of ϵ, the Hölder estimate of P_ϵ in the connected sub-region of Ω is bounded uniformly in ϵ. If the right-hand side of (2.3) is bounded independently of ϵ, the Lipschitz estimate of P_ϵ in the connected sub-region of Ω is bounded uniformly in ϵ.

Next we give one example to show that Hölder gradient estimate and L^p estimate of the second order derivatives of some elliptic solutions may not be bounded uniformly in ϵ. For any smooth function ξ in Y, define the left and the right limits, ξ_- and ξ_+, on ∂Y_m as $\xi_-(x) \equiv \lim_{\substack{x'\to 0\\ x+x'\in Y_m}} \xi(x+x')$

and $\xi_+(x) \equiv \lim_{\substack{x' \to 0 \\ x+x' \in Y_f}} \xi(x+x')$ for $x \in \partial Y_m$. We find a periodic function $\mathbb{X}_\epsilon^*$ in $\mathbb{R}^3$ with period Y as the solution of the following problem: In each cell Y, function $\mathbb{X}_\epsilon^*$ satisfies

$$\begin{cases} -\Delta\big(\mathbb{X}_\epsilon^* + y_1\big) = 0 & \text{in } Y_f, \\ -\epsilon^2 \Delta\big(\mathbb{X}_\epsilon^* + y_1\big) = 0 & \text{in } Y_m, \\ \nabla\big(\mathbb{X}_\epsilon^* + y_1\big)_+ \cdot \vec{\mathbf{n}}_y = \epsilon^2 \nabla\big(\mathbb{X}_\epsilon^* + y_1\big)_- \cdot \vec{\mathbf{n}}_y & \text{on } \partial Y_m, \\ \mathbb{X}_{\epsilon,+}^* = \mathbb{X}_{\epsilon,-}^* & \text{on } \partial Y_m, \\ \int_Y \mathbb{X}_\epsilon^* \, dy = 0, \end{cases}$$

where $\vec{\mathbf{n}}_y$ denotes the unit outward normal vector on ∂Y_m and y_1 is the first component of $y \in \mathbb{R}^3$. Clearly $\mathbb{X}_\epsilon^*$ is solvable and smooth [15]. Define $\widetilde{\mathbb{X}}_\epsilon(x) \equiv \epsilon \mathbb{X}_\epsilon^*(\frac{x}{\epsilon})$ in Ω. The function $\widetilde{\mathbb{X}}_\epsilon$ satisfies

$$\begin{cases} -\Delta(\widetilde{\mathbb{X}}_\epsilon + x_1) = 0 & \text{in } \Omega_f^\epsilon, \\ -\epsilon^2 \Delta(\widetilde{\mathbb{X}}_\epsilon + x_1) = 0 & \text{in } \Omega_m^\epsilon, \\ \nabla(\widetilde{\mathbb{X}}_\epsilon + x_1)_+ \cdot \vec{\mathbf{n}}^\epsilon = \epsilon^2 \nabla(\widetilde{\mathbb{X}}_\epsilon + x_1)_- \cdot \vec{\mathbf{n}}^\epsilon & \text{on } \partial\Omega_m^\epsilon, \\ \widetilde{\mathbb{X}}_{\epsilon,+} = \widetilde{\mathbb{X}}_{\epsilon,-} & \text{on } \partial\Omega_m^\epsilon, \end{cases}$$

with periodic boundary conditions on $\partial\Omega$. Here x_1 is the first component of $x \in \mathbb{R}^3$. Since $\nabla\widetilde{\mathbb{X}}_\epsilon(x) = \nabla\mathbb{X}_\epsilon^*(\frac{x}{\epsilon})$ and $\nabla^2\widetilde{\mathbb{X}}_\epsilon(x) = \frac{1}{\epsilon}\nabla^2\mathbb{X}_\epsilon^*(\frac{x}{\epsilon})$, we see that $[\nabla\widetilde{\mathbb{X}}_\epsilon]_{C^{0,\alpha}(\Omega_f^\epsilon)}$ for $\alpha \in (0,1)$ and $\|\nabla^2\widetilde{\mathbb{X}}_\epsilon\|_{L^s(\Omega_f^\epsilon)}$ for $s \in (1,\infty)$ are not bounded uniformly in ϵ.

3. Auxiliary result

First we recall an extension result.

Lemma 3.1. *(See [1,14].) For $1 \leqslant s < \infty$, there are a constant $d_1(Y_f, s)$ and a linear continuous extension operator $\Pi_\epsilon : W^{1,s}(\Omega_f^\epsilon) \to W^{1,s}(\Omega)$ such that if $\zeta \in W^{1,s}(\Omega_f^\epsilon)$, then*

$$\begin{cases} \Pi_\epsilon \zeta = \zeta & \text{in } \Omega_f^\epsilon \text{ almost everywhere}, \\ \|\Pi_\epsilon \zeta\|_{L^s(\Omega)} \leqslant d_1 \|\zeta\|_{L^s(\Omega_f^\epsilon)}, \\ \|\nabla \Pi_\epsilon \zeta\|_{L^s(\Omega)} \leqslant d_1 \|\nabla\zeta\|_{L^s(\Omega_f^\epsilon)}, \\ \Pi_\epsilon \zeta = g & \text{in } \Omega \text{ if } \zeta = g|_{\Omega_f^\epsilon} \text{ for some linear function } g \text{ in } \Omega. \end{cases}$$

Moreover, $\Pi_{\epsilon/r}\zeta(x) = \Pi_\epsilon g(rx)$ if $\zeta(x) = g(rx)$, $g \in L^s(\Omega_f^\epsilon)$, $\zeta \in L^s(\Omega_f^\epsilon/r)$, and $r > 0$.

It is known if $\mathbf{K}_\epsilon, \mathbf{k}_\epsilon \in [d_4, d_5]$ with $d_4 > 0$, $Q_\epsilon, F_\epsilon \in L^2(\Omega_f^\epsilon)$, and $q_\epsilon, f_\epsilon \in L^2(\Omega_m^\epsilon)$, then the H^1 solutions of (1.1) exist. Moreover, we have

Lemma 3.2. *If $\mathbf{K}_\epsilon, \mathbf{k}_\epsilon \in [d_4, d_5]$ with $d_4 > 0$, $Q_\epsilon, F_\epsilon \in L^2(\Omega_f^\epsilon)$, and $q_\epsilon, f_\epsilon \in L^2(\Omega_m^\epsilon)$, then the H^1 solution of (1.1) with $\int_\Omega \Pi_\epsilon P_\epsilon \, dx = 0$ satisfies*

$$\|P_\epsilon\|_{H^1(\Omega_f^\epsilon)} + \|p_\epsilon, \epsilon\nabla p_\epsilon\|_{L^2(\Omega_m^\epsilon)} \leqslant c\big(\|Q_\epsilon, F_\epsilon\|_{L^2(\Omega_f^\epsilon)} + \|q_\epsilon, f_\epsilon\|_{L^2(\Omega_m^\epsilon)}\big), \tag{3.1}$$

where c is a constant independent of ϵ.

Proof. If $P_\epsilon \mathcal{X}_{\Omega_f^\epsilon} + p_\epsilon \mathcal{X}_{\Omega_m^\epsilon}$ is an H^1 solution of (1.1), then $P_\epsilon \mathcal{X}_{\Omega_f^\epsilon} + p_\epsilon \mathcal{X}_{\Omega_m^\epsilon} + c$ is also an H^1 solution of (1.1) for any constant c. Adjust c so that $\int_\Omega \Pi_\epsilon (P_\epsilon + c)\,dx = 0$ (same notation for the adjusted solution). In this case, by Poincaré inequality and Lemma 3.1, the adjusted solution satisfies

$$\|P_\epsilon\|_{L^2(\Omega_f^\epsilon)} \leqslant \|\Pi_\epsilon P_\epsilon\|_{L^2(\Omega)} \leqslant c_1 \|\nabla \Pi_\epsilon P_\epsilon\|_{L^2(\Omega)} \leqslant c_2 \|\nabla P_\epsilon\|_{L^2(\Omega_f^\epsilon)}, \tag{3.2}$$

where c_1, c_2 are independent of ϵ. We also note, by Poincaré inequality and $(1.1)_4$,

$$\begin{aligned} \|p_\epsilon\|_{L^2(\Omega_m^\epsilon)} &\leqslant \|\Pi_\epsilon P_\epsilon\|_{L^2(\Omega_m^\epsilon)} + \|p_\epsilon - \Pi_\epsilon P_\epsilon\|_{L^2(\Omega_m^\epsilon)} \\ &\leqslant \|\Pi_\epsilon P_\epsilon\|_{L^2(\Omega)} + c_3 \epsilon \|\nabla p_\epsilon - \nabla \Pi_\epsilon P_\epsilon\|_{L^2(\Omega_m^\epsilon)}, \end{aligned} \tag{3.3}$$

where c_3 is independent of ϵ. By energy method and (3.2)–(3.3), we see that the adjusted solution satisfies (3.1). So this lemma holds. □

From now on, the H^1 solutions of (1.1) are required to satisfy $\int_\Omega \Pi_\epsilon P_\epsilon\,dx = 0$. Under (1) A1–A2, (2) $\|Q_\epsilon, F_\epsilon\|_{L^2(\Omega_f^\epsilon)} + \|q_\epsilon, f_\epsilon\|_{L^2(\Omega_m^\epsilon)}$ is bounded independently of ϵ, and (3) $Q_\epsilon \mathcal{X}_{\Omega_f^\epsilon}$ converges to Q in $L^2(\Omega)$ strongly, there is a subsequence of $\{P_\epsilon, Q_\epsilon, F_\epsilon, f_\epsilon\}$ (same notation for subsequence) satisfying, by Lemma 3.2 and compactness principle [11,14],

$$\begin{cases} \Pi_\epsilon P_\epsilon \to P_0 & \text{in } L^2(\Omega) \text{ strongly} \\ (\mathbf{K}_\epsilon \nabla P_\epsilon + Q_\epsilon)\mathcal{X}_{\Omega_f^\epsilon} \to \mathbf{K}^* \nabla P_0 + Q^* & \text{in } L^2(\Omega) \text{ weakly} \\ F_\epsilon \mathcal{X}_{\Omega_f^\epsilon} + f_\epsilon \mathcal{X}_{\Omega_m^\epsilon} \to F & \text{in } L^2(\Omega) \text{ weakly} \end{cases} \quad \text{as } \epsilon \to 0,$$

where $\mathcal{X}_{\Omega_f^\epsilon}$ (resp. $\mathcal{X}_{\Omega_m^\epsilon}$) is the characteristic function of Ω_f^ϵ (resp. Ω_m^ϵ), $\mathbf{K}^*$ is a positive definite matrix depending on $\mathbf{K}_\epsilon$, Y_f, and function Q^* depends on Q, $\mathbf{K}_\epsilon$. Moreover, the function $P_0 \in H^1(\Omega)$ satisfies

$$-\nabla \cdot \big(\mathbf{K}^* \nabla P_0 + Q^*\big) = F \quad \text{in } \Omega. \tag{3.4}$$

Let $\mathcal{G}(x-y)$ denote the fundamental solution of the Laplace equation, see §6.2 [7]. Define single-layer and double-layer potentials as, for any smooth function ζ on the boundary ∂D of a bounded smooth domain D,

$$\begin{cases} \mathcal{V}_{\partial D}(\zeta)(x) \equiv \displaystyle\int_{\partial D} \mathcal{G}(x-y)\zeta(y)\,d\sigma_y \\ \mathcal{T}_{\partial D}(\zeta)(x) \equiv \displaystyle\int_{\partial D} \partial_y \mathcal{G}(x-y)\vec{\mathbf{n}}_y \zeta(y)\,d\sigma_y \end{cases} \quad \text{for } x \in \partial D,$$

where $\vec{\mathbf{n}}_y$ is the unit vector outward normal to ∂D.

Lemma 3.3. *If D is a bounded smooth domain, then*:

1. *$\mathcal{V}_{\partial D}, \mathcal{T}_{\partial D}$ are pseudo-differential operators of order -1 on ∂D.*
2. *For any $|\beta| > 1/2$ and $\alpha \in (0,1)$, the linear operators*

$$\begin{cases} \mathcal{V}_{\partial D} : C^{0,\alpha}(\partial D) \to C^{1,\alpha}(\partial D), \\ \mathcal{T}_{\partial D} : C^{0,\alpha}(\partial D) \to C^{1,\alpha}(\partial D), \\ \beta I - \mathcal{T}_{\partial D} : C^{1,\alpha}(\partial D) \to C^{1,\alpha}(\partial D) \end{cases} \tag{3.5}$$

are bounded and $\beta I - \mathcal{T}_{\partial D}$ is invertible in $C^{1,\alpha}(\partial D)$.

3. *For any* $|\beta| > \frac{1}{2}$ *and* $s \in (2, \infty)$, *the linear operators*

$$\begin{cases} \mathcal{V}_{\partial D} : W^{1-\frac{1}{s},s}(\partial D) \to W^{2-\frac{1}{s},s}(\partial D), \\ \mathcal{T}_{\partial D} : W^{1-\frac{1}{s},s}(\partial D) \to W^{2-\frac{1}{s},s}(\partial D), \\ \beta I - \mathcal{T}_{\partial D} : W^{2-\frac{1}{s},s}(\partial D) \to W^{2-\frac{1}{s},s}(\partial D) \end{cases} \tag{3.6}$$

are bounded and $\beta I - \mathcal{T}_{\partial D}$ *is invertible in* $W^{2-\frac{1}{s},s}(\partial D)$.

Proof. By [7,13], $\mathcal{V}_{\partial D}$ and $\mathcal{T}_{\partial D}$ are pseudo-differential operators of order -1. By Theorem 2.5, Chapter XI [21], operators $\mathcal{V}_{\partial D}, \mathcal{T}_{\partial D}$, and $\beta I - \mathcal{T}_{\partial D}$ in (3.5) are bounded and linear. Tracing the proof of Theorem 4.6.5 [7], we see $\beta I - \mathcal{T}_{\partial D}$ is a Fredholm operator. Since $\beta I - \mathcal{T}_{\partial D}$ is invertible in L^2 space [9], it is one-to-one and bounded as well as has closed range in $C^{1,\alpha}(\partial D)$. By inverse mapping theorem [8], $\beta I - \mathcal{T}_{\partial D}$ is invertible in $C^{1,\alpha}(\partial D)$.

