Available online at www.sciencedirect.com

ScienceDirect

J. Math. Anal. Appl. 341 (2008) 1055–1067

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

On the total variation, topological entropy and sensitivity for interval maps

Jonq Juang [a], Shih-Feng Shieh [b,*]

[a] *Department of Applied Mathematics, National Chiao Tung University, Hsinchu, Taiwan, ROC*
[b] *Department of Mathematics, National Taiwan Normal University, Taipei, Taiwan, ROC*

Received 31 October 2007

Available online 17 November 2007

Submitted by Goong Chen

Abstract

A concept related to total variation termed $\mathcal{H}_1$ condition was recently proposed to characterize the chaotic behavior of an interval map f by Chen, Huang and Huang [G. Chen, T. Huang, Y. Huang, Chaotic behavior of interval maps and total variations of iterates, Internat. J. Bifur. Chaos 14 (2004) 2161–2186]. In this paper, we establish connections between $\mathcal{H}_1$ condition, sensitivity and topological entropy for interval maps. First, we introduce a notion of restrictiveness of a piecewise-monotone continuous interval map. We then prove that $\mathcal{H}_1$ condition of a piecewise-monotone continuous map implies the non-restrictiveness of the map. In addition, we also show that either $\mathcal{H}_1$ condition or sensitivity then gives the positivity of the topological entropy of f.

Keywords: Interval maps; Total variation; Topological entropy; Sensitivity

1. Introduction

The study of chaotic phenomena in dynamical systems has been a focus of attention both in mathematics and applied sciences. Even the chaotic behavior of one-dimensional interval maps turns out to be highly non-trivial and also extremely interesting, both from a mathematical and an applied point of view (see e.g., [2, Chapter 0]). However, like the term "chaos," its mathematical definition is not uniquely defined. Indeed there are many ways of quantitative measurement of the complex or chaotic nature of the dynamics. Just to name a few, they are sensitive dependence on initial data, topological entropy, Lyapunov exponents, homoclinic orbits, various concepts of fractal dimensional, absolutely continuous invariant measures. We will give the explicit definitions for sensitivity and topological entropy since they are relevant to our discussion here.

* Corresponding author.
E-mail addresses: jjuang@math.nctu.edu.tw (J. Juang), sfshieh@math.ntnu.edu.tw (S.-F. Shieh).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2007.11.013

Definition 1.1. (See e.g., [2, Definition 1.8.1].) Let X be a metric space with metric $d(\cdot,\cdot)$, and $f: X \to X$ be continuous. Let us say that f has sensitive dependence on initial data if there exists $\delta > 0$ such that for every $x_0 \in X$ and for every open set U containing x_0, there exist $y \in U$ and $n \in \mathbb{N}^+$ such that $d(f^n(y), f^n(x_0)) > \delta$.

For practical purpose, we will use a theorem of Misiurewicz and Szlenk [5] as the definition of the topological entropy of an interval map. For precise definition, see [2,4,6].

Definition 1.2. Let I be a closed interval and $f: I \to I$ be a piecewise-monotone continuous map. This means that f is continuous and that f has a finite number of turning points, i.e., points in the interior of [0, 1] where f has a local extremum. Such a map is called l-modal if f has precisely l turning points and if $f(\partial I) \subset \partial I$.

Definition 1.3. Let $f: I \to I$ be a continuous piecewise-monotone map. The lap number of f, denoted by $l(f)$, is the number of maximal intervals on which f is monotone. Then the topological entropy of f is defined to be

$$h_{\text{top}}(f) = \lim_{n\to\infty} \frac{\log(l(f^n))}{n}.$$

One other well-known result concerning the topological entropy of f is the following.

Theorem 1.1. *(See e.g., [4, Corollary 15.3.6].) If $f: I \to I$ is continuous, then $h_{\text{top}}(f) = 0$ if and only if the period of every periodic point of f is a power of two.*

Recently, Chen, Huang and Huang [1] introduced yet another new concept to characterize the chaotic behavior of interval maps. Specifically, they introduce the so-called $\mathcal{H}_1$ condition as follows.

