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More on the Magnus–Derek game

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ABSTRACT

We consider the so called Magnus–Derek game, which is a two-person game played on a round table with n positions. The two players are called Magnus and Derek. Initially there is a token placed at position 0. In each round Magnus chooses a positive integer $m \leq n/2$ as the distance of the targeted position from his current position for the token to move, and Derek decides a direction, clockwise or counterclockwise, to move the token. The goal of Magnus is to maximize the total number of positions visited, while Derek's is to minimize this number. If both players play optimally, we prove that Magnus, the maximizer, can achieve his goal in $O(n)$ rounds, which improves a previous result with $O(n \log n)$ rounds. Then we consider a modified version of the Magnus–Derek game, where one of the players reveals his moves in advance and the other player plays optimally. In this case we prove that it is NP-hard for Derek to achieve his goal if Magnus reveals his moves in advance. On the other hand, Magnus has an advantage to occupy all positions. We also consider the circumstance that both players play randomly, and we show that the expected time to visit all positions is $O(n \log n)$.

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1. Introduction

The Magnus–Derek game was first introduced by Nedev and Muthukrishnan [5]. The game is played on a round table with n positions and a token is placed at position 0 initially. For convenience, we label the positions with elements in $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$, clockwise consecutively. Suppose the current position is i. In a round, Magnus chooses a positive integer m, where $m \leq \frac{n}{2}$ for the token to move, and Derek choose a direction, either $+1$ (clockwise) or -1 (counterclockwise) for the token to move. Then the token is moved to position $(i+m) \bmod n$ or $(i-m) \bmod n$ according to Derek's decision. In the game, Magnus wants to visit as many positions as possible, while Derek wants to minimize the number of positions visited. This game can be used to model a mobile agent for distributed computing and network maintenance task. We refer to [5] for more related references.

Nedev [4], and Nedev and Muthukrishnan [5] showed that Magnus could visit all positions in $n-1$ steps if $n = 2^k$ for some nonnegative integer k, and for other cases, Magnus could visit $f^*(n) = (p-1)n/p$ positions, where p is the smallest odd prime factor for n. The round numbers needed for these cases are listed as follows:

- If $n = 2^k p$, where p is a prime and k is a nonnegative integer, then Magnus needs $O(p^2 + n)$ rounds.
- If n is a prime, then Magnus needs $O(n^2)$ rounds.
- Otherwise, Magnus needs $O(\frac{n^2}{p})$ rounds, where p is the smallest odd prime factor for n.

Later, Hurkens et al. [1] reduced the bound down to $O(n \log n)$ rounds and showed that Derek could always limit the number of visited positions to $f^*(n) = (p-1)n/p$. In this paper, we improve the bound on the rounds further to $O(n)$.

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Consider the situation in a ring network, Derek plays the role of an adversary and tries to reduce the visited positions in order to perform some malicious acts in the network, and Magnus plays the role of an agent in the network and tries to visit as many positions as possible to prevent malicious acts. We can modify the game in two ways: (1) Magnus predetermines a sequence of magnitudes, and Derek tries to design appropriate responses to minimize the number of positions that Magnus can visit. This is an open problem asked by Nedev and Muthukrishnan [5]. (2) Derek predetermines a sequence of directions, and Magnus tries to design appropriate response to visit as many positions as possible. In the first case, we prove that it is NP-hard [6] for Derek to minimize the number of positions that Magnus can visit and answer the above mentioned open question. For the second case, we show that Magnus can visit all of the positions. Furthermore, we consider the case that both players play randomly, that is, they choose their moves in every round uniformly at random. In this case, both players have no effective strategy and just adopt the random strategy. This is somewhat like performing a random walk on the n positions. We show that the expected number of rounds to visit all of the n positions is $O(n \log n)$, which is similar to the Coupon collection problem[3].