By theorem in §2.3.4 [22] and following the proof of Theorem 2.5, Chapter XI [21], operators $\mathcal{V}_{\partial D}$, $\mathcal{T}_{\partial D}$, and $\beta I - \mathcal{T}_{\partial D}$ in (3.6) are bounded linear operators. An analogous argument as that for (3.5) implies that $\beta I - \mathcal{T}_{\partial D}$ is invertible in $W^{2-\frac{1}{s},s}(\partial D)$. □

Now we consider the following problem

$$\begin{cases} -\nabla \cdot (\mathbf{K}\nabla U_\epsilon + \hat{Q}_\epsilon) = \hat{F}_\epsilon & \text{in } Y_f, \\ -\epsilon\nabla \cdot (\epsilon\mathbf{k}\nabla u_\epsilon + \hat{q}_\epsilon) = \hat{f}_\epsilon & \text{in } Y_m, \\ (\mathbf{K}\nabla U_\epsilon + \hat{Q}_\epsilon) \cdot \vec{\mathbf{n}}_y = \epsilon(\epsilon\mathbf{k}\nabla u_\epsilon + \hat{q}_\epsilon) \cdot \vec{\mathbf{n}}_y & \text{on } \partial Y_m, \\ U_\epsilon = u_\epsilon & \text{on } \partial Y_m, \end{cases} \tag{3.7}$$

where $\vec{\mathbf{n}}_y$ is the unit vector normal to ∂Y_m. Let $\mathbf{D}$ be a smooth domain satisfying $Y_m \subset \mathbf{D} \subset Y = Y_f \cup Y_m$ and $d_8 \equiv \min\{dist(Y_m, \partial\mathbf{D}), dist(\mathbf{D}, \partial Y)\} > 0$. If we define $\mathbf{D}_1 \equiv \{x \in Y_f \mid dist(x, Y_m) > \frac{d_8}{4}, dist(x, \partial Y) > \frac{d_8}{4}\}$, then $\partial\mathbf{D} \subset \mathbf{D}_1$.

Lemma 3.4. *If the following conditions hold*

1. $\mathbf{K}, \mathbf{k} \in [d_4, d_5]$ *with* $d_4 > 0$,
2. $\|\mathbf{K} - d_4\|_{C^{0,1-\frac{3}{r}}(Y_f)} + \|\mathbf{k} - d_4\|_{C^{0,1-\frac{3}{r}}(Y_m)} \leqslant cd_4$ *where* $r \in (3, \infty)$ *and* c *is a small number depending on* Y_f,
3. $\|U_\epsilon\|_{L^2(Y_f)} + \|\hat{Q}_\epsilon\|_{C^{0,1-\frac{3}{r}}(Y_f)} + \epsilon\|\hat{q}_\epsilon\|_{C^{0,1-\frac{3}{r}}(Y_m)} + \|\hat{F}_\epsilon \mathcal{X}_{Y_f} + \hat{f}_\epsilon \mathcal{X}_{Y_m}\|_{L^r(Y)}$ *is bounded independently of* ϵ *for* $r \in (3, \infty)$,

then the solutions of (3.7) *satisfy*

$$\|U_\epsilon\|_{C^{1,1-\frac{3}{r}}(\mathbf{D}\setminus Y_m)} + \epsilon^2\|u_\epsilon\|_{C^{1,1-\frac{3}{r}}(Y_m)} \leqslant c^*, \tag{3.8}$$

where c^* *is a constant depending on given data but independent of* ϵ.

Proof. Assume the coefficients and the solutions of (3.7) are smooth in Y_f and Y_m. Consider $(3.7)_1$ in Y_f. Theorem 8.17 of [11] implies, for $r \in (3, \infty)$,

$$\|U_\epsilon\|_{C^{1,1-\frac{3}{r}}(\mathbf{D}_1)} \leqslant \tilde{c}_1 \tag{3.9}$$

where $\tilde{c}_1$ is a constant depending on given data but independent of ϵ. Let $\hat{u}$ be a solution of

$$\begin{cases} -\epsilon\nabla\cdot\big(\epsilon d_4\nabla\hat{u}+\epsilon(\mathbf{k}-d_4)\nabla u_\epsilon+\hat{q}_\epsilon\big)=\hat{f}_\epsilon & \text{in } Y_m, \\ \hat{u}|_{\partial Y_m}=0, \end{cases} \tag{3.10}$$

and $\hat{U}$ a solution of

$$\begin{cases} -\nabla\cdot\big(d_4\nabla\hat{U}+(\mathbf{K}-d_4)\nabla U_\epsilon+\hat{Q}_\epsilon\big)=\hat{F}_\epsilon & \text{in } \mathbf{D}\setminus Y_m, \\ \hat{U}|_{\partial Y_m}=0, \\ \hat{U}-U_\epsilon|_{\partial\mathbf{D}}=0. \end{cases} \tag{3.11}$$

Then, by (3.9),

$$\begin{cases} \epsilon^2 d_4\|\hat{u}\|_{C^{1,1-\frac{3}{r}}(Y_m)}\leqslant\tilde{c}_1+\tilde{c}_2\epsilon^2\big\|(\mathbf{k}-d_4)\nabla u_\epsilon\big\|_{C^{0,1-\frac{3}{r}}(Y_m)}, \\ d_4\|\hat{U}\|_{C^{1,1-\frac{3}{r}}(\mathbf{D}\setminus Y_m)}\leqslant\tilde{c}_1+\tilde{c}_2\big\|(\mathbf{K}-d_4)\nabla U_\epsilon\big\|_{C^{0,1-\frac{3}{r}}(\mathbf{D}\setminus Y_m)}, \end{cases} \tag{3.12}$$

where $\tilde{c}_1$ is a constant depending on given data but independent of ϵ, and $\tilde{c}_2$ is a constant depending on Y_f. Define $\check{u}\equiv u_\epsilon-\hat{u}$ in Y_m and $\check{U}\equiv U_\epsilon-\hat{U}$ in $\mathbf{D}\setminus Y_m$. Eqs. (3.7) and (3.10)–(3.11) imply

$$\begin{cases} \Delta\check{u}=0 & \text{in } Y_m, \\ \Delta\check{U}=0 & \text{in } \mathbf{D}\setminus Y_m, \\ \check{U}|_{\partial Y_m}=\check{u}|_{\partial Y_m}, \\ \nabla\check{U}\cdot\vec{\mathbf{n}}_y|_{\partial Y_m}-\epsilon^2\nabla\check{u}\cdot\vec{\mathbf{n}}_y|_{\partial Y_m}=\mathcal{S}/d_4, \\ \check{U}|_{\partial\mathbf{D}}=0. \end{cases} \tag{3.13}$$

The function $\mathcal{S}$ in (3.13) satisfies, by (3.12) and $\partial\mathbf{D}\subset\mathbf{D}_1$,

$$\begin{aligned} \|\mathcal{S}\|_{C^{0,1-\frac{3}{r}}(\partial Y_m)}\leqslant{}&\tilde{c}_1+\tilde{c}_2\epsilon^2\big\|(\mathbf{k}-d_4)\nabla u_\epsilon\big\|_{C^{0,1-\frac{3}{r}}(Y_m)} \\ &+\tilde{c}_2\big\|(\mathbf{K}-d_4)\nabla U_\epsilon\big\|_{C^{0,1-\frac{3}{r}}(\mathbf{D}\setminus Y_m)}, \end{aligned} \tag{3.14}$$

where $\tilde{c}_1$ is a constant depending on given data but independent of ϵ, and $\tilde{c}_2$ is a constant depending on Y_f. By Green's formula, (3.13), and Theorem 6.5.1 [7], we see that

$$\begin{cases} \check{u}/2+\mathcal{T}_{\partial Y_m}(\check{u})=\mathcal{V}_{\partial Y_m}(\partial_{\mathbf{n}_y}\check{u}) \\ \check{U}/2-\mathcal{T}_{\partial Y_m}(\check{U})=-\mathcal{V}_{\partial Y_m}(\partial_{\mathbf{n}_y}\check{U})+\mathcal{V}_{\partial\mathbf{D}}(\partial_{\mathbf{n}_y}\check{U}|_{\partial\mathbf{D}}) \end{cases} \quad \text{on } \partial Y_m,$$

where $\partial_{\mathbf{n}_y}\check{U}|_{\partial\mathbf{D}}$ is the normal derivative of $\check{U}$ on $\partial\mathbf{D}$. Therefore,

$$\left(\frac{\epsilon^2+1}{2(1-\epsilon^2)}-\mathcal{T}_{\partial Y_m}\right)\check{u}=\frac{\mathcal{V}_{\partial\mathbf{D}}(\partial_{\mathbf{n}_y}\check{U}|_{\partial\mathbf{D}})}{1-\epsilon^2}-\frac{\mathcal{V}_{\partial Y_m}(\mathcal{S})}{(1-\epsilon^2)d_4} \quad \text{on } \partial Y_m. \tag{3.15}$$

Eqs. (3.9), (3.12), (3.15), Lemma 3.3, and [21] imply

$$\|\check{u}\|_{C^{1,1-\frac{3}{r}}(\partial Y_m)} \leqslant \tilde{c}_2 \big\| \big|\mathcal{T}_{\partial Y_m}(\check{u})\big| + \big|\mathcal{V}_{\partial Y_m}(\mathcal{S}/d_4)\big| + \big|\mathcal{V}_{\partial \mathbf{D}}(\partial_{\mathbf{n}_y}\check{U}|_{\partial \mathbf{D}})\big|\big\|_{C^{1,1-\frac{3}{r}}(\partial Y_m)}$$
$$\leqslant \tilde{c}_2 \big(\|\mathcal{S}/d_4\|_{C^{0,1-\frac{3}{r}}(\partial Y_m)} + \|\partial_{\mathbf{n}_y}\check{U}\|_{C^{0,1-\frac{3}{r}}(\partial \mathbf{D})}\big), \tag{3.16}$$

$$d_4\|\partial_{\mathbf{n}_y}\check{U}\|_{C^{0,1-\frac{3}{r}}(\partial \mathbf{D})} \leqslant \tilde{c}_1 + \tilde{c}_2 \big\|(\mathbf{K}-d_4)\nabla U_\epsilon\big\|_{C^{0,1-\frac{3}{r}}(\mathbf{D}\setminus Y_m)}, \tag{3.17}$$

where $\tilde{c}_1$ is a constant depending on given data but independent of ϵ and $\tilde{c}_2$ is a constant depending on Y_f. By (3.12), (3.14), and (3.16)–(3.17), we obtain

$$d_4\|U_\epsilon\|_{C^{1,1-\frac{3}{r}}(\mathbf{D}\setminus Y_m)} + \epsilon^2 d_4\|u_\epsilon\|_{C^{1,1-\frac{3}{r}}(Y_m)}$$
$$\leqslant \tilde{c}_1 + \tilde{c}_2\epsilon^2 \big\|(\mathbf{k}-d_4)\nabla u_\epsilon\big\|_{C^{0,1-\frac{3}{r}}(Y_m)} + c\big\|(\mathbf{K}-d_4)\nabla U_\epsilon\big\|_{C^{0,1-\frac{3}{r}}(\mathbf{D}\setminus Y_m)},$$

where $\tilde{c}_1$ is a constant depending on given data but independent of ϵ, and $\tilde{c}_2$ is a constant depending on Y_f. By the smallness assumption on $\mathbf{K}-d_4$ and $\mathbf{k}-d_4$, we obtain $d_4\|U_\epsilon\|_{C^{1,1-\frac{3}{r}}(\mathbf{D}\setminus Y_m)} + \epsilon^2 d_4\|u_\epsilon\|_{C^{1,1-\frac{3}{r}}(Y_m)} \leqslant \tilde{c}_1$, where $\tilde{c}_1$ is a constant depending on given data but independent of ϵ. So we prove (3.8) for the smooth coefficient case.

The estimate (3.8) for non-smooth coefficient case is directly from the estimate (3.8) for smooth coefficient case, approximation method, §16, Chapter 3 [15], and energy method. □

By a straightforward modification of the proof of Lemma 3.4, we see

Lemma 3.5. *If the following conditions hold*

1. $\mathbf{K}, \mathbf{k} \in [d_4, d_5]$ *with* $d_4 > 0$,
2. $\|\mathbf{K}-d_4\|_{C^{0,1-\frac{3}{r}}(Y_f)} + \|\mathbf{k}-d_4\|_{C^{0,1-\frac{3}{r}}(Y_m)} \leqslant cd_4$ *where* $r \in (3,\infty)$ *and* c *is a small number depending on* Y_f,
3. $\|U_\epsilon\|_{L^2(Y_f)} + \|\hat{Q}_\epsilon\|_{C^{0,1-\frac{3}{r}}(Y_f)} + \|\hat{q}_\epsilon\|_{C^{0,1-\frac{3}{r}}(Y_m)} + \|\hat{F}_\epsilon \mathcal{X}_{Y_f} + \epsilon^{-1}\hat{f}_\epsilon \mathcal{X}_{Y_m}\|_{L^r(Y)}$ *is bounded independently of* ϵ *for* $r \in (3,\infty)$,

then the solutions of (3.7) *satisfy*

$$\|U_\epsilon\|_{C^{1,1-\frac{3}{r}}(\mathbf{D}\setminus Y_m)} + \epsilon\|u_\epsilon\|_{C^{1,1-\frac{3}{r}}(Y_m)} \leqslant c^*,$$

where c^* *is a constant depending on given data but independent of* ϵ.