$\mathcal{H}_1$: Let $f: I \to I$ be continuous. For any closed subinterval $J \subseteq I$, $\lim_{n\to\infty} V_J(f^n) = \infty$. Here $V_J(g)$ denotes the total variation of a function g on an interval J.

We remark that though $\mathcal{H}_1$ condition is new, the concept of total variation treated as a characteristic of chaotic maps is certainly not new. For example, the following result can be found in [4].

Theorem 1.2. *(See e.g., [4, Corollary 15.2.14].) If $f: I \to I$ is a piecewise-monotone continuous map, then $\lim_{n\to\infty} \frac{1}{n} \log V_I(f^n) = h_{\text{top}}(f)$, where h_{top} is the topological entropy.*

Consequently, for such map f if the total variations of f^n on I grow exponential with repeat to n, then the topological entropy of f is positive. In such case, we shall say that the growth rate for the total variations of f on I is exponential. Among many results obtained in [1], one of them is as follows.

Theorem 1.3.

(i) *Let $f: I \to I$ be a continuous map. Then the sensitivity of f on initial data implies the $\mathcal{H}_1$ condition.*
(ii) *If, in addition, f is piecewise-monotone, then the converse of* (i) *holds true.*

In this paper, we wish to establish the connection between $\mathcal{H}_1$ condition and other well-known concepts that characterize the chaotic behavior of f. We begin with defining a horseshoe, or a saddle of a map f.

Definition 1.4. (See e.g., [4, Definition 15.1.10].) If $I \subset \mathbb{R}$ is a closed interval, $f: I \to \mathbb{R}$ is continuous and $a < c < b \in I$, then we say that $[a, b]$ is a horseshoe or a saddle for f if $[a, b] \subset f([a, c]) \cap f([c, b])$.

If a map has a horseshoe or a saddle, then f has positive topological entropy. In fact, we have the following.

Theorem 1.4. *(See e.g., [4, Corollary 15.1.11].) If $f: I \to \mathbb{R}$ has a horseshoe, then $h_{\text{top}}(f) \geqslant \log 2$.*

Definition 1.5. A continuous map f from I to I is said to be non-restrictive if there exist $k \in \mathbb{N}$, and two numbers p and q, $0 \leqslant p < q \leqslant 1$, such that the interval $[p, q]$ is a horseshoe for f^k. Otherwise, the map f is said to be restrictive.

We are now in a position to state our main result.

Main Theorem. *Let f be a piecewise-monotone continuous map satisfying $\mathcal{H}_1$ condition. Then f is non-restrictive.*

We conclude this introductory section with the following remarks.

(1) We prove the Main Theorem by contradiction. That is, we assume that f is piecewise-monotone continuous, restrictive satisfying $\mathcal{H}_1$ condition. We then need to construct finitely renormalizable intervals $\{I_i\}_{i=1}^n$ from an m-modal map that has the following properties:
   (i) Dynamics (under f) of the intervals $\{I_i\}_{i=1}^n$ behaves like an adding machine (see Theorem 2.1).
   (ii) Along the path of a sequence $w = (w_1, w_2, \ldots, w_n, \ldots) \in \Sigma_2$, space of all one sided sequences with symbols in $\{1, 2\}$, the length of I_{w^n} is shrinking to zero as $n \to \infty$. Here $w^n = (w_1, w_2, \ldots, w_n)$ (see Theorem 2.2).
(2) We have made use of Theorem 1.2 to prove our Main Theorem. In [1], a counterexample is given so as to show that the converse of Theorem 1.3(i) is not true. This is to say the assumption that f is continuous piecewise-monotone is needed in proving our Main Theorem.
(3) Combining our Main Theorem, Definition 1.5 and Theorems 1.1–1.4, one readily conclude that the following results.

Theorem 1.5. *Let f be a piecewise-monotone continuous map satisfying $\mathcal{H}_1$ condition or sensitivity of f on initial data. Then the following holds.*

(i) $h_{\text{top}}(f) > 0$.
(ii) *f has a periodic point whose period is not a power of* 2.
(iii) *The growth rate for the total variation of f^n on I is exponential.*

2. Main results

We need the following hyperbolicity result, which plays an important role in our subsequent analysis.