Throughout this paper, we assume that both players know the factors of n and all of the arithmetic operations are under $\mathbb{Z}_n$ unless stated otherwise. We organize the rest of the paper as follows. In Section 2, we prove that Magnus can visit the maximum number of possible positions in $O(n)$ rounds. In Sections 3 and 4, we investigate how a player can achieve the best possible result when he knows his rival's moves beforehand. In Section 5, we consider both players playing randomly.

2. Visit $f^*(n)$ positions in $O(n)$ rounds

In this section we give a new strategy for Magnus to visit $f^*(n)$ positions in $O(n)$ rounds. Previous results show that when n is prime this problem can be the hardest. For this case, Nedev and Muthukrishnan [5] showed that Magnus could visit $f^*(n)$ positions in $O(n^2)$ rounds. Hurkens et al. [1] reduced it to $O(n \log n)$ rounds. We show that Magnus only needs $O(n)$ rounds to visit $f^*(n)$ positions. We adopt the idea of Hurkens et al. with some modification to obtain a better bound. We first focus on the case when n is an odd prime and then extend it for general n.

Let A and B be two subsets of $\mathbb{Z}_n$, and define $A + B = \{a + b \mid a \in A, b \in B\}$.

Definition 1. Let $n \geq 3$ be an odd integer. For any two elements $a, b \in \mathbb{Z}_n$, the midpoint of a and b, denoted as $Mid(a, b)$, is $(a+b)/2$ if $a + b$ is even; $(a + b + n)/2 \bmod n$ otherwise. If S is a subset of $\mathbb{Z}_n$, define $\mathrm{MID}(S) = \{Mid(a, b) \mid a, b \in S\}$, $\mathrm{SUM}(S) = \{a + b \mid a, b \in S\}$ and $SUM^k(S) = \{a + b \mid a, b \in SUM^{k-1}(S)\}$.

By the definition we have the following immediate fact.

Fact 1. *If S is a proper subset of $\mathbb{Z}_n$ and $SUM(S) = \mathbb{Z}_n$, then any $x \in \mathbb{Z}_n$ is the midpoint of some elements $a, b \in S$, i.e., $x = Mid(a, b)$ for a, b with $a + b = 2x$.*

The following theorem is a very useful tool in our proofs.

Theorem 2 (*Cauchy-Davenport [2]*)**.** *If p is a prime, and A, B are two non-empty subsets of $\mathbb{Z}_p$, then*

$$|A + B| \geq \min\{p, |A| + |B| - 1\}.$$

Now we are ready to prove our result.

Lemma 3. *Assume S_0 is a subset of $\mathbb{Z}_n$ and $\lceil \frac{n}{2^{k-1}} \rceil \geq |S_0| > \lceil \frac{n}{2^k} \rceil$ for some k, where n is a prime and $1 \leq k \leq \log n$. Let $S_i = SUM(S_{i-1})$ for $i \geq 1$. Then $S_k = SUM^k(S_0) = \mathbb{Z}_n$.*

Proof. We prove the lemma by induction on k, where k satisfies $\lceil \frac{n}{2^{k-1}} \rceil \geq |S_0| > \lceil \frac{n}{2^k} \rceil$.

Basis: When $k = 1$, we have $|S_0| > \lceil \frac{n}{2} \rceil$. By Theorem 2, we have $|S_0 + S_0| \geq \min\{n, |S_0| + |S_0| - 1\} \geq n$, so $S_1 = SUM(S_0) = S_0 + S_0 = \mathbb{Z}_n$.

Inductive Step: Assume the lemma is true for $k = m - 1$, that is, if $S_0 > \lceil \frac{n}{2^k} \rceil = \lceil \frac{n}{2^{m-1}} \rceil$, then $S_{m-1} = \mathbb{Z}_n$. Now we consider the case for $k = m$. We have $|S_0| > \lceil \frac{n}{2^m} \rceil$, which implies $|S_0| \geq \lceil \frac{n}{2^m} \rceil + 1$. Then $|S_1| = |SUM(S_0)|$. If $S_1 = \mathbb{Z}_n$, then we are done. Suppose not. By Theorem 2, we have $|S_1| \geq 2|S_0| - 1 \geq 2(\lceil \frac{n}{2^m} \rceil + 1) - 1 = 2\lceil \frac{n}{2^m} \rceil + 1 > \lceil \frac{n}{2^{m-1}} \rceil$. By the induction hypothesis, we have $S_m = SUM^{m-1}(S_1) = \mathbb{Z}_n$. Thus, it holds for the case $k = m$. □

