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NSC Project Report on "Dynamics in Spatially Extended Systems"

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In this project, we studied the dynamics for general lattice dynamical systems as well as discrete-time reaction-diffusion systems and neural networks. The goal is to find the correspondence between solutions, patterns and the dynamics on large finite lattice and the ones on infinite lattice for the same system (as described in componentwise manner). Certain solutions and patterns for systems of interests on infinite lattice are, in fact, easier to construct than the ones for systems on finite lattice with boundary conditions.

What we have achieved so far is to establish theory on convergence of dynamics and chaotic behaviors etc., in systems on finite lattice as well as a methodology of constructing stationary solutions or patterns on finite lattice and infinite lattice. It has been presented in an article that the global attractors for discrete-time reaction-diffusion systems on finite lattices approach the global attractor for the same system on the infinite lattice as lattice size tends to infinity. Nevertheless, one does not know how the ingredients of the attractors vary in the approximations. These remain to be investigated.

We have completed several papers during the period of this project. Most of them are concerned with neural networks models. We did come up with some ideas to treat discrete-time reaction-diffusion systems in the aspect of stationary patterns, spatial entropy, effect of boundary condition on patterns and entropy. The results will be released later.

We attach the introduction for one of the papers in this report.

Dynamics for Discrete-Time Cellular Neural Networks

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Abstract

This presentation investigates the dynamics of discrete-time cellular neural networks (DT-CNN). In contrast to that classical neural networks are mostly gradient-like systems, DT-CNN possesses both complete stability and chaotic behaviors as different parameters are considered. An energy-like function which decreases along orbits of DT-CNN as well as the existence of a globally attracting set are derived. Complete stability can then be concluded, with further analysis on the sets on which the energy function is constant. The formations of saturated stationary patterns for DT-CNN are shown to be analogous to the ones in continuous-time CNN. Thus, DT-CNN shares similar properties with continuous-time CNN. By confirming the existence of snap-back repellers, hence transversal homoclinic orbits, we also conclude that DT-CNN with certain parameters exhibits chaotic dynamics, according to the theorem by Marotto.

keywords: Cellular neural network, Pattern formation, Complete stability, Homoclinic orbits, Snap-back repeller, Chaos

1 Introduction

Cellular neural network (CNN) is a large aggregation of analogue circuits. It was first proposed by Chua and Yang in 1988. A CNN assembly consists of arrays of

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identical elementary processing units called cells. The cells are only connected to their nearest neighbors. This local connectivity makes CNN very suitable for VLSI implementation. The equations for two-dimension layout of CNN are given by

$$C\frac{dx_{ij}(t)}{dt} = -\frac{1}{R}x_{ij}(t) + \sum_{(k,\ell)\in N_r(i,j)} [a_{ij,k\ell}h(x_{k\ell}(t)) + b_{ij,k\ell}u_{k\ell}] + I, \tag{1.1}$$

where x_{ij}, $y_{ij} = h(x_{ij})$ are the state and output voltage of the specified CNN cell at site (i, j), respectively; $u_{k\ell}$ is the controlling input. $N_r(i, j)$ represents the neighborhood of (i, j) with radius r (a positive integer). CNN are characterized by the bias I, the template set A and B which consist of real numbers $a_{ij,k\ell}$ and $b_{ij,k\ell}$, respectively. $a_{ij,k\ell}$ represents the linear feedback, $b_{ij,k\ell}$ the linear control. The standard output function h is a piecewise-linear mapping defined by $h(\xi) = \frac{1}{2}(|\xi + 1| - |\xi - 1|)$. C is the linear capacitor and R is the linear resistor. R and C are often set to $R = 1$ and $C = 1$, for convenience of discussion. A complete set of the CNN model requires imposing boundary condition for the cells on the boundary of the assembly, see [Shih, 2000]. This presentation will not take into account the boundary conditions, since the approach and methodology used herein prevail even with consideration of boundary conditions. Eq. (1.1) is a continuous-time model, we thus call it CT-CNN.