Lemma 2.1. *(See [1, Proposition 4.1(v)].) Let $f : I \to I$ be piecewise-monotone continuous and satisfy $\mathcal{H}_1$ condition. Then so does f^n for any $n \in \mathbb{N}$.*

Lemma 2.2. *Let $f : I \to I$ be piecewise-monotone continuous and satisfy $\mathcal{H}_1$ condition. Suppose x_0 is a fixed point of f in I. Then there exists a neighborhood $(x_0 - \epsilon, x_0 + \epsilon)$ of x_0 such that $f(x) > x$ or $f(x) < x_0$ for $x \in (x_0, x_0 + \epsilon)$. Moreover, for $x \in (x_0 - \epsilon, x_0)$, $f(x) > x_0$ or $f(x) < x$. This property also holds if x_0 is a left or right endpoint of I.*

Proof. Since the proof of the lemma lies in the same spirit as that of Proposition 4.1(iv) in [1], we skip the proof. □

Remark 2.1.

(1) Let $P = (p, f(p))$ be a fixed point, and let $B_r(P)$ be a disk with center at P and radius r. Set

$$I_{p,r} = \{(x, y) \in \mathbb{R}^2\colon\ p > y > x\} \cap B_r(P),$$
$$II_{p,r} = \{(x, y) \in \mathbb{R}^2\colon\ y < x < p\} \cap B_r(P),$$
$$III_{p,r} = \{(x, y) \in \mathbb{R}^2\colon\ x > y > p\} \cap B_r(P),$$
$$IV_{p,r} = \{(x, y) \in \mathbb{R}^2\colon\ y > x > p\} \cap B_r(P),$$

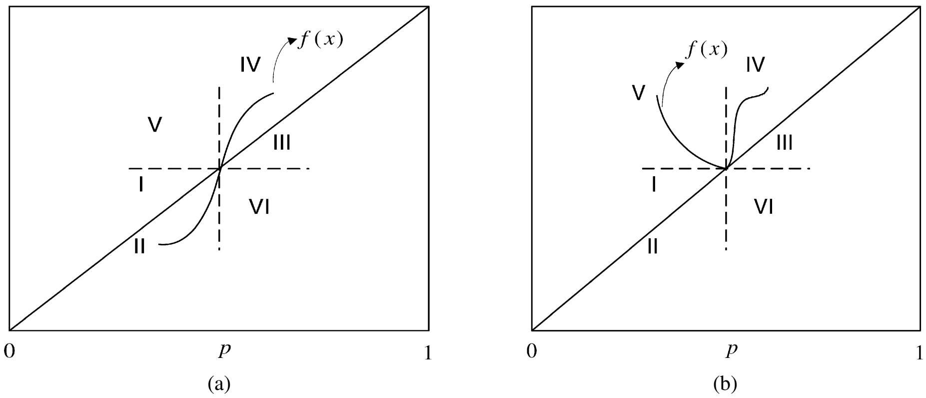


Fig. 1. (a) f is increasing near p. (b) p is also a turning point.

$$V_{p,r} = \{(x, y) \in \mathbb{R}^2\colon\ x < p,\ y > p\} \cap B_r(P),$$
$$VI_{p,r} = \{(x, y) \in \mathbb{R}^2\colon\ x > p,\ y < p\} \cap B_r(P).$$

The assertions of the lemma are then equivalent to saying that the graph of f must only stay in the regions $II_{p,r}$, $IV_{p,r}$, $V_{p,r}$ and $VI_{p,r}$ for some $r > 0$, and that it can never stay in regions $I_{p,r}$ and $III_{p,r}$.

(2) For a better visualization of the lemma, see Fig. 1.

Definition 2.1. Let p be a periodic point of period n. If f^n is decreasing on a neighborhood of p, then p is said to be of A type.