Theorem 4. *If n is a prime, then Magnus can visit $f^*(n) = n - 1$ positions in $2n$ rounds.*

Proof. Let C_0 be the set of unvisited positions, which is $\mathbb{Z}_n$ initially. By Lemma 3, we know $SUM(C_0) = \mathbb{Z}_n$ if $|C_0| > n/2$. By Fact 1, it implies that any position can be the middle point of 2 unvisited positions in C_0. Thus, as long as $|C_0| > n/2$, Magnus can occupy a new position in each round.

In general for $\lceil \frac{n}{2^{k-1}} \rceil \geq |C_0| > \lceil \frac{n}{2^k} \rceil$, $1 \leq k \leq \log n$, we claim that Magnus can occupy a new position in C_0 in every k rounds. The theorem follows by the claim, since

$$\sum_{k=1}^{\log n} k \left\lfloor \frac{n}{2^k} \right\rfloor \leq 2n.$$

We have shown the basis case ($k = 1$) of the claim. Now assume the claim holds up to $k - 1$. Now consider the case when $\lceil \frac{n}{2^{k-1}} \rceil \geq |C_0| > \lceil \frac{n}{2^k} \rceil$. Let $C_1 = MID(C_0)$. Note that $|C_1| = |C_0 + C_0| > \lceil \frac{n}{2^{k-1}} \rceil$. It is clear that $SUM^{k-1}(C_1) = \mathbb{Z}_n$, by

Lemma 3. By the induction hypothesis, we know Magnus can visit a new position in C_1 in every $k-1$ rounds. Then from a position in C_1, Magnus can visit a new position in C_0 in another round, since every element in C_1 is the middle point of two elements $a, b \in C_0$, where if $a = b$, then $a, b \in C_1$, which implies Magnus may visit a new position in C_0 in at most k rounds. This completes the proof of the claim. The remaining one unvisited position is not reachable for Magnus when Derek plays optimally. Thus the theorem holds. □

As in [5], we use $C(l, d, s) = \{s + i \cdot d \mid 0 \le i < l\}$ to denote a set of l positions starting from s and the distance between each pair of adjacent positions in the set is d.

Suppose that $n = mp$ is an odd positive integer and p is the smallest prime factor of n. Let $C_j = C(m, p, j) \subset \mathbb{Z}_n, j \in \mathbb{Z}_p$. We have the following general property.

Lemma 5. *Let $S_0 = C_i \cup R$ for some $i \in \mathbb{Z}_p$, where $R \subset \mathbb{Z}_n$ and $R \cap C_i = \emptyset$, and $S_i = SUM\,(S_{i-1})$ for $i \ge 1$. If $\lceil \frac{p}{2^{k-1}} \rceil \ge l > \lceil \frac{p}{2^k} \rceil$ for some k, where $1 \le k \le \log p$ and l is the number of $C_j, j \ne i$, intersecting with R, then $S_{k+1} = \mathbb{Z}_n$.*

Proof. For convenience, let $\mathcal{C}$ be the collection $\{C_j | j \ne i, C_j \cap R \ne \emptyset\}$ and $|\mathcal{C}| = l$. Let $S' = \{j \mid C_j \in \mathcal{C}\}$. By Lemma 3, we have $SUM^k(S') = \mathbb{Z}_p$. Note that $\{a\} + C_i = C_{(a+i) \bmod p} \subseteq SUM(\{a\} \cup C_i)$. $SUM^k(S') = \mathbb{Z}_p$ implies that $\mathbb{Z}_n \subseteq SUM^{k+1}(S_0)$. Thus $S_{k+1} = SUM^{k+1}(S_0) = \mathbb{Z}_n$. □

Let u be an odd integer. Hurkens et al. [1] (Lemma 3.2) proved that: *if Magnus has a strategy to visit $f^*(u)$ positions in $g(u)$ rounds, then, for any integer n with u as its largest odd factor, Magnus has a strategy to visit $f^*(n)$ positions in $g(u) + n - u$ rounds.* Thus to prove a linear upper bound on the round number, it suffices to focus on odd integers.