Lemma 3.6. *If the following conditions hold*

1. $\mathbf{K}, \mathbf{k} \in [d_4, d_5]$ *with* $d_4 > 0$,
2. $\|\mathbf{K}-d_4\|_{W^{1,r}(Y_f)} + \|\mathbf{k}-d_4\|_{W^{1,r}(Y_m)} \leqslant cd_4$ *where* $r \in (3,\infty)$ *and* c *is a small number depending on* Y_f,
3. $\|U_\epsilon\|_{L^2(Y_f)} + \|\hat{Q}_\epsilon\|_{W^{1,r}(Y_f)} + \|\hat{q}_\epsilon\|_{W^{1,r}(Y_m)} + \|\hat{F}_\epsilon \mathcal{X}_{Y_f} + \epsilon^{-1}\hat{f}_\epsilon \mathcal{X}_{Y_m}\|_{L^r(Y)}$ *is bounded independently of* ϵ *for* $r \in (3,\infty)$,

then the solutions of (3.7) *satisfy*

$$\|U_\epsilon\|_{W^{2,r}(\mathbf{D}\setminus Y_m)} + \epsilon\|u_\epsilon\|_{W^{2,r}(Y_m)} \leqslant c^*,$$

where c^* *is a constant depending on given data but independent of* ϵ.

Lemmas 3.4, 3.5, 3.6 are proved if Y_m is a connected set. An analogous argument also proves Lemmas 3.4, 3.5, 3.6 if Y_m is the union of several non-overlapping connected sets.

4. Uniform Hölder estimate

In this section we prove Theorem 2.1. For convenience, let us assume $\overline{B(0,1)} \subset \Omega$.

Lemma 4.1. *Under* A1–A2, *for any* $\delta > 0$, *there are constant* $\theta \in (0,1)$ (*depending on* $\delta, \mathbf{K}^*, Y_f$) *and constant* $\epsilon_0 \in (0,1)$ (*depending on* θ, δ, d_4, d_5) *such that if* $\mathbf{K}_\nu$, $P_{\epsilon,\nu}$, $Q_{\epsilon,\nu}$, $F_{\epsilon,\nu}$, $\mathbf{k}_\nu$, $p_{\epsilon,\nu}$, $q_{\epsilon,\nu}$, $f_{\epsilon,\nu}$ *satisfy*

$$\begin{cases} -\nabla\cdot(\mathbf{K}_\nu\nabla P_{\epsilon,\nu} + Q_{\epsilon,\nu}) = F_{\epsilon,\nu} & \text{in } B(0,1)\cap\Omega_f^\nu, \\ -\epsilon\nabla\cdot(\epsilon\mathbf{k}_\nu\nabla p_{\epsilon,\nu} + q_{\epsilon,\nu}) = \epsilon f_{\epsilon,\nu} & \text{in } B(0,1)\cap\Omega_m^\nu, \\ (\mathbf{K}_\nu\nabla P_{\epsilon,\nu} + Q_{\epsilon,\nu})\cdot\vec{\mathbf{n}}^\nu = \epsilon(\epsilon\mathbf{k}_\nu\nabla p_{\epsilon,\nu} + q_{\epsilon,\nu})\cdot\vec{\mathbf{n}}^\nu & \text{on } B(0,1)\cap\partial\Omega_m^\nu, \\ P_{\epsilon,\nu} = p_{\epsilon,\nu} & \text{on } B(0,1)\cap\partial\Omega_m^\nu, \end{cases} \tag{4.1}$$

and

$$\max\Big\{\|P_{\epsilon,\nu}\|_{L^2(B(0,1)\cap\Omega_f^\nu)}, \epsilon\|p_{\epsilon,\nu}\|_{L^2(B(0,1)\cap\Omega_m^\nu)}, \|q_{\epsilon,\nu}\|_{L^{3+\delta}(B(0,1)\cap\Omega_m^\nu)},$$
$$\epsilon_0^{-1}\|Q_{\epsilon,\nu}\mathcal{X}_{\Omega_f^\nu}, F_{\epsilon,\nu}\mathcal{X}_{\Omega_f^\nu} + \nu f_{\epsilon,\nu}\mathcal{X}_{\Omega_m^\nu}\|_{L^{3+\delta}(B(0,1))}\Big\} \leqslant 1, \tag{4.2}$$

then, for any $\epsilon \leqslant \nu \leqslant \epsilon_0$,

$$\begin{cases} \displaystyle\fint_{B(0,\theta)} \left|\Pi_\nu P_{\epsilon,\nu} - (\Pi_\nu P_{\epsilon,\nu})_{0,\theta}\right|^2 dx \leqslant \theta^{2\mu}, \\ \displaystyle\fint_{B(0,\theta)\cap\Omega_m^\nu} \epsilon^2\left|p_{\epsilon,\nu} - (\Pi_\nu P_{\epsilon,\nu})_{0,\theta}\right|^2 dx \leqslant \theta^{2\mu}. \end{cases} \tag{4.3}$$

Here $\mathbf{K}^*$ *is the positive definite matrix in* (3.4), $\vec{\mathbf{n}}^\nu$ *is the unit vector normal to* $\partial\Omega_m^\nu$, d_4 *and* d_5 *are defined in* A2, $\mu \equiv 1 - \frac{3}{3+\delta}$, *and* Π_ν *is the extension operator defined in Lemma* 3.1.

Proof. Let $\mathcal{L}_0 \equiv -\nabla\cdot(\mathbf{K}^*\nabla)$ denote a differential operator, where $\mathbf{K}^*$ is the positive definite matrix in (3.4). If $\mathcal{L}_0 P_0 = 0$ and if μ' satisfies $\mu < \mu' < 1$, then

$$\fint_{B(0,\theta)} \left|P_0 - (P_0)_{0,\theta}\right|^2 dx \leqslant \theta^{2\mu'} \fint_{B(0,1)} P_0^2\, dx \tag{4.4}$$

for θ sufficiently small. This is due to Theorem 1.2 on p. 70 [10] and that $\mathcal{L}_0$-harmonic functions are bounded in $C^2(B(0,\theta))$, for some $\theta < 1$, uniformly by their L^2 norm [10,11]. Fix a value θ and we claim $(4.3)_1$. If not, there is a sequence $\{\mathbf{K}_\nu, P_{\epsilon,\nu}, Q_{\epsilon,\nu}, F_{\epsilon,\nu}, \mathbf{k}_\nu, p_{\epsilon,\nu}, q_{\epsilon,\nu}, f_{\epsilon,\nu}\}$ satisfying (4.1) and

$$\begin{cases} \max\Big\{\|P_{\epsilon,\nu}\|_{L^2(B(0,1)\cap\Omega_f^\nu)}, \epsilon\|p_{\epsilon,\nu}\|_{L^2(B(0,1)\cap\Omega_m^\nu)}, \|q_{\epsilon,\nu}\|_{L^{3+\delta}(B(0,1)\cap\Omega_m^\nu)}\Big\} \leqslant 1, \\ \displaystyle\lim_{\epsilon\leqslant\nu\to 0} \|Q_{\epsilon,\nu}, F_{\epsilon,\nu}\|_{L^{3+\delta}(B(0,1)\cap\Omega_f^\nu)} + \|\nu f_{\epsilon,\nu}\|_{L^{3+\delta}(B(0,1)\cap\Omega_m^\nu)} = 0, \\ \displaystyle\fint_{B(0,\theta)} \left|\Pi_\nu P_{\epsilon,\nu} - (\Pi_\nu P_{\epsilon,\nu})_{0,\theta}\right|^2 dx > \theta^{2\mu}. \end{cases} \tag{4.5}$$

By Lemma 3.1 and compactness principle, we can extract a subsequence (same notation for subsequence) such that, as $\epsilon \leqslant \nu \to 0$,

$$\begin{cases} \Pi_\nu P_{\epsilon,\nu} \to P_0 & \text{in } L^2\big(B(0,\theta)\big) \text{ strongly}, \\ \mathbf{K}_\nu \mathcal{X}_{\Omega_f^\nu} \nabla P_{\epsilon,\nu} \to \mathbf{K}^* \nabla P_0 & \text{in } L^2\big(B(0,\theta)\big) \text{ weakly}, \\ Q_{\epsilon,\nu}\mathcal{X}_{\Omega_f^\nu}, F_{\epsilon,\nu}\mathcal{X}_{\Omega_f^\nu}, \nu f_{\epsilon,\nu}\mathcal{X}_{\Omega_m^\nu} \to 0 & \text{in } L^2\big(B(0,1)\big) \text{ strongly}. \end{cases} \tag{4.6}$$

We note that P_0 satisfies $\mathcal{L}_0 P_0 = 0$. Eqs. (4.4)–(4.6) imply, for θ small enough (depending on δ, $\mathbf{K}^*$, Y_f),

$$\begin{aligned} \theta^{2\mu} &\leqslant \lim_{\epsilon \leqslant \nu \to 0} \fint_{B(0,\theta)} \big|\Pi_\nu P_{\epsilon,\nu} - (\Pi_\nu P_{\epsilon,\nu})_{0,\theta}\big|^2\,dx \\ &= \lim_{\epsilon \leqslant \nu \to 0} \fint_{B(0,\theta)} |\Pi_\nu P_{\epsilon,\nu}|^2\,dx - \left|\fint_{B(0,\theta)} \Pi_\nu P_{\epsilon,\nu}\right|^2 = \fint_{B(0,\theta)} P_0^2\,dx - \left|\fint_{B(0,\theta)} P_0\right|^2 \\ &= \fint_{B(0,\theta)} \big|P_0 - (P_0)_{0,\theta}\big|^2\,dx \leqslant \theta^{2\mu'} \fint_{B(0,1)} P_0^2\,dx < \theta^{2\mu}. \end{aligned}$$

So we get $\theta^{2\mu} < \theta^{2\mu}$, which is impossible. Therefore we prove $(4.3)_1$.

Define $\hat{P} \equiv \theta^{-\mu}(\Pi_\nu P_{\epsilon,\nu} - (\Pi_\nu P_{\epsilon,\nu})_{0,\theta})$ and $\hat{p} \equiv \theta^{-\mu}(p_{\epsilon,\nu} - (\Pi_\nu P_{\epsilon,\nu})_{0,\theta})$. Then $(4.1)_{2,4}$ imply, for any smooth function ζ with support in $\nu(Y_m + j) \subset B(0,\theta) \cap \Omega_m^\nu$ for some $j \in \mathbb{Z}^3$,

$$\begin{aligned} &\epsilon^2 \int_{\nu(Y_m+j)} (\hat{p} - \hat{P}) \nabla \cdot (\mathbf{k}_\nu \nabla \zeta)\,dx \\ &\quad = \int_{\nu(Y_m+j)} \big(\epsilon^2 \mathbf{k}_\nu \nabla \hat{P} + \epsilon\theta^{-\mu} q_{\epsilon,\nu}\big)\nabla\zeta\,dx - \int_{\nu(Y_m+j)} \epsilon\theta^{-\mu} f_{\epsilon,\nu}\zeta\,dx. \end{aligned} \tag{4.7}$$

If ζ is the solution of

$$\begin{cases} \nabla \cdot (\mathbf{k}_\nu \nabla \zeta) = \hat{p} - \hat{P} & \text{in } \nu(Y_m + j) \\ \zeta = 0 & \text{on } \nu(\partial Y_m + j) \end{cases} \quad \text{for } j \in \mathbb{Z}^3, \tag{4.8}$$

then

$$c_1 \nu^{-1} \|\zeta\|_{L^2(\nu(Y_m+j))} \leqslant \|\nabla\zeta\|_{L^2(\nu(Y_m+j))} \leqslant c_2 \nu \|\hat{p} - \hat{P}\|_{L^2(\nu(Y_m+j))}, \tag{4.9}$$

where c_1, c_2 are independent of ν. If we take the solution ζ of (4.8) as the test function in (4.7), then (4.7) and (4.9) imply

$$\begin{aligned} \epsilon^2 \|\hat{p} - \hat{P}\|^2_{L^2(\nu(Y_m+j))} &\leqslant c\big\|\nu\epsilon\mathbf{k}_\nu \nabla\hat{P} + \nu\theta^{-\mu} q_{\epsilon,\nu}\big\|^2_{L^2(\nu(Y_m+j))} \\ &\quad + c\big\|\nu^2\theta^{-\mu} f_{\epsilon,\nu}\big\|^2_{L^2(\nu(Y_m+j))}, \end{aligned} \tag{4.10}$$

where c is independent of ϵ, ν. Therefore, by $(4.3)_1$ and (4.10),

$$\theta^{-2\mu} \fint_{B(0,\theta)\cap\Omega_m^\nu} \epsilon^2 \left|p_{\epsilon,\nu} - (\Pi_\nu P_{\epsilon,\nu})_{0,\theta}\right|^2 dx$$

$$= \fint_{B(0,\theta)\cap\Omega_m^\nu} |\epsilon\hat{p}|^2\, dx \leqslant c\theta^{-3} \int_{B(0,\theta)\cap\Omega_m^\nu} |\epsilon\hat{p} - \epsilon\hat{P}|^2\, dx + c\epsilon^2 \fint_{B(0,\theta)} |\hat{P}|^2\, dx$$

$$\leqslant c\theta^{-3} \int_{B(0,\theta)\cap\Omega_m^\nu} \left(\left|\nu\epsilon\mathbf{k}_\nu\nabla\hat{P} + \nu\theta^{-\mu}q_{\epsilon,\nu}\right|^2 + \left|\nu^2\theta^{-\mu}f_{\epsilon,\nu}\right|^2\right) dx + c\epsilon^2, \tag{4.11}$$

where c is independent of ϵ, ν. By (4.2) and energy method, $\|P_{\epsilon,\nu}\|_{H^1(B(0,\theta)\cap\Omega_f^\nu)}$ is bounded by a constant depending on θ and given data. Note $\epsilon \leqslant \nu \leqslant \epsilon_0$. If ϵ_0 is small enough, the right-hand side of (4.11) is smaller than 1. So $(4.3)_2$ follows. □