Proposition 2.1. *(See [1, Theorem 5.1].) Let $f : I \to I$ be a piecewise-monotone continuous map satisfying $\mathcal{H}_1$ condition. Then f has an interior period two point which is of A type. Consequently, f has interior period 2^n point which is of A type for any $n \in \mathbb{N}$.*

Proof. We first prove that f has an interior fixed point which is of A type. Since the graph of f near 0 cannot stay in region $III_{0,r}$, $f(0^+) > 0$. Thus, the first interior fixed point must be either of type A or a local minimum point. If all the interior fixed points are local minimum points, then 1 must be a fixed point and the graph of f near 1 stays in region $I_{1,r}$ for some $r > 0$, a contradiction. Hence, f must have an interior fixed point which is of A type. Let p be the first such fixed point. Consider the graph of f^2 near p, we see that it must stay in region $II_{p,r}$ for some $r > 0$. Hence, all the fixed points of f^2 on $(0, p)$ must be of type A or local maximum points. If all such points are local maximum points, then $f^2(0) = 0$, and the graph of f^2 near 0 stays in region $III_{0,r}$ for some $r > 0$, a contradiction. Hence, f^2 must have a fixed point of A type on $(0, p)$. The proof of the proposition is thus complete. □

Lemma 2.3. *Let $f : I \to I$ be a piecewise-monotone continuous map satisfying $\mathcal{H}_1$ condition. Let x_0 and x_1, $x_0 < x_1$, be two consecutive fixed points of f^{2^n} for some n. Then there exists one turning point $c \in (x_0, x_1)$. Moreover, let $c_{\min}$ and $c_{\max}$ be the smallest and largest turning points in (x_0, x_1), respectively. Then*

$$f^{2^n}(c_{\max}) > x_1 \quad \text{if } f^{2^n}(x) > x \text{ for } x \in (x_0, x_1),$$

and

$$f^{2^n}(c_{\min}) < x_0 \quad \text{if } f^{2^n}(x) < x \text{ for } x \in (x_0, x_1).$$

Proof. Suppose to the contrary that there is no turning point in (x_0, x_1). Then f^{2^n} must be monotonically increasing on (x_0, x_1), a contradiction to Lemma 2.2. By the assumption that x_0 and x_1 are two consecutive fixed points of f^{2^n}, we then have two possibilities: (i) $f^{2^n}(x) > x$ for all $x \in (x_0, x_1)$, (ii) $f^{2^n}(x) < x$ for all $x \in (x_0, x_1)$. For case (i), since the graph of f^{2^n} near x_1 cannot stay in region $I_{x_1,r}$ for some $r > 0$, we thus conclude that $c_{\max}$ exists and $f^{2^n}(c_{\max}) > x_1$. The case (ii) can be treated similarly. □

Lemma 2.4. *Let $f : I \to I$ be a piecewise-monotone continuous map that satisfies $\mathcal{H}_1$ condition. Then the following holds true.*

(i) *The map f^2 must have a fixed point of A type.*
(ii) *There exists a subinterval $J \subset I$ such that $f^2 : J \to J$ is an l-modal map or f^2 has a horseshoe on J.*

Proof. Let $g = f^2$. It follows from Proposition 2.1 that there are at least two interior fixed points of g. We next show that g must be decreasing on a neighborhood N_p of a certain fixed point p of g. To see this, let $\{p_1, p_2, \dots, p_l\}$ be the set of all interior fixed points of g. Here, we suppose that $p_1 < p_2 < \cdots < p_l$. Note that the graph of g on $(0, p_1)$ cannot stay in regions $III_{0,r}$ and $I_{p_1,r}$ for some $r > 0$. Hence, g is decreasing on $(p_1 - \delta_1, p_1)$ for some $\delta_1 > 0$. Suppose p_1 is not a turning point. Then we are done. If p_1 is a turning point, then g must also be decreasing on $(p_2 - \delta_2, p_2)$ for some $\delta_2 > 0$. Now, suppose that all p_i, $1 \leqslant i \leqslant l$, are turning points. Then the graph of g on $(p_l, 1)$ must stay above the line $y = x$ and $g(1) = 1$. This means that the graph of g on $(p_l, 1)$ is in region $I_{1,r}$ for some $r > 0$, which is not possible. Thus, g must be decreasing on some neighborhood of a certain interior fixed point. Without loss of generality, we shall assume p_1 be such fixed point. Then the graph of g on (p_1, p_2) stays below the line $y = x$. It then follows from Lemma 2.3 that there is one point $q_1 \in (p_1, p_2)$ such that $g(q_1) = g(p_1) = p_1$. Let $J = [p_1, q_1]$. Then $g = f^2 : J \to J$ is as claimed. We thus complete the proof of the lemma. □