Theorem 6. *Let $n = mp$ be an odd integer, where p is the smallest prime factor of n. Then there is a strategy for Magnus to visit $f^*(n) = (p-1)n/p$ positions in at most $3n$ rounds.*

Proof. Let $C_i = C(n/p, p, i), i \in \mathbb{Z}_p$, and S_0 be the unvisited positions, which is $\mathbb{Z}_n$ initially. Note that when Derek plays optimally, he can always keep one of C_i's, say C_0, from Magnus' visiting [5]. By Lemma 5, we know $SUM^{k+1}(S_0) = \mathbb{Z}_n$ as long as S_0 intersects with t C_i's other than C_0 and $\lceil \frac{p}{2^{k-1}} \rceil \ge t > \lceil \frac{p}{2^k} \rceil$. As in the proof of Theorem 4, it implies Magnus can visit a new position in S_0 within $k+1$ rounds. The smaller the t is, the more rounds Magnus needs to visit a new position. The best strategy for Derek is to force Magnus to visit C_i one after another in order to make t smaller.

Therefore, it takes at most

$$\sum_{k=1}^{\log p} (k+1) \left\lfloor \frac{p}{2^k} \right\rfloor (n/p) \le \sum_k (k+1) \left(\frac{n}{2^k} \right) \le 3n \text{ rounds.} \quad \square$$

3. When Derek knows the moves of Magnus

In this section we consider a variant of the game, where Magnus reveals all of his moves $m_1, m_2, \ldots, m_r$ to Derek. The goal of Derek is to design a sequence of directions $d_1, d_2, \ldots, d_r$ such that the number of positions Magnus can visit is minimal. We prove that it is NP-hard for Derek to obtain a sequence of directions to achieve his goal. The proof is done by reducing the Partition problem, which is well-known NP-complete [6], to the decision version of this problem. We give some helpful definitions as follows.

Definition 2 (*Partition Problem: [6]*)**.** Given a multi-set of positive integers S, determine whether it can be partitioned into two disjoint subsets S_1 and S_2 such that $\sum_{x \in S_1} x = \sum_{y \in S_2} y$.

Definition 3 (*Derek Problem*)**.** Given two positive integers n and r, a sequence of r integers $M = (m_1, m_2, \ldots, m_r) \in Z_n^r$ and an integer K, determine whether there is a sequence $D = (d_1, d_2, \ldots, d_r) \in \{-1, 1\}^r$ such that $S_D = \{x | x \equiv \sum_{i=1}^{j} d_i m_i \pmod{n}, 1 \le j \le r\}$ has $|\{0\} \bigcup S_D| \le K$.

The above decision problem implies that it is NP-hard for Derek to minimize the number positions for Magnus to visit.

Theorem 7. Derek problem *is NP-complete.*

Proof. It is clear that the Derek problem is in NP, since we can verify the answer in polynomial time. Next we give a reduction from the Partition problem to the Derek problem.

Consider an instance of the Partition problem with a multiset $S = \{x_1, x_2, \ldots, x_t\}$. Let $L = 1 + \sum_{x \in S} x$, $p_i = 2^i L$ and $q_i = p_i + x_i$, for $1 \le i \le t$. Then to construct an instance of the Derek problem, we let $M = (2^{t+2}L, q_1, q_2, \ldots, q_t, 2^{t+2}L, p_1, p_2, \ldots, p_t) = (m_1, m_2, \ldots, m_{2t+2})$, $n = 2^{t+5}L$ and $r = K = 2t + 2$.