In this investigation, we shall study the discrete-time cellular neural network (DT-CNN). DT-CNN admits rich dynamics including properties analogous to the ones in CT-CNN. The model is described by the following difference equation.

$$x_{ij}(t+1) = \mu x_{ij}(t) + \sum_{(k,\ell)\in N_r(i,j)} [\widetilde{a}_{ij,k\ell}h(x_{k\ell}(t)) + \widetilde{b}_{ij,k\ell}u_{k\ell}] + I_{ij}, \tag{1.2}$$

where t is an integer and (i, j) belongs to a $n_1 \times n_2$ lattice. (1.2) can be derived from a delta-operator based CNN. If one collects from a continuous-time signal $\mathbf{x}(t)$ a discrete-time sequence $\mathbf{x}[k] = \mathbf{x}(kT)$, the delta operator

$$\delta\mathbf{x}[k] = \frac{\mathbf{x}[k+1] - \mathbf{x}[k]}{T}$$

is an approximation of the derivative of $\mathbf{x}(t)$. Indeed, $\lim_{T\to 0} \delta\mathbf{x}[k] = \dot{\mathbf{x}}(t)|_{t=kT}$. In this case, $\mu = 1 - T/\tau$, where T is the sampling period, and $\tau = RC$. The parameters $\widetilde{a}_{ij,k\ell}, \widetilde{b}_{ij,k\ell}$ in (1.2) correspond to $a_{ij,k\ell}$, $b_{ij,k\ell}$ in (1.1) under sampling, see [Hänggi et. al., 1999]. If (1.2) is considered in conjunction with (1.1), then T is required to satisfy $\tau \geq T$ to avoid aliasing effects. Under this situation, $0 \leq \mu \leq 1$ and CT-CNN is the limiting case of delta-operator based CNN with $T \to 0$. If the delta-operator based CNN is considered by itself, then there is no restriction on T, and thus no restrictions on μ in (1.2). On the other hand, a sample-data based CNN has been introduced in [Harrer & Nossek, 1992]. Such a network corresponds to the limiting case of delta-operator based CNN as $T \to 1$. The readers are referred to [Hänggi et. al., 1999] and the reference therein for an account of unifying results on the above-mentioned models. On the other hand, Euler's difference scheme for (1.1) takes the form

$$x_{ij}(t+1) = (1 - \frac{\Delta t}{RC})x_{ij}(t) + \frac{\Delta t}{C}\left(\sum_{k\in N_r(i,j)} a_{ij,k\ell}h(x_{k\ell}(t)) + b_{ij,k\ell}u_{k\ell} + I\right). \quad (1.3)$$

The parameters in (1.2) and (1.3) are related by $\mu = 1 - \frac{\Delta t}{RC}$, $\widetilde{a}_{ij,k\ell} = \frac{\Delta t}{C}a_{ij,k\ell}$, $\widetilde{b}_{ij,k\ell} = \frac{\Delta t}{C}b_{ij,k\ell}$ and $I_{ij} = \frac{\Delta t}{C}I$.

A lattice of any dimension with finitely many sites can be re-indexed into a one-dimensional manner. Thus, CNN of any dimension can be reformulated into a one-dimensional setting, *cf.* [Shih & Weng, 2002]. Restated, the DT-CNN (1.2) in two-dimensional layout can be written into a one-dimensional form as

$$x_i(t+1) = \mu x_i(t) + \sum_{k=1}^{n} \omega_{ik}h(x_k(t)) + z_i, \quad (1.4)$$

where $i = 1, \cdots, n$, $n = n_1 \cdot n_2$, and ω_{ik}, z_i correspond to $\widetilde{a}_{ij,k\ell}$ and $(\widetilde{b}_{ij,k\ell}u_{k\ell} + I_{ij})$ respectively. This expression suppresses local connectivity among cells of DT-CNN. However, it is more concise for our presentation. We shall study DT-CNN in the form (1.4) for most of this presentation. In the discussions of pattern formation in the last section, the expression of DT-CNN exhibiting local connectivity (such as (1.2)) will be adopted.