Lemma 2.5. *Let $f : I \to I$ be a piecewise-monotone continuous map satisfying $\mathcal{H}_1$ condition. Suppose p is an interior period 2^n point which is of A type. Set $p_k = f^k(p)$, $1 \leqslant k \leqslant 2^n$, $p_0 = p$. Then p_k is also of type A for any $1 \leqslant k \leqslant 2^n$.*

Proof. We first prove that p_k cannot be a turning point of f^{2^n}. If so, then p_l with $l \geqslant k$ must also be a turning point of f^{2^n}, which contradicts with the fact that $p_0 = p = p_{2^n}$. It then suffices to show that p_1 is of A type for f^{2^n}. We first assume that f is decreasing on a neighborhood U of p, say $U = [p - \epsilon, p + \epsilon]$ for some $\epsilon > 0$. We may also let f^{2^n} be also decreasing on U. Thus

$$f(x) < f(y) \quad \text{whenever } x > y,\ x, y \in U, \tag{2.1a}$$

and

$$f^{2^n}(x) < f^{2^n}(y) \quad \text{whenever } x > y,\ x, y \in U. \tag{2.1b}$$

Note that $f^i(U)$ is a neighborhood of p_i, since each p_i, $1 \leqslant i \leqslant 2^n$, is not a turning point. If necessary, we can make ϵ so small that f is still decreasing on $f^{2^n}(U)$. Letting $a, b \in f(U)$ with $a > b$, we have that there exist x and y, $x < y$, such that $f(x) = a$ and $f(y) = b$. Upon using (2.1), we see that

$$f\bigl(f^{2^n}(x)\bigr) = f^{2^n}\bigl(f(x)\bigr) = f^{2^n}(a) > f^{2^n}\bigl(f(y)\bigr) = f^{2^n}(b) \quad \text{whenever } a < b.$$

Thus p_1 is a fixed point of A type for f^{2^n}. The case that f is increasing on a neighborhood of p can be similarly treated. □

Remark 2.2. If f is smooth, then the proof of the lemma can be done by noting that

$$\bigl(f^{2^n}\bigr)'(p_k) = \bigl(f^{2^n}\bigr)'(p_0).$$

We next turn our attention to the notion of restrictiveness.

Remark 2.3. If f is restrictive then so is f^{2^n}.

Let $f : I \to I$ be a map satisfying the following:

$$\text{(i)} \quad f \text{ is restrictive and satisfies } \mathcal{H}_1 \text{ condition}, \tag{2.2a}$$

$$\text{(ii)} \quad f(0) = f(1) = 0. \tag{2.2b}$$

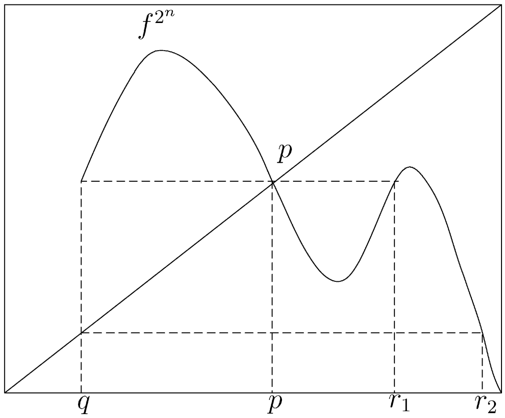


Fig. 2.