Lemma 4.2. *Under* A1–A2, *for any $\delta > 0$, there are constant $\theta \in (0,1)$ (depending on $\delta, \mathbf{K}^*, Y_f$) and constant $\epsilon_0 \in (0,1)$ (depending on θ, δ, d_4, d_5) such that if $\mathbf{K}_\epsilon, P_\epsilon, Q_\epsilon, F_\epsilon, \mathbf{k}_\epsilon, p_\epsilon, q_\epsilon, f_\epsilon$ satisfy*

$$\begin{cases} -\nabla\cdot(\mathbf{K}_\epsilon\nabla P_\epsilon + Q_\epsilon) = F_\epsilon & \text{in } B(0,1)\cap\Omega_f^\epsilon, \\ -\epsilon\nabla\cdot(\epsilon\mathbf{k}_\epsilon\nabla p_\epsilon + q_\epsilon) = f_\epsilon & \text{in } B(0,1)\cap\Omega_m^\epsilon, \\ (\mathbf{K}_\epsilon\nabla P_\epsilon + Q_\epsilon)\cdot\vec{\mathbf{n}}^\epsilon = \epsilon(\epsilon\mathbf{k}_\epsilon\nabla p_\epsilon + q_\epsilon)\cdot\vec{\mathbf{n}}^\epsilon & \text{on } B(0,1)\cap\partial\Omega_m^\epsilon, \\ P_\epsilon = p_\epsilon & \text{on } B(0,1)\cap\partial\Omega_m^\epsilon, \end{cases} \tag{4.12}$$

then, for all $\epsilon \leqslant \epsilon_0$ and k satisfying $\epsilon/\theta^k \leqslant \epsilon_0$,

$$\begin{cases} \fint_{B(0,\theta^k)} \left|\Pi_\epsilon P_\epsilon - (\Pi_\epsilon P_\epsilon)_{0,\theta^k}\right|^2 dx \leqslant \theta^{2k\mu} J_\epsilon^2, \\ \fint_{B(0,\theta^k)\cap\Omega_m^\epsilon} \epsilon^2\left|p_\epsilon - (\Pi_\epsilon P_\epsilon)_{0,\theta^k}\right|^2 dx \leqslant \theta^{2k\mu} J_\epsilon^2. \end{cases} \tag{4.13}$$

Here $\mathbf{K}^*$ is the matrix in (3.4), *d_4 and d_5 are defined in* A2, *$\mu \equiv 1 - \frac{3}{3+\delta}$, and*

$$J_\epsilon \equiv \|P_\epsilon\mathcal{X}_{\Omega_f^\epsilon} + \epsilon p_\epsilon\mathcal{X}_{\Omega_m^\epsilon}\|_{L^2(B(0,1))} + \|q_\epsilon\|_{L^{3+\delta}(B(0,1)\cap\Omega_m^\epsilon)}$$
$$+ \epsilon_0^{-1}\|Q_\epsilon\mathcal{X}_{\Omega_f^\epsilon}, F_\epsilon\mathcal{X}_{\Omega_f^\epsilon} + f_\epsilon\mathcal{X}_{\Omega_m^\epsilon}\|_{L^{3+\delta}(B(0,1))}. \tag{4.14}$$

Proof. We assume $J_\epsilon < \infty$, otherwise it is clear. This is done by induction on k. For $k=1$, we define $\hat{P}_\epsilon \equiv \frac{P_\epsilon}{J_\epsilon}$, $\hat{Q}_\epsilon \equiv \frac{Q_\epsilon}{J_\epsilon}$, $\hat{F}_\epsilon \equiv \frac{F_\epsilon}{J_\epsilon}$, $\hat{p}_\epsilon \equiv \frac{p_\epsilon}{J_\epsilon}$, $\hat{q}_\epsilon \equiv \frac{q_\epsilon}{J_\epsilon}$, $\hat{f}_\epsilon \equiv \frac{f_\epsilon}{J_\epsilon}$. Then $\mathbf{K}_\epsilon, \hat{P}_\epsilon, \hat{Q}_\epsilon, \hat{F}_\epsilon, \mathbf{k}_\epsilon, \hat{p}_\epsilon, \hat{q}_\epsilon, \hat{f}_\epsilon$ satisfy (4.12) and

$$\max\left\{\|\hat{P}_\epsilon\|_{L^2(B(0,1)\cap\Omega_f^\epsilon)}, \epsilon\|\hat{p}_\epsilon\|_{L^2(B(0,1)\cap\Omega_m^\epsilon)}, \|\hat{q}_\epsilon\|_{L^{3+\delta}(B(0,1)\cap\Omega_m^\epsilon)},\right.$$
$$\left.\epsilon_0^{-1}\|\hat{Q}_\epsilon\mathcal{X}_{\Omega_f^\epsilon}, \hat{F}_\epsilon\mathcal{X}_{\Omega_f^\epsilon} + \hat{f}_\epsilon\mathcal{X}_{\Omega_m^\epsilon}\|_{L^{3+\delta}(B(0,1))}\right\} \leqslant 1.$$

By Lemma 4.1 (in this case $\nu = \epsilon \leqslant \epsilon_0$),

$$\begin{cases} \displaystyle\fint_{B(0,\theta)} \left|\Pi_\epsilon \hat{P}_\epsilon - (\Pi_\epsilon \hat{P}_\epsilon)_{0,\theta}\right|^2 dx \leqslant \theta^{2\mu}, \\ \displaystyle\fint_{B(0,\theta)\cap\Omega_m^\epsilon} \epsilon^2 \left|\hat{p}_\epsilon - (\Pi_\epsilon \hat{P}_\epsilon)_{0,\theta}\right|^2 dx \leqslant \theta^{2\mu}. \end{cases}$$

This implies (4.13) for $k = 1$ case. Suppose (4.13) holds for some k satisfying $\epsilon/\theta^k \leqslant \epsilon_0$, we define

$$\begin{cases} \hat{\mathbf{K}}_\epsilon(x) \equiv \mathbf{K}_\epsilon(\theta^k x) \\ \hat{P}_\epsilon(x) \equiv J_\epsilon^{-1}\theta^{-k\mu}\left(P_\epsilon(\theta^k x) - (\Pi_\epsilon P_\epsilon)_{0,\theta^k}\right) \\ \hat{Q}_\epsilon(x) \equiv J_\epsilon^{-1}\theta^{k(1-\mu)} Q_\epsilon(\theta^k x) \\ \hat{F}_\epsilon(x) \equiv J_\epsilon^{-1}\theta^{k(2-\mu)} F_\epsilon(\theta^k x) \end{cases} \quad \text{in } B(0,1)\cap\Omega_f^\epsilon/\theta^k,$$

$$\begin{cases} \hat{\mathbf{k}}_\epsilon(x) \equiv \mathbf{k}_\epsilon(\theta^k x) \\ \hat{p}_\epsilon(x) \equiv J_\epsilon^{-1}\theta^{-k\mu}\left(p_\epsilon(\theta^k x) - (\Pi_\epsilon P_\epsilon)_{0,\theta^k}\right) \\ \hat{q}_\epsilon(x) \equiv J_\epsilon^{-1}\theta^{k(1-\mu)} q_\epsilon(\theta^k x) \\ \hat{f}_\epsilon(x) \equiv J_\epsilon^{-1}\theta^{k(2-\mu)} f_\epsilon(\theta^k x) \end{cases} \quad \text{in } B(0,1)\cap\Omega_m^\epsilon/\theta^k.$$

Then they satisfy

$$\begin{cases} -\nabla\cdot(\hat{\mathbf{K}}_\epsilon\nabla\hat{P}_\epsilon + \hat{Q}_\epsilon) = \hat{F}_\epsilon & \text{in } B(0,1)\cap\Omega_f^\epsilon/\theta^k, \\ -\epsilon\nabla\cdot(\epsilon\hat{\mathbf{k}}_\epsilon\nabla\hat{p}_\epsilon + \hat{q}_\epsilon) = \hat{f}_\epsilon & \text{in } B(0,1)\cap\Omega_m^\epsilon/\theta^k, \\ (\hat{\mathbf{K}}_\epsilon\nabla\hat{P}_\epsilon + \hat{Q}_\epsilon)\cdot\vec{\mathbf{n}} = \epsilon(\epsilon\hat{\mathbf{k}}_\epsilon\nabla\hat{p}_\epsilon + \hat{q}_\epsilon)\cdot\vec{\mathbf{n}} & \text{on } B(0,1)\cap\partial\Omega_m^\epsilon/\theta^k, \\ \hat{P}_\epsilon = \hat{p}_\epsilon & \text{on } B(0,1)\cap\partial\Omega_m^\epsilon/\theta^k, \end{cases}$$

where $\vec{\mathbf{n}}$ is the unit vector normal to $\partial\Omega_m^\epsilon/\theta^k$. By induction,

$$\max\left\{\|\hat{P}_\epsilon\|_{L^2(B(0,1)\cap\Omega_f^\epsilon/\theta^k)}, \epsilon\|\hat{p}_\epsilon\|_{L^2(B(0,1)\cap\Omega_m^\epsilon/\theta^k)}, \|\hat{q}_\epsilon\|_{L^{3+\delta}(B(0,1)\cap\Omega_m^\epsilon/\theta^k)}, \right.$$
$$\left.\epsilon_0^{-1}\|\hat{Q}_\epsilon\mathcal{X}_{\Omega_f^\epsilon/\theta^k}, \hat{F}_\epsilon\mathcal{X}_{\Omega_f^\epsilon/\theta^k} + \theta^{-k}\hat{f}_\epsilon\mathcal{X}_{\Omega_m^\epsilon/\theta^k}\|_{L^{3+\delta}(B(0,1))}\right\} \leqslant 1.$$

Note $\epsilon \leqslant \epsilon/\theta^k \leqslant \epsilon_0$. By Lemma 4.1 (in this case $\nu = \epsilon/\theta^k$),

$$\begin{cases} \displaystyle\fint_{B(0,\theta)} \left|\Pi_{\epsilon/\theta^k}\hat{P}_\epsilon - (\Pi_{\epsilon/\theta^k}\hat{P}_\epsilon)_{0,\theta}\right|^2 dx \leqslant \theta^{2\mu}, \\ \displaystyle\fint_{B(0,\theta)\cap\Omega_m^\epsilon/\theta^k} \epsilon^2\left|\hat{p}_\epsilon - (\Pi_{\epsilon/\theta^k}\hat{P}_\epsilon)_{0,\theta}\right|^2 dx \leqslant \theta^{2\mu}. \end{cases} \tag{4.15}$$

By Lemma 3.1,

$$\fint_{B(0,\theta)} \left|\Pi_{\epsilon/\theta^k}\hat{P}_\epsilon - (\Pi_{\epsilon/\theta^k}\hat{P}_\epsilon)_{0,\theta}\right|^2 dx = \fint_{B(0,\theta^{k+1})} \frac{|\Pi_\epsilon P_\epsilon - (\Pi_\epsilon P_\epsilon)_{0,\theta^{k+1}}|^2}{J_\epsilon^2\theta^{2k\mu}}\,dx, \tag{4.16}$$

$$\fint_{B(0,\theta)\cap\Omega_m^\epsilon/\theta^k} \left|\hat{p}_\epsilon - (\Pi_{\epsilon/\theta^k}\hat{P}_\epsilon)_{0,\theta}\right|^2 dx = \fint_{B(0,\theta^{k+1})\cap\Omega_m^\epsilon} \frac{|p_\epsilon - (\Pi_\epsilon P_\epsilon)_{0,\theta^{k+1}}|^2}{J_\epsilon^2\theta^{2k\mu}}\,dx. \tag{4.17}$$

Eqs. (4.15)–(4.17) imply Eq. (4.13) for $k+1$ case. □

Lemma 4.3. *Under* A1–A3, *for any* $\delta > 0$, *there is* $\epsilon_0 > 0$ (*depending on* δ, d_4, d_5) *such that the solutions of* (1.1) *satisfy, for all* $\epsilon \leqslant \epsilon_0$,

$$\begin{aligned}
&[P_\epsilon]_{C^{0,\mu}(B(0,\frac{1}{2})\cap\Omega_f^\epsilon)} + \epsilon^{1-\mu}|||\nabla P_\epsilon|||_{C^{0,\mu}(B(0,\frac{1}{2})\cap\Omega_f^\epsilon)} + \epsilon^{3-\mu}|||\nabla p_\epsilon|||_{C^{0,\mu}(B(0,\frac{1}{2})\cap\Omega_m^\epsilon)} \\
&\quad\leqslant c\big(J_\epsilon + \epsilon^{1-\mu}|||Q_\epsilon|||_{C^{0,\mu}(\Omega_f^\epsilon)} + \epsilon^{2-\mu}|||q_\epsilon|||_{C^{0,\mu}(\Omega_m^\epsilon)}\big),
\end{aligned} \tag{4.18}$$

$$\begin{aligned}
&[P_\epsilon]_{C^{0,\mu}(B(0,\frac{1}{2})\cap\Omega_f^\epsilon)} + \epsilon^{1-\mu}|||\nabla P_\epsilon|||_{C^{0,\mu}(B(0,\frac{1}{2})\cap\Omega_f^\epsilon)} + \epsilon^{2-\mu}|||\nabla p_\epsilon|||_{C^{0,\mu}(B(0,\frac{1}{2})\cap\Omega_m^\epsilon)} \\
&\quad\leqslant c\big(J_\epsilon + \epsilon^{1-\mu}|||Q_\epsilon|||_{C^{0,\mu}(\Omega_f^\epsilon)} + \epsilon^{1-\mu}|||q_\epsilon|||_{C^{0,\mu}(\Omega_m^\epsilon)}\big).
\end{aligned} \tag{4.19}$$

Here d_4 *and* d_5 *are defined in* A2, $\mu \equiv 1 - \frac{3}{3+\delta}$, J_ϵ *is defined in* (4.14), *and the constant* c *is independent of* ϵ.