This presentation aims to explore dynamical features of the DT-CNN, including complete stability, chaotic behaviors, and pattern formation. By complete stability, we mean that every orbit tends to a steady state solution (fixed point of the DT-CNN herein) as time tends to infinity. Complete stability for the CT-CNN has been studied in [Chua & Yang, 1988; Lin & Shih, 1999; Shih, 2001]. The basic assumption for such a result is the symmetry of the coupling weights, that is, $W := [\omega_{ik}]$ is a symmetric matrix. The situation is much more complicated in DT-CNN. We first derive the existence of a trapping region, and a Lyapunov function V. Because of the saturation part of the output function h, there is a large portion of phase space on which V is constant. We then analyze the dynamics on these regions and conclude complete stability for the DT-CNN, under an additional assumption. We shall also illustrate that formations of saturated stationary solutions and patterns for DT-CNN are analogous to the ones in CT-CNN.

The second goal of this work is to study the snap-back repellers of the DT-CNN. A fixed point $\overline{\mathbf{x}}$ of a map F is said to be a *snap-back repeller* of F if there exists a positive real number r and a point $\mathbf{x}_0 \in B(\overline{\mathbf{x}}; r)$ with $\mathbf{x}_0 \neq \overline{\mathbf{x}}$ such that all eigenvalues of $DF(\mathbf{x})$ exceed unity in norm for all $\mathbf{x} \in B(\overline{\mathbf{x}}; r)$ and $F^m(\mathbf{x}_0) = \overline{\mathbf{x}}$ with $det(DF^m(\mathbf{x}_0)) \neq 0$ for some positive integer m. Such $\mathbf{x}_0$ is called a *snap-back point*. It is obvious that the existence of snap-back repeller implies the existence of a transversal homoclinic orbit. In 1975, Li and Yorke [1975] proved that period three implies chaos as well as certain sensitive dependence on initial conditions for one-dimensional mappings. It was extended into multi-dimensional maps by Marotto [1978].

Theorem 1.1 (Marotto). *If F has a snap-back repeller, then the dynamical system $\mathbf{x} \to F(\mathbf{x})$ is chaotic in the following sense:*
(1) There exists a positive integer m_0 such that F has p-periodic points for every integer $p \geq m_0$.
(2) There exists a scrambled set, that is, an uncountable set L containing no periodic points such that the following pertains:
(a) $F(L) \subset L$; *(b) for every $\mathbf{y} \in L$ and any periodic point $\mathbf{x}$ of F,* $\limsup_{k\to\infty} \|F^k(\mathbf{y}) - F^k(\mathbf{x})\| > 0$;
(c) for every $\mathbf{x}, \mathbf{y} \in L$ with $\mathbf{x} \neq \mathbf{y}$, $\limsup_{k\to\infty} \|F^k(\mathbf{y}) - F^k(\mathbf{x})\| > 0$.
(3) There exists a uncountable subset L_0 of L such that for every $\mathbf{x}, \mathbf{y} \in L_0$,

$$\liminf_{k\to\infty} \|F^k(\mathbf{y}) - F^k(\mathbf{x})\| = 0.$$

Marotto concluded that snap-back repellers imply chaos in the sense of the above theorem. The snap-back repeller, which is suitable for the non-invertible map with repelling fixed points, is a powerful technique for proving chaos in multi-dimensional maps. Guckenheimer and Holmes [1983] showed that if a diffeomorphism has a transversal homoclinic orbit to the hyperbolic fixed point $\overline{\mathbf{x}}$, then F is locally topologically equivalent to a sub-shift of finite type. This theorem only applies to diffeomorphisms. Marotto [1979] also applied his theorem to study the transversal homoclinic orbits for the Hénon map.

Classical continuous-time neural networks do not exhibit chaotic behaviors. It is certainly not the case for discrete-time neural networks, see, for example, [Chen &

Shih, 2002] on a study of the so-called transiently chaotic neural network. Recently, Sbitnev and Chua have studied the local activity criteria and its application to non-homogeneous spatiotemporal patterns for the DT-CNN. Other mathematical studies in CT-CNN include investigations of pattern formations and spatial entropy, in [Juang & Lin, 2000; Shih, 1998; Shih, 2000], and travelling wave solutions [Hsu & Lin, 2000].

In Section 2, we shall address the dynamics and demonstrate complete stability and chaotic behaviors of DT-CNN. Rigorous justifications as well as numerical illustrations for these dynamics are arranged in Section 3. In Section 4, we briefly show that formations of saturated stationary patterns for the DT-CNN can be established as in CT-CNN.