Note that the reduction can be done in polynomial time in terms of the size of S. Observe that $\sum_{i=1}^{t}(p_i + q_i) < 2^{t+2}L$, $\sum_{i=1}^{k} p_i < p_{k+1}$ and $\sum_{i=1}^{k} q_i < q_{k+1}$, for $1 \le k < t$. Since $\sum_i |m_i| < n/2$, we can ignore the modular operation of Z_n in our proof. WLOG, Derek can always set $d_1 = 1$.

If S can be partitioned into two disjoint sets with equal sum, it implies that there is a sequence $D' = (d'_1, d'_2, \ldots, d'_t) \in \{-1, 1\}^t$ such that $\sum_{i=1}^{t} d'_i x_i = 0$. Then we can find a vector $D = (d_1, d_2, \ldots, d_r) \in \{-1, 1\}^r$, with $d_1 = 1, d_{i+1} = d'_i$ for

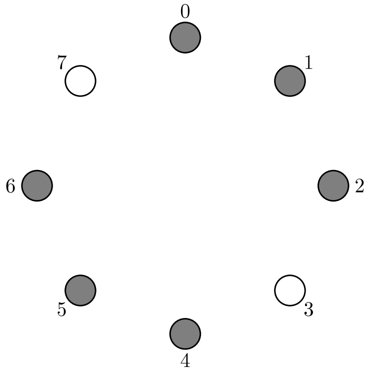

Fig. 1. An example for 3-**balanced** with $n = 8$, where the gray nodes are occupied.

$1 \leq i \leq t$ and $d_j = -d_{j-t-1}$ for $t + 2 \leq j \leq 2t + 2$. Thus we have $\sum_{i=1}^{2t+2} d_i m_i = \sum_{i=1}^{t} (p_i - p_i) + \sum_{i=1}^{t} d'_i x_i = 0$, and it implies $|\{0\} \bigcup S_D| \leq 2t + 2$, since there are at most $2t + 2$ distinct prefix sums.

Conversely, if S cannot be partitioned into two subsets of equal sum, that is, for arbitrary $D' \in \{-1, 1\}^t$, $\sum_{i=1}^{t} d'_i x_i \neq 0$. We claim that there are $2t + 2$ distinct prefix sums for M. Suppose there are integers k' and k such that $1 \leq k' < k \leq 2t + 2$ and $\sum_{i=1}^{k'} d_i m_i = \sum_{i=1}^{k} d_i m_i$. It implies $\sum_{i=k'+1}^{k} d_i m_i = 0$, which is impossible by the above observations. I.e. there always exists an m_j, $k' < k \leq k$, appearing in the summation $\sum_{i=k'+1}^{k} d_i m_i$ such that $2m_j > \sum_{i=k'+1}^{k} m_i$. Thus we conclude that all the prefix sums are distinct.

Next we show that $\sum_{i=1}^{r} d_i m_i \neq 0$. Note that $\sum_{i=1}^{r} d_i m_i \equiv \sum_{i=1}^{t} d_{i+1} x_i \pmod{L} \not\equiv 0$, since S cannot be partitioned evenly and $-L < \sum_{i=1}^{t} d_{i+1} x_i < L$. This fact implies that $(d_1 m_1, \ldots, d_{2t+2} m_{2t+2})$ has distinct prefix sums and none of them is 0. So we know that for any D we have $|S_D| = 2t + 2$ and $0 \notin S_D$. Therefore, $|\{0\} \bigcup S_D| = 2t + 3$. □

4. When Magnus knows Derek's moves

In this case, Magnus actually has an advantage over Derek. Derek gives all his moves first, and Magnus will try to find a set of magnitudes such that he can visit as many positions as possible. Assume there are n positions on the round table. Let $d_1, d_2, \ldots, d_k$ be the sequence given by Derek. For all $1 \leq i \leq k$, d_i will be either $+1$ (clockwise) or -1 (counterclockwise). The sequence of magnitudes from Magnus is denoted as $m_1, m_2, \ldots, m_k$. Let $k = n - 1$ and we have the following:

Proposition 1. (a) *If n is even, Magnus can always occupy all n positions regardless of $d_1, d_2, \ldots, d_{n-1}$.*
(b) *If n is odd and Magnus can choose any magnitude in the set $\{1, \ldots, \lceil \frac{n}{2} \rceil\}$, then Magnus can visit all n positions regardless of $d_1, d_2, \ldots, d_{n-1}$.*

We give two different strategies for even n and odd n, respectively. We determine the moves $m_1, m_2, \ldots, m_{n-1}$ by observing the pattern of occupied positions.