Proof. We denote by c a constant independent of ϵ. Lemma 4.2 implies that the solutions of (1.1) satisfy

$$\begin{cases} \fint_{B(0,r)} \left|\Pi_\epsilon P_\epsilon - (\Pi_\epsilon P_\epsilon)_{0,r}\right|^2 dx \leqslant cr^{2\mu}J_\epsilon^2 \\ \fint_{B(0,r)\cap\Omega_m^\epsilon} \epsilon^2\left|p_\epsilon - (\Pi_\epsilon P_\epsilon)_{0,r}\right|^2 dx \leqslant cr^{2\mu}J_\epsilon^2 \end{cases} \quad \text{for } r \geqslant \epsilon/\epsilon_0. \tag{4.20}$$

Case 1. To show (4.18), we define

$$\hat{J}_\epsilon \equiv J_\epsilon + \epsilon^{1-\mu}|||Q_\epsilon|||_{C^{0,\mu}(\Omega_f^\epsilon)} + \epsilon^{2-\mu}|||q_\epsilon|||_{C^{0,\mu}(\Omega_m^\epsilon)},$$

$$\begin{cases} \hat{\mathbf{K}}_\epsilon(x) \equiv \mathbf{K}_\epsilon(\epsilon x) \\ \hat{P}_\epsilon(x) \equiv \hat{J}_\epsilon^{-1}\epsilon^{-\mu}\big(P_\epsilon(\epsilon x) - (\Pi_\epsilon P_\epsilon)_{0,2\epsilon/\epsilon_0}\big) \\ \hat{Q}_\epsilon(x) \equiv \hat{J}_\epsilon^{-1}\epsilon^{1-\mu}Q_\epsilon(\epsilon x) \\ \hat{F}_\epsilon(x) \equiv \hat{J}_\epsilon^{-1}\epsilon^{2-\mu}F_\epsilon(\epsilon x) \end{cases} \quad \text{in } B\left(0, \frac{2}{\epsilon_0}\right) \cap \Omega_f^\epsilon/\epsilon,$$

$$\begin{cases} \hat{\mathbf{k}}_\epsilon(x) \equiv \mathbf{k}_\epsilon(\epsilon x) \\ \hat{p}_\epsilon(x) \equiv \hat{J}_\epsilon^{-1}\epsilon^{-\mu}\big(p_\epsilon(\epsilon x) - (\Pi_\epsilon P_\epsilon)_{0,2\epsilon/\epsilon_0}\big) \\ \hat{q}_\epsilon(x) \equiv \hat{J}_\epsilon^{-1}\epsilon^{1-\mu}q_\epsilon(\epsilon x) \\ \hat{f}_\epsilon(x) \equiv \hat{J}_\epsilon^{-1}\epsilon^{2-\mu}f_\epsilon(\epsilon x) \end{cases} \quad \text{in } B\left(0, \frac{2}{\epsilon_0}\right) \cap \Omega_m^\epsilon/\epsilon.$$

Then they satisfy

$$\begin{cases} -\nabla\cdot(\hat{\mathbf{K}}_\epsilon\nabla\hat{P}_\epsilon+\hat{Q}_\epsilon)=\hat{F}_\epsilon & \text{in } B\left(0,\dfrac{2}{\epsilon_0}\right)\cap\Omega_f^\epsilon/\epsilon, \\ -\epsilon\nabla\cdot(\epsilon\hat{\mathbf{k}}_\epsilon\nabla\hat{p}_\epsilon+\hat{q}_\epsilon)=\hat{f}_\epsilon & \text{in } B\left(0,\dfrac{2}{\epsilon_0}\right)\cap\Omega_m^\epsilon/\epsilon, \\ (\hat{\mathbf{K}}_\epsilon\nabla\hat{P}_\epsilon+\hat{Q}_\epsilon)\cdot\vec{\mathbf{n}}=\epsilon(\epsilon\hat{\mathbf{k}}_\epsilon\nabla\hat{p}_\epsilon+\hat{q}_\epsilon)\cdot\vec{\mathbf{n}} & \text{on } B\left(0,\dfrac{2}{\epsilon_0}\right)\cap\partial\Omega_m^\epsilon/\epsilon, \\ \hat{P}_\epsilon=\hat{p}_\epsilon & \text{on } B\left(0,\dfrac{2}{\epsilon_0}\right)\cap\partial\Omega_m^\epsilon/\epsilon, \end{cases} \tag{4.21}$$

where $\vec{\mathbf{n}}$ is the unit vector normal to $\partial\Omega_m^\epsilon/\epsilon$. Taking $r=\frac{2\epsilon}{\epsilon_0}$ in (4.20), we see

$$\begin{aligned} &\|\hat{P}_\epsilon\mathcal{X}_{\Omega_f^\epsilon/\epsilon}+\epsilon\hat{p}_\epsilon\mathcal{X}_{\Omega_m^\epsilon/\epsilon}\|_{L^2(B(0,\frac{2}{\epsilon_0}))}+\|\hat{Q}_\epsilon\|_{C^{0,\mu}(B(0,\frac{2}{\epsilon_0})\cap\Omega_f^\epsilon/\epsilon)} \\ &\quad+\|\epsilon\hat{q}_\epsilon\|_{C^{0,\mu}(B(0,\frac{2}{\epsilon_0})\cap\Omega_m^\epsilon/\epsilon)}+\|\hat{F}_\epsilon\mathcal{X}_{\Omega_f^\epsilon/\epsilon}+\hat{f}_\epsilon\mathcal{X}_{\Omega_m^\epsilon/\epsilon}\|_{L^{3+\delta}(B(0,\frac{2}{\epsilon_0}))}\leqslant c. \end{aligned}$$

By Lemma 3.4,

$$\|\hat{P}_\epsilon\|_{C^{1,\mu}(B(0,\frac{1}{\epsilon_0})\cap\Omega_f^\epsilon/\epsilon)}+\epsilon^2\|\hat{p}_\epsilon\|_{C^{1,\mu}(B(0,\frac{1}{\epsilon_0})\cap\Omega_m^\epsilon/\epsilon)}\leqslant c. \tag{4.22}$$

Eq. (4.22) then implies

$$\fint_{B(0,r)}\left|\Pi_\epsilon P_\epsilon-(\Pi_\epsilon P_\epsilon)_{0,r}\right|^2dx\leqslant cr^{2\mu}\hat{J}_\epsilon^2\quad\text{for } r\leqslant\epsilon/\epsilon_0. \tag{4.23}$$

Eq. (4.18) follows from (4.20), (4.22), (4.23), and Theorem 1.2 on p. 70 [10].

Case 2. To show (4.19), we follow the idea of Case 1. Define

$$\hat{J}_\epsilon\equiv J_\epsilon+\epsilon^{1-\mu}|||Q_\epsilon|||_{C^{0,\mu}(\Omega_f^\epsilon)}+\epsilon^{1-\mu}|||q_\epsilon|||_{C^{0,\mu}(\Omega_m^\epsilon)}$$

and define $\hat{\mathbf{K}}_\epsilon, \hat{P}_\epsilon, \hat{Q}_\epsilon, \hat{F}_\epsilon, \hat{\mathbf{k}}_\epsilon, \hat{p}_\epsilon, \hat{q}_\epsilon, \hat{f}_\epsilon$ exactly same as those in Case 1. Then they satisfy (4.21) and

$$\begin{aligned} &\|\hat{P}_\epsilon\mathcal{X}_{\Omega_f^\epsilon/\epsilon}+\epsilon\hat{p}_\epsilon\mathcal{X}_{\Omega_m^\epsilon/\epsilon}\|_{L^2(B(0,\frac{2}{\epsilon_0}))}+\|\hat{Q}_\epsilon\|_{C^{0,\mu}(B(0,\frac{2}{\epsilon_0})\cap\Omega_f^\epsilon/\epsilon)} \\ &\quad+\|\hat{q}_\epsilon\|_{C^{0,\mu}(B(0,\frac{2}{\epsilon_0})\cap\Omega_m^\epsilon/\epsilon)}+\left\|\hat{F}_\epsilon\mathcal{X}_{\Omega_f^\epsilon/\epsilon}+\epsilon^{-1}\hat{f}_\epsilon\mathcal{X}_{\Omega_m^\epsilon/\epsilon}\right\|_{L^{3+\delta}(B(0,\frac{2}{\epsilon_0}))}\leqslant c. \end{aligned}$$

By Lemma 3.5,

$$\|\hat{P}_\epsilon\|_{C^{1,\mu}(B(0,\frac{1}{\epsilon_0})\cap\Omega_f^\epsilon/\epsilon)}+\epsilon\|\hat{p}_\epsilon\|_{C^{1,\mu}(B(0,\frac{1}{\epsilon_0})\cap\Omega_m^\epsilon/\epsilon)}\leqslant c. \tag{4.24}$$

Eq. (4.24) implies (4.23) holds for Case 2. Eq. (4.19) follows from (4.20), (4.24), and [10]. □

Assume A1–A4 hold. Because of the periodicity assumption A4 and the periodic boundary condition, one can extend the equations in (1.1) to a larger domain $\widetilde{\Omega}$ so that the original boundary $\partial\Omega$ is in the interior region of the new domain $\widetilde{\Omega}$. Then arguing as above, we see that Lemma 4.3 also holds around the boundary $\partial\Omega$. Then by Lemma 3.2 and the interior estimate of Lemma 4.3, we obtain the estimates (2.1), (2.2). So we prove Theorem 2.1.

5. Uniform Lipschitz estimate

In this section we prove Theorem 2.2. By A4 and periodic boundary conditions, the solution of (1.1) with $\int_\Omega \Pi_\epsilon P_\epsilon\,dx = 0$ satisfies, for $i = 1, 2, 3$,

$$\begin{cases} -\nabla\cdot\big(\mathbf{K}_\epsilon\nabla P_\epsilon^{\dagger_i} + \mathbf{K}_\epsilon^{\dagger_i}\nabla P_\epsilon(x+5\epsilon\vec{e}_i) + Q_\epsilon^{\dagger_i}\big) = F_\epsilon^{\dagger_i} & \text{in } \Omega_f^\epsilon, \\ -\epsilon\nabla\cdot\big(\epsilon\mathbf{k}_\epsilon\nabla p_\epsilon^{\dagger_i} + \epsilon\mathbf{k}_\epsilon^{\dagger_i}\nabla p_\epsilon(x+5\epsilon\vec{e}_i) + q_\epsilon^{\dagger_i}\big) = f_\epsilon^{\dagger_i} & \text{in } \Omega_m^\epsilon, \\ \big(\mathbf{K}_\epsilon\nabla P_\epsilon^{\dagger_i} + \mathbf{K}_\epsilon^{\dagger_i}\nabla P_\epsilon(x+5\epsilon\vec{e}_i) + Q_\epsilon^{\dagger_i}\big)\cdot\vec{\mathbf{n}}^\epsilon & \\ \quad = \epsilon\big(\epsilon\mathbf{k}_\epsilon\nabla p_\epsilon^{\dagger_i} + \epsilon\mathbf{k}_\epsilon^{\dagger_i}\nabla p_\epsilon(x+5\epsilon\vec{e}_i) + q_\epsilon^{\dagger_i}\big)\cdot\vec{\mathbf{n}}^\epsilon & \text{on } \partial\Omega_m^\epsilon, \\ P_\epsilon^{\dagger_i} = p_\epsilon^{\dagger_i} & \text{on } \partial\Omega_m^\epsilon, \\ \displaystyle\int_\Omega \Pi_\epsilon P_\epsilon^{\dagger_i}\,dx = 0, & \end{cases} \tag{5.1}$$

with periodic boundary conditions on $\partial\Omega$. See Section 2 for the definition of functions $\mathbf{K}_\epsilon^{\dagger_i}$, $P_\epsilon^{\dagger_i}$, $Q_\epsilon^{\dagger_i}$, $F_\epsilon^{\dagger_i}$, $\mathbf{k}_\epsilon^{\dagger_i}$, $p_\epsilon^{\dagger_i}$, $q_\epsilon^{\dagger_i}$, $f_\epsilon^{\dagger_i}$. Lemma 3.2 and (2.2) of Theorem 2.1 imply

Lemma 5.1. *Under* A1–A4, *the solution of* (5.1) *satisfies, for* $i = 1, 2, 3$,

$$\begin{aligned} &\big\|P_\epsilon^{\dagger_i}\big\|_{C^{0,\mu}(\Omega_f^\epsilon)} + \epsilon^{1-\mu}\big|\big|\big|\nabla P_\epsilon^{\dagger_i}\big|\big|\big|_{C^{0,\mu}(\Omega_f^\epsilon)} + \epsilon^{2-\mu}\big|\big|\big|\nabla p_\epsilon^{\dagger_i}\big|\big|\big|_{C^{0,\mu}(\Omega_m^\epsilon)} \\ &\quad\leqslant c\big(\big\|Q_\epsilon^{\dagger_i}, F_\epsilon^{\dagger_i}, \mathbf{K}_\epsilon^{\dagger_i}\nabla P_\epsilon\big\|_{L^{3+\delta}(\Omega_f^\epsilon)} + \big\|q_\epsilon^{\dagger_i}, f_\epsilon^{\dagger_i}, \epsilon\mathbf{k}_\epsilon^{\dagger_i}\nabla p_\epsilon\big\|_{L^{3+\delta}(\Omega_m^\epsilon)} \\ &\qquad + \epsilon^{1-\mu}\big|\big|\big|Q_\epsilon^{\dagger_i}, \mathbf{K}_\epsilon^{\dagger_i}\nabla P_\epsilon\big|\big|\big|_{C^{0,\mu}(\Omega_f^\epsilon)} + \big|\big|\big|\epsilon^{1-\mu}q_\epsilon^{\dagger_i}, \epsilon^{2-\mu}\mathbf{k}_\epsilon^{\dagger_i}\nabla p_\epsilon\big|\big|\big|_{C^{0,\mu}(\Omega_m^\epsilon)}\big), \end{aligned}$$

where $\delta > 0$, $\mu \equiv 1 - \frac{3}{3+\delta}$ *and the constant* c *is independent of* ϵ.