Let

$$F(n) = \{0 < p < 1 \mid p \text{ is a period } 2^n \text{ point of } f \text{ and is also of A type}\}. \tag{2.3}$$

For fixed n, let $p \in F(n)$. We define $q(p)$, $r_1(p)$ and $r_2(p)$ as follows (see Fig. 2):

$$q(p) = \max\{x < p \mid f^{2^n}(x) = p\}, \tag{2.4a}$$

$$r_1(p) = \{x > p \mid f^{2^n}(x) = p\}, \tag{2.4b}$$

and

$$r_2(p) = \{x > p \mid f^{2^n}(x) = q(p)\}. \tag{2.4c}$$

We then set the following notations:

$$I_1^n(p) = [p, \min\{r_1(p), r_2(p)\}] := [p, r(p)], \tag{2.5}$$

$$F_p = \text{the set of fixed points of } f^{2^{n+1}} \text{ on } I_1^n(p) = \{p = p_1 < p_2 < \cdots < p_m\}, \tag{2.6a}$$

$$Fix(f^{2^{n+1}}) = \text{the set of fixed points of } f^{2^{n+1}} \text{ on } [0, 1]. \tag{2.6b}$$

Remark 2.4. Since f satisfies (2.2), the existence of $q(p)$ and $r_2(p)$ is guaranteed. However, $r_1(p)$ might not exist. Moreover, it is clear that $f^{2^{n+1}}(x) > p$ for $x \in (p, r(p))$.

In the following lemmas, we shall establish that the graphs of f^k on $I_1^n(p)$, $1 \leqslant k < 2^{n+1}$, are pretty much similar.

Lemma 2.6. *Let f satisfy* (2.2) *and $p \in F(n)$, where $F(n)$ is given in* (2.3)*. For any $1 \leqslant k < 2^{n+1}$, $f^k(p) = \max f^k(F_p)$ or $f^k(p) = \min f^k(F_p)$.*

Proof. Assume that $m > 2$, where m is the cardinality of F. If $m = 2$, then we are done. Let $\tilde{p}_i = f^k(p_i)$. Suppose the assertion of the lemma is not true. Then, for some $1 \leqslant i \leqslant m-1$, we have $\tilde{p}_i < \tilde{p}_1 = \tilde{p} < \tilde{p}_{i+1}$ or $\tilde{p}_{i+1} < \tilde{p}_1 = \tilde{p} < \tilde{p}_i$. Hence, there exists $x \in (p_i, p_{i+1})$ such that $f^k(x) = f^k(p)$. Thus,

$$f^{2^{n+1}}(x) = f^{2^{n+1}}(p) = p.$$

Hence, either $f^{2^n}(x) = p$ or $f^{2^n}(x) = q(p)$. However, $x < p_{i+1} \leqslant r(x) \leqslant r_1(p)$, a contradiction. $\square$

Lemma 2.7. *Let f satisfy* (2.2) *and $n \in \mathbb{N}$ be fixed. Then the following assertions hold for any $1 \leqslant k \leqslant 2^{n+1}$.*

(i) $[\min f^k(F_p), \max f^k(F_p)] \cap Fix(f^{2^{n+1}}) = f^k(F_p)$.
(ii) *If $f^k(p) = \min f^k(F_p)$ (respectively $f^k(p) = \max f^k(F_p)$), then $f^{2^{n+1}}(x) > f^k(p)$ (respectively $f^{2^{n+1}}(x) < f^k(p)$) for $x \in (f^k(p), \max f^k(F_p))$ (respectively $x \in (\min f^k(F_p), f^k(p))$).*