Definition 4. We call the occupied positions on the round table k-**balanced**, if the occupied positions consist of two disjoint sets of consecutive positions, i.e., $S_0 = \{j, \ldots, j + k - 1\}$ and $S_1 = \{j + n/2, \ldots, j + n/2 + k - 1\}$ for some k and $j \in \mathbb{Z}_n$, and the token is sitting at one of the four end positions: $j, j + k - 1, j + n/2$ and $j + n/2 + k - 1$.

Without loss of generality, we assume position 0 is in S_0. The strategy is: at round i, if i is odd, then $m_i = n/2$; otherwise, if the position at (*current position* + d_i) is not occupied then $m_i = 1$, else $m_i = i/2$. During the even rounds, the set of occupied positions holding the token will be extended with a newly occupied position. While during the odd rounds, the set of occupied positions without the token will be extended with a newly occupied position and thus the balanced invariant property is maintained. It is clear that Magnus can occupy all positions (Fig. 1).

For odd n, the strategy is even simpler. Here Magnus is allowed to choose magnitude from $\{1, \ldots, \lceil \frac{n}{2} \rceil\}$. The pattern of occupied positions is slightly different. The strategy is: at round i, if $d_i = +1$, then $m_i = \lfloor n/2 \rfloor$; otherwise $m_i = \lceil n/2 \rceil$. In fact, independent of d_i, it moves to the same position from the current position. Starting from position 0, it moves to positions ($i * \lfloor n/2 \rfloor \bmod n$), in order $i = 1, \ldots, n - 1$, where all positions are distinct. Thus, Magnus can occupy all positions in this case as well.

5. When Derek and Magnus play randomly

Here, we consider the case when both players play randomly. The token will visit the positions on the circle randomly. Assume that the token is at position i, Magnus chooses m uniformly from $\{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$, and Derek chooses the direction d uniformly from $\{1, -1\}$. Let $p_{i,j}$ be the probability that the token is moved from position i to position j. For any $i, j \in \mathbb{Z}_n$ and

$i \neq j$, let $\ell \leq \lfloor n/2 \rfloor$ be the distance between i and j. Then $\Pr[m = \ell] = 1/\lfloor \frac{n}{2} \rfloor$ and $\Pr[d =$ the direction from i to $j] = 1/2$. If n is odd, then $p_{i,j} = 1/\lfloor \frac{n}{2} \rfloor \times 1/2 = 1/\frac{n-1}{2} \times 1/2 = 1/(n-1)$. Thus, for odd n, we have:

$$p_{i,j} := \begin{cases} 0 & \text{if } i = j, \\ \frac{1}{n-1} & \text{otherwise.} \end{cases}$$

Similarly, for even n, we have:

$$p_{i,j} := \begin{cases} 0 & \text{if } i = j, \\ 2/n & \text{if } j \equiv i + \frac{n}{2} \bmod n, \\ 1/n & \text{otherwise.} \end{cases}$$

We are interested in the cover time, which is the number of rounds needed to visit all positions. We show that the number of rounds needed is $\Theta(n \log n)$. Define $c_{(i,i+1)}$, $i \in \mathbb{Z}_n$, to be the number of rounds needed to change from a state with i positions visited to a state with $i+1$ positions visited. Since the token is at position 0 initially, we denote the cover time C_n as

$$C_n = \sum_{i=1}^{n-1} c_{(i,i+1)},$$

and the expected cover time is

$$E[C_n] = \sum_{i=1}^{n-1} E[c_{(i,i+1)}].$$

Lemma 8. (a) *When n is odd,* $E[c_{(i,i+1)}] = \frac{n-1}{n-i}$; (b) *When n is even,* $\frac{n}{n-i+1} \leq E[c_{(i,i+1)}] \leq \frac{n}{n-i}$.