Let Γ_i, $i = 1, 2, 3$, denote one of the faces of the cube $5Y$, that is,

$$\Gamma_i \equiv \big\{x = (x_1, x_2, x_3) \in \partial(5Y)\colon\ x_i = 0\big\}.$$

Lemma 5.2. *If* $\zeta_i \in W^{2-\frac{1}{3+\delta},3+\delta}(\Gamma_i)$, $\eta_i \in W^{1-\frac{1}{3+\delta},3+\delta}(\Gamma_i)$ *satisfy, for* $i = 1, 2, 3$,

$$\begin{cases} \zeta_1(0,0,x_3) - \zeta_2(0,0,x_3) - \zeta_1(0,5,x_3) + \zeta_2(5,0,x_3) = 0, \\ \zeta_2(x_1,0,5) + \zeta_3(x_1,0,0) - \zeta_2(x_1,0,0) - \zeta_3(x_1,5,0) = 0, \\ \zeta_1(0,x_2,5) + \zeta_3(0,x_2,0) - \zeta_1(0,x_2,0) - \zeta_3(5,x_2,0) = 0, \\ \zeta_3(0,0,0) + \zeta_1(0,5,5) + \zeta_2(0,0,5) - \zeta_1(0,5,0) - \zeta_3(5,5,0) - \zeta_2(0,0,0) = 0, \end{cases} \tag{5.2}$$

$$\begin{cases} \eta_1(0,0,x_3) - \partial_{x_1}\zeta_2(0,0,x_3) - \eta_1(0,5,x_3) + \partial_{x_1}\zeta_2(5,0,x_3) = 0, \\ \partial_{x_1}\zeta_2(0,0,5) + \partial_{x_1}\zeta_3(0,0,0) - \partial_{x_1}\zeta_2(0,0,0) - \partial_{x_1}\zeta_3(0,5,0) = 0, \\ \eta_1(0,x_2,5) + \partial_{x_1}\zeta_3(0,x_2,0) - \eta_1(0,x_2,0) - \partial_{x_1}\zeta_3(5,x_2,0) = 0, \\ \partial_{x_1}\zeta_3(0,0,0) + \eta_1(0,5,5) + \partial_{x_1}\zeta_2(0,0,5) - \eta_1(0,5,0) \\ \quad - \partial_{x_1}\zeta_3(5,5,0) - \partial_{x_1}\zeta_2(0,0,0) = 0, \end{cases} \tag{5.3}$$

$$\begin{cases} \partial_{x_2}\zeta_1(0,0,x_3) - \eta_2(0,0,x_3) - \partial_{x_2}\zeta_1(0,5,x_3) + \eta_2(5,0,x_3) = 0, \\ \eta_2(x_1,0,5) + \partial_{x_2}\zeta_3(x_1,0,0) - \eta_2(x_1,0,0) - \partial_{x_2}\zeta_3(x_1,5,0) = 0, \\ \partial_{x_2}\zeta_1(0,0,5) + \partial_{x_2}\zeta_3(0,0,0) - \partial_{x_2}\zeta_1(0,0,0) - \partial_{x_2}\zeta_3(5,0,0) = 0, \\ \partial_{x_2}\zeta_3(0,0,0) + \partial_{x_2}\zeta_1(0,5,5) + \eta_2(0,0,5) - \partial_{x_2}\zeta_1(0,5,0) \\ \quad - \partial_{x_2}\zeta_3(5,5,0) - \eta_2(0,0,0) = 0, \end{cases} \tag{5.4}$$

$$\begin{cases} \partial_{x_3}\zeta_1(0,0,0) - \partial_{x_3}\zeta_2(0,0,0) - \partial_{x_3}\zeta_1(0,5,0) + \partial_{x_3}\zeta_2(5,0,0) = 0, \\ \partial_{x_3}\zeta_1(0,0,5) - \partial_{x_3}\zeta_2(0,0,5) - \partial_{x_3}\zeta_1(0,5,5) + \partial_{x_3}\zeta_2(5,0,5) = 0, \\ \partial_{x_3}\zeta_2(x_1,0,5) + \eta_3(x_1,0,0) - \partial_{x_3}\zeta_2(x_1,0,0) - \eta_3(x_1,5,0) = 0, \\ \partial_{x_3}\zeta_1(0,x_2,5) + \eta_3(0,x_2,0) - \partial_{x_3}\zeta_1(0,x_2,0) - \eta_3(5,x_2,0) = 0, \\ \eta_3(0,0,0) + \partial_{x_3}\zeta_1(0,5,5) + \partial_{x_3}\zeta_2(0,0,5) - \partial_{x_3}\zeta_1(0,5,0) \\ \quad - \eta_3(5,5,0) - \partial_{x_3}\zeta_2(0,0,0) = 0, \end{cases} \tag{5.5}$$

where $x_i \in [0,5]$ and $\delta > 0$, then there is a continuous function ψ defined on $5Y$ such that

$$\begin{cases} \psi(x + 5\vec{e}_i) - \psi(x) = \zeta_i(x) \\ \partial_{x_i}\psi(x + 5\vec{e}_i) - \partial_{x_i}\psi(x) = \eta_i(x) \end{cases} \quad \text{on } \Gamma_i, \tag{5.6}$$

$$\|\psi\|_{W^{2,3+\delta}(5Y)} \leqslant c \sum_{i=1}^{3} \left(\|\zeta_i\|_{W^{2-\frac{1}{3+\delta},3+\delta}(\Gamma_i)} + \|\eta_i\|_{W^{1-\frac{1}{3+\delta},3+\delta}(\Gamma_i)} \right), \tag{5.7}$$

where c is a constant.

Proof of Lemma 5.2 will be given in Section 6. One may note that if ψ is a smooth function in $5Y$, then ζ_i, η_i (for $i = 1,2,3$) defined as (5.6) satisfy conditions (5.2)–(5.5). In a cube $5Y + j \subset \Omega/\epsilon$ for some $j \in \mathbb{Z}^3$, we define, in $(5Y + j) \cap \Omega_f^\epsilon/\epsilon$,

$$\hat{\mathbf{K}}_\epsilon(x) \equiv \mathbf{K}_\epsilon(\epsilon x), \qquad \hat{P}_\epsilon(x) \equiv P_\epsilon(\epsilon x), \qquad \hat{Q}_\epsilon(x) \equiv \epsilon Q_\epsilon(\epsilon x), \qquad \hat{F}_\epsilon(x) \equiv \epsilon^2 F_\epsilon(\epsilon x),$$

and, in $(5Y + j) \cap \Omega_m^\epsilon/\epsilon$,

$$\hat{\mathbf{k}}_\epsilon(x) \equiv \mathbf{k}_\epsilon(\epsilon x), \qquad \hat{p}_\epsilon(x) \equiv p_\epsilon(\epsilon x), \qquad \hat{q}_\epsilon(x) \equiv \epsilon q_\epsilon(\epsilon x), \qquad \hat{f}_\epsilon(x) \equiv \epsilon^2 f(\epsilon x).$$

By (1.1), if we let $\ell_i(x) \equiv P_\epsilon^{\dagger_i}(\epsilon x)$ for $i = 1,2,3$, then

$$\begin{cases} -\nabla \cdot (\hat{\mathbf{K}}_\epsilon \nabla \hat{P}_\epsilon + \hat{Q}_\epsilon) = \hat{F}_\epsilon & \text{in } (5Y + j) \cap \Omega_f^\epsilon/\epsilon, \\ -\epsilon \nabla \cdot (\epsilon \hat{\mathbf{k}}_\epsilon \nabla \hat{p}_\epsilon + \hat{q}_\epsilon) = \hat{f}_\epsilon & \text{in } (5Y + j) \cap \Omega_m^\epsilon/\epsilon, \\ (\hat{\mathbf{K}}_\epsilon \nabla \hat{P}_\epsilon + \hat{Q}_\epsilon) \cdot \vec{\mathbf{n}}_y = \epsilon(\epsilon \hat{\mathbf{k}}_\epsilon \nabla \hat{p}_\epsilon + \hat{q}_\epsilon) \cdot \vec{\mathbf{n}}_y & \text{on } (5Y + j) \cap \partial\Omega_m^\epsilon/\epsilon, \\ \hat{P}_\epsilon = \hat{p}_\epsilon & \text{on } (5Y + j) \cap \partial\Omega_m^\epsilon/\epsilon, \\ \hat{P}_\epsilon(x + 5\vec{e}_i) - \hat{P}_\epsilon(x) = 5\epsilon \ell_i(x) & \text{on } \Gamma_i + j \text{ for } i = 1,2,3, \\ \partial_i \hat{P}_\epsilon(x + 5\vec{e}_i) - \partial_i \hat{P}_\epsilon(x) = 5\epsilon \partial_i \ell(x) & \text{on } \Gamma_i + j \text{ for } i = 1,2,3. \end{cases} \tag{5.8}$$

Lemma 5.3. *There exists a function $\mathcal{U}_\epsilon \in W^{2,3+\delta}(5Y + j)$ for $j \in \mathbb{Z}^3$ satisfying*

$$\begin{cases} \mathcal{U}_\epsilon(x + 5\vec{e}_i) - \mathcal{U}_\epsilon(x) = \ell_i(x) \\ \partial_i \mathcal{U}_\epsilon(x + 5\vec{e}_i) - \partial_i \mathcal{U}_\epsilon(x) = \partial_i \ell_i(x) \end{cases} \quad \text{on } \Gamma_i + j \text{ for } i = 1,2,3,$$

$$\|\mathcal{U}_\epsilon\|_{W^{2,3+\delta}(5Y+j)} \leqslant c\left(\sum_{i=1}^{3}\|\ell_i\|_{L^\infty((7Y+j-\vec{1})\cap\Omega_f^\epsilon/\epsilon)} + \epsilon|||F_\epsilon|||_{L^{3+\delta}(\epsilon(7Y+j-\vec{1})\cap\Omega_f^\epsilon)} + \sum_{i=1}^{3}\left|\left|\left| Q_\epsilon, \epsilon\mathbf{K}_\epsilon^{\dagger_i}\nabla P_\epsilon\right|\right|\right|_{W^{1,3+\delta}(\epsilon(7Y+j-\vec{1})\cap\Omega_f^\epsilon)}\right).$$

Here $\ell_i(x) \equiv P_\epsilon^{\dagger_i}(\epsilon x)$, $\vec{1} \equiv (1,1,1)$ *is a vector with all components* 1, $\delta > 0$, *and* c *is a constant independent of* ϵ.

Proof. By $(5.1)_1$ and regularity result [11],

$$\sum_{i=1}^{3}\left(\|\ell_i\|_{W^{2-\frac{1}{3+\delta},3+\delta}(\Gamma_i+j)} + \|\partial_i\ell_i\|_{W^{1-\frac{1}{3+\delta},3+\delta}(\Gamma_i+j)}\right) \leqslant c\epsilon|||F_\epsilon|||_{L^{3+\delta}(\epsilon(7Y+j-\vec{1})\cap\Omega_f^\epsilon)} + c\sum_{i=1}^{3}\left(\|\ell_i\|_{L^\infty((7Y+j-\vec{1})\cap\Omega_f^\epsilon/\epsilon)} + \left|\left|\left| Q_\epsilon, \epsilon\mathbf{K}_\epsilon^{\dagger_i}\nabla P_\epsilon\right|\right|\right|_{W^{1,3+\delta}(\epsilon(7Y+j-\vec{1})\cap\Omega_f^\epsilon)}\right),$$

where c is a constant independent of ϵ. If we define $\zeta_i(x) \equiv \ell_i(x+j)$ and $\eta_i \equiv \partial_i\ell_i(x+j)$ for $i=1,2,3$, then ζ_i and η_i satisfy (5.2)–(5.5) of Lemma 5.2. So we obtain $\psi(x)$ by Lemma 5.2. This lemma follows if we take $\mathcal{U}_\epsilon(x+j) = \psi(x)$. $\square$

If $\check{P}_\epsilon\mathcal{X}_{(5Y+j)\cap\Omega_f^\epsilon/\epsilon} + \check{p}_\epsilon\mathcal{X}_{(5Y+j)\cap\Omega_m^\epsilon/\epsilon} \equiv \hat{P}_\epsilon\mathcal{X}_{(5Y+j)\cap\Omega_f^\epsilon/\epsilon} + \hat{p}_\epsilon\mathcal{X}_{(5Y+j)\cap\Omega_m^\epsilon/\epsilon} - 5\epsilon\mathcal{U}_\epsilon$, then, by (5.8),

$$\begin{cases} -\nabla\cdot(\hat{\mathbf{K}}_\epsilon\nabla\check{P}_\epsilon + 5\epsilon\hat{\mathbf{K}}_\epsilon\nabla\mathcal{U}_\epsilon + \hat{Q}_\epsilon) = \hat{F}_\epsilon & \text{in } (5Y+j)\cap\Omega_f^\epsilon/\epsilon, \\ -\epsilon\nabla\cdot\left(\epsilon\hat{\mathbf{k}}_\epsilon\nabla\check{p}_\epsilon + 5\epsilon^2\hat{\mathbf{k}}_\epsilon\nabla\mathcal{U}_\epsilon + \hat{q}_\epsilon\right) = \hat{f}_\epsilon & \text{in } (5Y+j)\cap\Omega_m^\epsilon/\epsilon, \\ (\hat{\mathbf{K}}_\epsilon\nabla\check{P}_\epsilon + 5\epsilon\hat{\mathbf{K}}_\epsilon\nabla\mathcal{U}_\epsilon + \hat{Q}_\epsilon)\cdot\vec{\mathbf{n}}_y & \\ \quad = \epsilon\left(\epsilon\hat{\mathbf{k}}_\epsilon\nabla\check{p}_\epsilon + 5\epsilon^2\hat{\mathbf{k}}_\epsilon\nabla\mathcal{U}_\epsilon + \hat{q}_\epsilon\right)\cdot\vec{\mathbf{n}}_y & \text{on } (5Y+j)\cap\partial\Omega_m^\epsilon/\epsilon, \\ \check{P}_\epsilon = \check{p}_\epsilon & \text{on } (5Y+j)\cap\partial\Omega_m^\epsilon/\epsilon, \\ \check{P}_\epsilon(x+5\vec{e}_i) - \check{P}_\epsilon(x) = 0 & \text{on } \Gamma_i + j \text{ for } i=1,2,3, \\ \partial_i\check{P}_\epsilon(x+5\vec{e}_i) - \partial_i\check{P}_\epsilon(x) = 0 & \text{on } \Gamma_i + j \text{ for } i=1,2,3. \end{cases} \tag{5.9}$$