Proof. (a) Suppose that there are $n-i$ unvisited positions. The probability to visit one of the unvisited positions is $p_i = \frac{n-i}{n-1}$. Note that $c_{(i,i+1)}$ is a geometric random variable with parameter p_i, and thus

$$E[c_{(i,i+1)}] = \frac{1}{p_i} = \frac{n-1}{n-i}.$$

(b) Assume the token is at position x. For even n, position $x + \frac{n}{2} \bmod n$ has a greater chance to be visited. If $x + \frac{n}{2} \bmod n$ has been visited, then the probability to visit a new position is

$$p_i = \frac{n-i}{n}.$$

If $x + \frac{n}{2} \bmod n$ has not been visited, then the probability to visit a new position is

$$p_i = \frac{1}{n} \times (n-i-1) + \frac{2}{n} = \frac{n-i+1}{n}.$$

To bound the value of $E[c_{(i,i+1)}]$, we know that the above cases can happen, and we let p_i^* be the probability to visit a new position, where $\frac{n-i}{n} \leq p_i^* \leq \frac{n-i+1}{n}$. Note that p_i* depends on the current position and is well bounded. Let c' and c'' be two geometric random variables with parameter $\frac{n-i}{n}$ and $\frac{n-i+1}{n}$, respectively. Then we have

$$\frac{n}{n-i+1} = E[c''] \leq E[c_{(i,i+1)}] \leq E[c'] = \frac{n}{n-i}. \quad \square$$

Since we know the range of $E[c_{i,i+1}]$ for all $i \in \{1, \ldots, n-1\}$, we can bound the expected cover time. We show that $E[C_n] = \Theta(n \log n)$ with the following theorem.

Theorem 9. (a) *When n is odd,* $E[C_n] = (n-1)H_{n-1}$, *where* $H_n = \sum_{i=1}^{n} \frac{1}{i}$; (b) *When n is even,* $nH_n - n \leq E[C_n] \leq nH_n - 1$.

Proof. (a) From part (a) of Lemma 8, $E[c_{(i,i+1)}] = \frac{n-1}{n-i}$. Hence,

$$E[C_n] = \sum_{i=1}^{n-1} E[c_{(i,i+1)}] = \sum_{i=1}^{n-1} \frac{n-1}{n-i} = (n-1) \sum_{i=1}^{n-1} \frac{1}{i} = (n-1)H_{n-1}.$$

(b) From part (b) of Lemma 8, $\frac{n}{n-i+1} \leq E[c_{(i,i+1)}] \leq \frac{n}{n-i}$ and $c_{1,2} = 1$. Hence,

$$\sum_{i=1}^{n-1} \frac{n}{n-i+1} \leq E[C_n] \leq \sum_{i=1}^{n-1} \frac{n}{n-i}.$$

We know that

$$\sum_{i=1}^{n-1} \frac{n}{n-i+1} = n\sum_{i=2}^{n} \frac{1}{i} = n\left(\sum_{i=1}^{n} \frac{1}{i} - 1\right) = nH_n - n,$$

and

$$\sum_{i=1}^{n-1} \frac{n}{n-i} = \sum_{i=1}^{n-1} \frac{n}{i} = n\sum_{i=1}^{n} \frac{1}{i} - \frac{n}{n} = nH_n - 1.$$

Since $H_n = \Theta(\log n)$, we know that the expected cover time is $\Theta(n \log n)$. □

6. Concluding remarks

In this paper we have answered two open questions in [5,1], i.e. we prove that (1) Magnus can visit the maximum number of positions in $O(n)$ rounds; (2) It is NP-hard for Derek to find an optimal strategy with Magnus' moves revealed in advance. Several other questions raised in [5] remain open.

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