By Lemmas 3.6, 5.1, 5.3, periodic assumption A4, and [11], the solution of (5.9) with

$$\int_{(5Y+j)\cap\Omega_f^\epsilon/\epsilon} \check{P}_\epsilon\,dx = 0$$

exists uniquely and satisfies

$$\|\hat{P}_\epsilon\|_{W^{2,3+\delta}((3Y+j+\vec{1})\cap\Omega_f^\epsilon/\epsilon)} + \sup_{\substack{k\in\mathbb{Z}^3 \\ Y_m+k\subset(3Y+j+\vec{1})\cap\Omega_m^\epsilon/\epsilon}} \epsilon\|\hat{p}_\epsilon\|_{W^{2,3+\delta}(Y_m+k)} \leqslant c\epsilon\Big(\|\nabla\mathcal{U}_\epsilon\|_{W^{1,3+\delta}(5Y+j)} + \epsilon|||F_\epsilon|||_{L^{3+\delta}(\epsilon(5Y+j)\cap\Omega_f^\epsilon)} + |||f_\epsilon|||_{L^{3+\delta}(\epsilon(5Y+j)\cap\Omega_m^\epsilon)} + |||Q_\epsilon|||_{W^{1,3+\delta}(\epsilon(5Y+j)\cap\Omega_f^\epsilon)} + \sup_{\substack{k\in\mathbb{Z}^3 \\ \epsilon(Y_m+k)\subset\Omega_m^\epsilon}} |||q_\epsilon|||_{W^{1,3+\delta}(\epsilon(Y_m+k))}\Big)$$

$$\leqslant c\epsilon \sum_{i=1}^{3}\Big(\| Q_\epsilon^{\dagger_i}, F_\epsilon^{\dagger_i}, \mathbf{K}_\epsilon^{\dagger_i}\nabla P_\epsilon \|_{L^{3+\delta}(\Omega_f^\epsilon)} + \| q_\epsilon^{\dagger_i}, f_\epsilon^{\dagger_i}, \epsilon\mathbf{k}_\epsilon^{\dagger_i}\nabla p_\epsilon \|_{L^{3+\delta}(\Omega_m^\epsilon)}$$

$$+ \big\| Q_\epsilon, \epsilon\mathbf{K}_\epsilon^{\dagger_i}\nabla P_\epsilon \big\|_{W^{1,3+\delta}(\epsilon(7Y+j-\vec{1})\cap\Omega_f^\epsilon)} + \epsilon^{1-\mu}\big\| Q_\epsilon^{\dagger_i}, \mathbf{K}_\epsilon^{\dagger_i}\nabla P_\epsilon \big\|_{C^{0,\mu}(\Omega_f^\epsilon)}$$

$$+ \sup_{\substack{k\in\mathbb{Z}^3 \\ \epsilon(Y_m+k)\subset\Omega_m^\epsilon}} \big(\| q_\epsilon \|_{W^{1,3+\delta}(\epsilon(Y_m+k))} + \big\| \epsilon^{1-\mu} q_\epsilon^{\dagger_i}, \epsilon^{2-\mu}\mathbf{k}_\epsilon^{\dagger_i}\nabla p_\epsilon \big\|_{C^{0,\mu}(\epsilon(Y_m+k))}\big)$$

$$+ \epsilon \| F_\epsilon \|_{L^{3+\delta}(\epsilon(7Y+j-\vec{1})\cap\Omega_f^\epsilon)} + \| f_\epsilon \|_{L^{3+\delta}(\epsilon(5Y+j)\cap\Omega_m^\epsilon)}\Big), \tag{5.10}$$

where c is a constant independent of ϵ.

Same reasoning as that at the end of Section 4, by the periodic boundary condition, we extend the equations in (1.1) to a larger domain $\widetilde{\Omega}$ so that the boundary $\partial\Omega$ is inside the new domain $\widetilde{\Omega}$. Then by the interior estimate (5.10), we obtain the estimate (2.3) in the whole domain Ω. So we prove Theorem 2.2.

6. Proof of Lemma 5.2

Let $D \equiv [0,5]\times[0,5]$, $\widetilde{\Gamma}_1 \equiv \{0\}\times[0,5]$, $\widetilde{\Gamma}_2 \equiv [0,5]\times\{0\}$. $\vec{e}_i$ $(i=1,2)$ is the unit vector in coordinate direction x_i in $\mathbb{R}^2$. By trace theorem in [19], approximation method, and a modification of the reasoning in [6], we have the following result:

Lemma 6.1. *If ψ is continuous on ∂D, $\psi \in W^{2-\frac{2}{3+\delta},3+\delta}(\widetilde{\Gamma}_i \cup \widetilde{\Gamma}_i + 5\vec{e}_i)$, and $\Psi_i \in W^{1-\frac{2}{3+\delta},3+\delta}(\widetilde{\Gamma}_i \cup \widetilde{\Gamma}_i + 5\vec{e}_i)$ for $i=1,2$, and if ψ and Ψ_i satisfy compatibility condition on ∂D (that is, $\frac{d}{dx}\psi\vec{\mathbf{t}} + \sum_{i=1}^{2}\mathcal{X}_{\widetilde{\Gamma}_i\cup\widetilde{\Gamma}_i+5\vec{e}_i}\Psi_i\vec{e}_i$ is a continuous vector function on ∂D), then ψ can be extended to domain D such that $\partial_{x_i}\psi|_{\widetilde{\Gamma}_i\cup\widetilde{\Gamma}_i+5\vec{e}_i} = \Psi_i$ and*

$$\|\psi\|_{W^{2-\frac{1}{3+\delta},3+\delta}(D)} \leqslant c\sum_{i=1}^{2}\big(\|\psi\|_{W^{2-\frac{2}{3+\delta},3+\delta}(\widetilde{\Gamma}_i\cup\widetilde{\Gamma}_i+5\vec{e}_i)} + \|\Psi_i\|_{W^{1-\frac{2}{3+\delta},3+\delta}(\widetilde{\Gamma}_i\cup\widetilde{\Gamma}_i+5\vec{e}_i)}\big).$$

Here $\delta > 0$, $\vec{\mathbf{t}}$ is the tangential unit vector on ∂D, and $\mathcal{X}_{\widetilde{\Gamma}_i\cup\widetilde{\Gamma}_i+5\vec{e}_i}$ is the characteristic function on $\widetilde{\Gamma}_i \cup \widetilde{\Gamma}_i + 5\vec{e}_i$.

Definition 6.1. For given functions ψ on $\partial(5Y)$ and Ψ_i on $\Gamma_i \cup \Gamma_i + 5\vec{e}_i$ for $i=1,2,3$, we say that ψ and Ψ_i satisfy compatibility condition on $\partial(5Y)$ if $\nabla_\tau\psi + \sum_{i=1}^{3}\mathcal{X}_{\Gamma_i\cup\Gamma_i+5\vec{e}_i}\Psi_i\vec{e}_i$ is a continuous vector function on $\partial(5Y)$. Here $\nabla_\tau\psi$ is the tangential derivative vector (that is, $\nabla_\tau\psi = \sum_{j\neq i}\frac{\partial\psi}{\partial x_j}\vec{e}_j$ on $\Gamma_i \cup \Gamma_i + 5\vec{e}_i$ for $i=1,2,3$) and $\mathcal{X}_{\Gamma_i\cup\Gamma_i+5\vec{e}_i}$ is the characteristic function on $\Gamma_i \cup \Gamma_i + 5\vec{e}_i$.

Now we prove Lemma 5.2. This includes four steps. The first three steps are to find ψ defined on $\partial(5Y)$ and Ψ_i defined on $\Gamma_i \cup \Gamma_i + 5\vec{e}_i$ for $i=1,2,3$ such that they satisfy compatibility condition on $\partial(5Y)$. The function Ψ_i can be regarded as $\partial_{x_i}\psi$ for $i=1,2,3$.

Step 1. Find $\psi, \Psi_1, \Psi_2, \Psi_3$ on the vertices of $5Y$. It is not difficult to see that if

$$\begin{cases} g_1 - g_5 - g_2 + g_8 = 0, \\ g_3 - g_6 - g_4 + g_7 = 0, \\ g_5 + g_9 - g_6 - g_{10} = 0, \\ g_1 + g_9 - g_3 - g_{11} = 0, \\ g_9 + g_2 + g_5 - g_4 - g_{12} - g_6 = 0 \end{cases} \tag{6.1}$$

is satisfied, then the following system

$$\begin{pmatrix} -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \\ g_6 \\ g_7 \\ g_8 \\ g_9 \\ g_{10} \\ g_{11} \\ g_{12} \end{pmatrix} \tag{6.2}$$

is solvable. In order to obtain $(5.6)_1$, ψ on the vertices of $5Y$ has to satisfy (6.2) with

$$\begin{cases} (f_1, f_2, \ldots, f_8) = \big(\psi(0,0,5), \psi(0,5,5), \psi(5,5,5), \psi(5,0,5), \\ \qquad \psi(0,0,0), \psi(0,5,0), \psi(5,5,0), \psi(5,0,0)\big), \\ (g_1, g_2, \ldots, g_{12}) = \big(\zeta_1(0,0,5), \zeta_1(0,5,5), \zeta_1(0,0,0), \zeta_1(0,5,0), \\ \qquad \zeta_2(0,0,5), \zeta_2(0,0,0), \zeta_2(5,0,0), \zeta_2(5,0,5), \\ \qquad \zeta_3(0,0,0), \zeta_3(0,5,0), \zeta_3(5,0,0), \zeta_3(5,5,0)\big). \end{cases}$$

Condition (6.1) holds because of (5.2). Eqs. (6.1)–(6.2) imply ψ on the vertices of $5Y$ is solvable.

As mentioned above, Ψ_1 is regarded as $\partial_{x_1}\psi$. Function Ψ_1 on the vertices of $5Y$ has to satisfy (6.2) with

$$\begin{cases} (f_1, f_2, \ldots, f_8) = \big(\Psi_1(0,0,5), \Psi_1(0,5,5), \Psi_1(5,5,5), \Psi_1(5,0,5), \\ \qquad \Psi_1(0,0,0), \Psi_1(0,5,0), \Psi_1(5,5,0), \Psi_1(5,0,0)\big), \\ (g_1, g_2, \ldots, g_{12}) = \big(\eta_1(0,0,5), \eta_1(0,5,5), \eta_1(0,0,0), \eta_1(0,5,0), \\ \qquad \partial_{x_1}\zeta_2(0,0,5), \partial_{x_1}\zeta_2(0,0,0), \partial_{x_1}\zeta_2(5,0,0), \partial_{x_1}\zeta_2(5,0,5), \\ \qquad \partial_{x_1}\zeta_3(0,0,0), \partial_{x_1}\zeta_3(0,5,0), \partial_{x_1}\zeta_3(5,0,0), \partial_{x_1}\zeta_3(5,5,0)\big). \end{cases}$$

The first four equations in (6.2) are from $(5.6)_2$ for $i = 1$. The next four equations in (6.2) are the horizontal difference of $\partial_{x_1}\psi$. For example, the difference between $\partial_{x_1}\psi(0,5,5)$ and $\partial_{x_1}\psi(0,0,5)$ is $\partial_{x_1}\zeta_2(0,0,5)$, the difference between $\partial_{x_1}\psi(0,5,0)$ and $\partial_{x_1}\psi(0,0,0)$ is $\partial_{x_1}\zeta_2(0,0,0)$, and so on. The last four equations in (6.2) are the vertical difference of $\partial_{x_1}\psi$. For example, the difference between $\partial_{x_1}\psi(0,0,5)$ and $\partial_{x_1}\psi(0,0,0)$ is $\partial_{x_1}\zeta_3(0,0,0)$, the difference between $\partial_{x_1}\psi(0,5,5)$ and $\partial_{x_1}\psi(0,5,0)$ is $\partial_{x_1}\zeta_3(0,5,0)$, and so on. Condition (5.3) implies that condition (6.1) holds. Eqs. (6.1)–(6.2) imply function Ψ_1 on the vertices of $5Y$ is solvable.

Again we regard Ψ_2 as $\partial_{x_2}\psi$. Function Ψ_2 on the vertices of $5Y$ needs to satisfy (6.2) with

$$\begin{cases} (f_1, \ldots, f_8) = \big(\Psi_2(0,0,5), \Psi_2(0,5,5), \Psi_2(5,5,5), \Psi_2(5,0,5), \\ \qquad \Psi_2(0,0,0), \Psi_2(0,5,0), \Psi_2(5,5,0), \Psi_2(5,0,0)\big), \\ (g_1, \ldots, g_{12}) = \big(\partial_{x_2}\zeta_1(0,0,5), \partial_{x_2}\zeta_1(0,5,5), \partial_{x_2}\zeta_1(0,0,0), \partial_{x_2}\zeta_1(0,5,0), \\ \qquad \eta_2(0,0,5), \eta_2(0,0,0), \eta_2(5,0,0), \eta_2(5,0,5), \\ \qquad \partial_{x_2}\zeta_3(0,0,0), \partial_{x_2}\zeta_3(0,5,0), \partial_{x_2}\zeta_3(5,0,0), \partial_{x_2}\zeta_3(5,5,0)\big). \end{cases}$$

The first four equations in (6.2) are the horizontal difference of $\partial_{x_2}\psi$. For example, the difference between $\partial_{x_2}\psi(5,0,5)$ and $\partial_{x_2}\psi(0,0,5)$ is $\partial_{x_2}\zeta_1(0,0,5)$, the difference between $\partial_{x_2}\psi(5,0,0)$ and $\partial_{x_2}\psi(0,0,0)$ is $\partial_{x_2}\zeta_1(0,0,0)$, and so on. The middle four equations in (6.2) are from $(5.6)_2$ for $i=2$. The last four equations in (6.2) are the vertical difference of $\partial_{x_2}\psi$. For example, the difference between $\partial_{x_2}\psi(0,0,5)$ and $\partial_{x_2}\psi(0,0,0)$ is $\partial_{x_2}\zeta_3(0,0,0)$, and so on. Condition (5.4) implies (6.1). Eqs. (6.1)–(6.2) imply Ψ_2 on the vertices of $5Y$ is solvable.

Function Ψ_3 on the vertices of $5Y$ has to satisfy (6.2) with

$$\begin{cases} (f_1,\dots,f_8)=\big(\Psi_3(0,0,5),\Psi_3(0,5,5),\Psi_3(5,5,5),\Psi_3(5,0,5),\\ \qquad \Psi_3(0,0,0),\Psi_3(0,5,0),\Psi_3(5,5,0),\Psi_3(5,0,0)\big),\\ (g_1,\dots,g_{12})=\big(\partial_{x_3}\zeta_1(0,0,5),\partial_{x_3}\zeta_1(0,5,5),\partial_{x_3}\zeta_1(0,0,0),\partial_{x_3}\zeta_1(0,5,0),\\ \qquad \partial_{x_3}\zeta_2(0,0,5),\partial_{x_3}\zeta_2(0,0,0),\partial_{x_3}\zeta_2(5,0,0),\partial_{x_3}\zeta_2(5,0,5),\\ \qquad \eta_3(0,0,0),\eta_3(0,5,0),\eta_3(5,0,0),\eta_3(5,5,0)\big). \end{cases}$$

The first eight equations in (6.2) are the horizontal difference of $\partial_{x_3}\psi$. The last four equations in (6.2) are from $(5.6)_2$ for $i=3$. Condition (5.5) implies (6.1). Eqs. (6.1)–(6.2) imply Ψ_3 on the vertices of $5Y$ is solvable.

Step 2. Find $\psi,\Psi_1,\Psi_2,\Psi_3$ on the edges of $5Y$. We note the system

$$\begin{pmatrix} -1 & 1 & 0 & 0\\ 0 & 0 & -1 & 1\\ 1 & 0 & -1 & 0\\ 0 & 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} f_1\\ f_2\\ f_3\\ f_4\end{pmatrix}=\begin{pmatrix} g_1\\ g_2\\ g_3\\ g_4\end{pmatrix} \tag{6.3}$$

is solvable if $g_1-g_2+g_3-g_4=0$. In a horizontal square D with vertices $a=(5,0,x_3), b=(5,5,x_3), c=(0,0,x_3)$, and $d=(0,5,x_3)$ for $x_3\in(0,5)$, we find ψ,Ψ_1,Ψ_2 on the vertices a,b,c,d of ∂D by solving

$$\begin{pmatrix} -1 & 1 & 0 & 0\\ 0 & 0 & -1 & 1\\ 1 & 0 & -1 & 0\\ 0 & 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} \psi(a) & \Psi_1(a) & \Psi_2(a)\\ \psi(b) & \Psi_1(b) & \Psi_2(b)\\ \psi(c) & \Psi_1(c) & \Psi_2(c)\\ \psi(d) & \Psi_1(d) & \Psi_2(d)\end{pmatrix}=\begin{pmatrix} \zeta_2(a) & \partial_{x_1}\zeta_2(a) & \eta_2(a)\\ \zeta_2(c) & \partial_{x_1}\zeta_2(c) & \eta_2(c)\\ \zeta_1(c) & \eta_1(c) & \partial_{x_2}\zeta_1(c)\\ \zeta_1(d) & \eta_1(d) & \partial_{x_2}\zeta_1(d)\end{pmatrix}. \tag{6.4}$$

Eq. (6.4) is solvable because of $(5.2)_1$, $(5.3)_1$, $(5.4)_1$, and (6.3). The constant 4×4 matrix in the left-hand side of (6.4) has rank 3. From Step 1, we see that ψ,Ψ_1,Ψ_2 on the vertices a,b,c,d of ∂D when $x_3\in\{0,5\}$ also satisfy $(5.2)_1$, $(5.3)_1$, $(5.4)_1$. So we can find smooth functions ψ,Ψ_1,Ψ_2 defined on the four line segments $\{(5,0,x_3),\ x_3\in[0,5]\}$, $\{(5,5,x_3),\ x_3\in[0,5]\}$, $\{(0,0,x_3),\ x_3\in[0,5]\}$, $\{(0,5,x_3),\ x_3\in[0,5]\}$ such that (6.4) holds. Moreover, at the end points of the four line segments, $\partial_{x_3}\psi(x_1,x_2,x_3)=\Psi_3(x_1,x_2,x_3)$ for $x_i\in\{0,5\}$ for $i=1,2,3$.

In a vertical square D with vertices $a=(x_1,0,5)$, $b=(x_1,5,5)$, $c=(x_1,0,0)$, and $d=(x_1,5,0)$ for $x_1\in(0,5)$, we find ψ,Ψ_2,Ψ_3 on the vertices a,b,c,d of ∂D by solving

$$\begin{pmatrix} -1 & 1 & 0 & 0\\ 0 & 0 & -1 & 1\\ 1 & 0 & -1 & 0\\ 0 & 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} \psi(a) & \Psi_2(a) & \Psi_3(a)\\ \psi(b) & \Psi_2(b) & \Psi_3(b)\\ \psi(c) & \Psi_2(c) & \Psi_3(c)\\ \psi(d) & \Psi_2(d) & \Psi_3(d)\end{pmatrix}=\begin{pmatrix} \zeta_2(a) & \eta_2(a) & \partial_{x_3}\zeta_2(a)\\ \zeta_2(c) & \eta_2(c) & \partial_{x_3}\zeta_2(c)\\ \zeta_3(c) & \partial_{x_2}\zeta_3(c) & \eta_3(c)\\ \zeta_3(d) & \partial_{x_2}\zeta_3(d) & \eta_3(d)\end{pmatrix}. \tag{6.5}$$

Eq. (6.5) is solvable by $(5.2)_2$, $(5.4)_2$, $(5.5)_3$, and (6.3). The constant 4×4 matrix in (6.5) has rank 3. From Step 1, we see that the values of ψ,Ψ_2,Ψ_3 on the vertices a,b,c,d of ∂D when $x_1\in\{0,5\}$ satisfy $(5.2)_2$, $(5.4)_2$, $(5.5)_3$. So we can find smooth functions ψ, Ψ_2, Ψ_3 defined on the four line segments

$\{(x_1,0,5),\ x_1\in(0,5)\}$, $\{(x_1,5,5),\ x_1\in(0,5)\}$, $\{(x_1,0,0),\ x_1\in(0,5)\}$, $\{(x_1,5,0),\ x_1\in(0,5)\}$ such that (6.5) holds. Moreover, at the end points of the four line segments, $\partial_{x_1}\psi(x_1,x_2,x_3)=\Psi_1(x_1,x_2,x_3)$ for $x_i\in\{0,5\}$ for $i=1,2,3$.

In a vertical square D with vertices $a=(0,x_2,5)$, $b=(5,x_2,5)$, $c=(0,x_2,0)$, and $d=(5,x_2,0)$ for $x_2\in(0,5)$, we find ψ,Ψ_1,Ψ_3 on the vertices a,b,c,d of ∂D by solving

$$\begin{pmatrix} -1 & 1 & 0 & 0\\ 0 & 0 & -1 & 1\\ 1 & 0 & -1 & 0\\ 0 & 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} \psi(a) & \Psi_1(a) & \Psi_3(a)\\ \psi(b) & \Psi_1(b) & \Psi_3(b)\\ \psi(c) & \Psi_1(c) & \Psi_3(c)\\ \psi(d) & \Psi_1(d) & \Psi_3(d) \end{pmatrix}=\begin{pmatrix} \zeta_1(a) & \eta_1(a) & \partial_{x_3}\zeta_1(a)\\ \zeta_1(c) & \eta_1(c) & \partial_{x_3}\zeta_1(c)\\ \zeta_3(c) & \partial_{x_1}\zeta_3(c) & \eta_3(c)\\ \zeta_3(d) & \partial_{x_1}\zeta_3(d) & \eta_3(d) \end{pmatrix}. \tag{6.6}$$

Eq. (6.6) is solvable by $(5.2)_3$, $(5.3)_3$, $(5.5)_4$, and (6.3). The constant matrix in (6.6) has rank 3 and the values of ψ, Ψ_1, Ψ_3 at $x_2\in\{0,5\}$ satisfy $(5.2)_3$, $(5.3)_3$, $(5.5)_4$ by Step 1. One can find smooth functions ψ,Ψ_1,Ψ_3 on the four line segments $\{(0,x_2,5),\ x_2\in(0,5)\}$, $\{(5,x_2,5),\ x_2\in(0,5)\}$, $\{(0,x_2,0),\ x_2\in(0,5)\}$, $\{(5,x_2,0),\ x_2\in(0,5)\}$ such that (6.6) holds. Moreover, at the end points of the four line segments, $\partial_{x_2}\psi(x_1,x_2,x_3)=\Psi_2(x_1,x_2,x_3)$ for $x_i\in\{0,5\}$ for $i=1,2,3$.

Step 3. Find $\psi,\Psi_1,\Psi_2,\Psi_3$ on the surface of $\partial(5Y)$. In the square Γ_3 with vertices $a=(0,0,0)$, $b=(0,5,0)$, $c=(5,0,0)$, and $d=(5,5,0)$, we see that, by Step 2, (1) ψ is continuous on $\partial\Gamma_3$, (2) $\Psi_1|_{\overline{ab}\cup\overline{cd}}$, $\Psi_2|_{\overline{ac}\cup\overline{bd}}$, and ψ on each line segment $\overline{ab}$, $\overline{cd}$, $\overline{ac}$, $\overline{bd}$ are smooth, and (3) ψ,Ψ_1,Ψ_2 satisfy compatibility conditions on boundary $\partial\Gamma_3$. By Lemma 6.1, one can extend ψ to the square Γ_3 (same notation for the extended function) such that $\partial_{x_1}\psi|_{\overline{ab}\cup\overline{cd}}=\Psi_1|_{\overline{ab}\cup\overline{cd}}$, $\partial_{x_2}\psi|_{\overline{ac}\cup\overline{bd}}=\Psi_2|_{\overline{ac}\cup\overline{bd}}$, and

$$\|\psi\|_{W^{2-\frac{1}{3+\delta},3+\delta}(\Gamma_3)}\leqslant c\sum_{i=1}^{3}\bigl(\|\zeta_i\|_{W^{2-\frac{1}{3+\delta},3+\delta}(\Gamma_i)}+\|\eta_i\|_{W^{1-\frac{1}{3+\delta},3+\delta}(\Gamma_i)}\bigr). \tag{6.7}$$

By Step 2, we also see that Ψ_3 is continuous on $\partial\Gamma_3$ and smooth on each line segment $\overline{ab},\overline{cd},\overline{ac},\overline{bd}$. By trace theorem [12] and approximation method, we can extend Ψ_3 to the whole square Γ_3 (same notation for the extended function) and

$$\|\Psi_3\|_{W^{1-\frac{1}{3+\delta},3+\delta}(\Gamma_3)}\leqslant c\sum_{i=1}^{3}\bigl(\|\zeta_i\|_{W^{2-\frac{1}{3+\delta},3+\delta}(\Gamma_i)}+\|\eta_i\|_{W^{1-\frac{1}{3+\delta},3+\delta}(\Gamma_i)}\bigr). \tag{6.8}$$

Let $\Gamma_3+5\vec{e}_3$ be a square with vertices $a_1=(0,0,5), b_1=(0,5,5), c_1=(5,0,5), d_1=(5,5,5)$ and define $\psi(x+5\vec{e}_3)\equiv\psi(x)+\zeta_3(x)$ for $x\in\Gamma_3$. By Step 2, we see that ψ,Ψ_1,Ψ_2 satisfy $\partial_{x_1}\psi|_{\overline{a_1b_1}\cup\overline{c_1d_1}}=\Psi_1|_{\overline{a_1b_1}\cup\overline{c_1d_1}}$, $\partial_{x_2}\psi|_{\overline{a_1c_1}\cup\overline{b_1d_1}}=\Psi_2|_{\overline{a_1c_1}\cup\overline{b_1d_1}}$, and compatibility conditions on the boundary $\partial\Gamma_3+5\vec{e}_3$. Define $\Psi_3(x+5\vec{e}_3)\equiv\Psi_3(x)+\eta_3(x)$ for $x\in\Gamma_3$ and obtain Ψ_3 on the square $\Gamma_3+5\vec{e}_3$. We also see ψ,Ψ_3 on $\Gamma_3+5\vec{e}_3$ satisfy (6.7) and (6.8). By a similar reasoning as above, we can construct functions ψ,Ψ_1,Ψ_2 on the other faces of $5Y$ in such a way that functions $\psi,\Psi_1,\Psi_2,\Psi_3$ have the following properties:

1. $\psi\in W^{2-\frac{1}{3+\delta},3+\delta}(\Gamma_i\cup\Gamma_i+5\vec{e}_i)$, $\Psi_i\in W^{1-\frac{1}{3+\delta},3+\delta}(\Gamma_i\cup\Gamma_i+5\vec{e}_i)$ for $i=1,2,3$, and ψ is a continuous function on $\partial(5Y)$.
2. $\psi,\Psi_1,\Psi_2,\Psi_3$ satisfy compatibility condition on $\partial(5Y)$.
3. $\psi,\Psi_i,\zeta_i,\eta_i$ satisfy

$$\begin{cases} \psi(x+5\vec{e}_i)-\psi(x)=\zeta_i(x)\\ \Psi_i(x+5\vec{e}_i)-\Psi_i(x)=\eta_i(x) \end{cases}\quad \text{on } \Gamma_i \text{ for } i=1,2,3.$$

4. There is a constant c such that

$$\sum_{i=1}^{3}\left(\|\psi\|_{W^{2-\frac{1}{3+\delta},3+\delta}(\Gamma_i\cup\Gamma_i+5\vec{e}_i)}+\|\Psi_i\|_{W^{1-\frac{1}{3+\delta},3+\delta}(\Gamma_i\cup\Gamma_i+5\vec{e}_i)}\right) \leqslant c\sum_{i=1}^{3}\left(\|\zeta_i\|_{W^{2-\frac{1}{3+\delta},3+\delta}(\Gamma_i)}+\|\eta_i\|_{W^{1-\frac{1}{3+\delta},3+\delta}(\Gamma_i)}\right). \tag{6.9}$$

Step 4. By [6] and (6.9), we can extend ψ to $5Y$ such that (1) $\partial_{x_i}\psi|_{\Gamma_i\cup\Gamma_i+5\vec{e}_i}=\Psi_i$ for $i=1,2,3$ and (2) Eqs. (5.6)–(5.7) hold. So Lemma 5.2 holds.

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