Proof. Set $A = [\min f^k(F_p), \max f^k(F_p)] \cap Fix(f^{2^{n+1}})$. Then A must contain $f^k(F_p)$. If $A \neq f^k(F_p)$, then there exists a fixed point q of $f^{2^{n+1}}$ for which $q \notin f^k(F_p)$ and $\min f^k(F_p) < q < \max f^k(F_p)$. Thus, there exists $x \in (p_1, p_m)$ such that $f^k(x) = q$. Hence, $f^{2^{n+1}}(x) = f^{2^{n+1}-k}(q) \notin f^{2^{n+1}-k}(f^k(F_p)) = f^{2^{n+1}}(F_p) = F_p$. Since $f^{2^{n+1}-k}(q)$ is also a fixed point of $f^{2^{n+1}}$, we have that $f^{2^{n+1}}(x) \notin I_1^n(p)$. Upon using Remark 2.4, we then conclude that $f^{2^{n+1}}(x) > r(p)$. This implies that $f^{2^{n+1}}$ has a horseshoe on $I_1^n(p)$, a contradiction to our assumption that f is restrictive. For the second part of the lemma, we only illustrate the case that $f^k(p) = \min f^k(F_p)$. Assume there is an $x \in (f^k(p), \max f^k(F_p))$ for which $f^{2^{n+1}}(x) = f^k(p)$. It then follows that there exists $x' \in (p_1, p_n)$ such that $f^k(x') = x$. And so,

$$p = f^{2^{n+1}-k}\big(f^k(p)\big) = f^{2^{n+1}-k}\big(f^{2^{n+1}}(x)\big) = f^{2^{n+2}}(x').$$

However, this is not possible unless $f^{2^{n+1}}$ has a horseshoe on $I_1^n(p)$. We have just completed the proof of the lemma. □

Lemma 2.8. *Let f satisfy* (2.2) *and $p \in F(n)$. For any $1 \leqslant k < 2^{n+1}$, if $f^k(p) = \max f^k(F_p)$ (respectively $f^k(p) = \min f^k(F_p)$), and let $\min f^k(F_p) = f^k(f_p) =: \tilde{p}_i$ (respectively $\max f^k(F_p) = f^k(f_p) =: \tilde{p}_i$) for some i, then $\tilde{p}_i$ is either a fixed point of A type for $f^{2^{n+1}}$ or $f^{2^{n+1}}(\tilde{p}_i)$ is a local minimum (respectively local maximum).*

Proof. Let $F_1(p) = \{x \in F_p \mid f^{2^{n+1}}$ is increasing on a neighborhood of $x\}$, $F_2(p) = \{x \in F_p \mid x$ is of A type for $f^{2^{n+1}}\}$, and $F_3(p) = \{x \in F_p \mid x$ is a turning point of $f^{2^{n+1}}\}$. Since $f^{2^{n+1}}(p) = f^{2^{n+1}}(r(p)) = p$, $Card(F_1(p)) = Card(F_2(p))$. Using Lemmas 2.5–2.7, we see that, for all $1 \leqslant k \leqslant 2^{n+1}$,

$$Card\big(F_1\big(f^k(p)\big)\big) = Card\big(F_2\big(f^k(p)\big)\big) = Card\big(F_1(p)\big) \tag{2.7a}$$

and

$$Card\big(F_3(p)\big) = Card\big(F_3\big(f^k(p)\big)\big). \tag{2.7b}$$

Since $f^k(p)$ is a fixed point of A type for f^{2^n},

$$f^k(p) \in F_1\big(f^k(p)\big). \tag{2.8}$$

Let $f^k(p) = \max f^k(F_p)$. Now, suppose $\tilde{p}_i \in F_1(p)$. Upon using (2.7) and (2.8), we then conclude that there exist $\tilde{p}_j = f^k(p_j)$, $\tilde{p}_l = f^k(p_l)$, $1 \leqslant j, l \leqslant m$, for which $\tilde{p}_j, \tilde{p}_l \in F_2(p)$ and $(\tilde{p}_j, \tilde{p}_l) \cap F_1(p) = \emptyset$, see Fig. 3.

However, to connect the graph of $f^{2^{n+1}}$ between $\tilde{p}_j$ and $\tilde{p}_l$, it must cross the line $y = x$. Hence, there must exist $\tilde{p}_k \in F_1(p)$, a contradiction. Suppose $\tilde{p}_i \in F_3(p)$ and $\tilde{p}_i$ is a local maximum point. It then follows from (2.7) and (2.8) that there exists $\tilde{p}_{i'} \in F_2(p)$, where $\tilde{p}_{i'} \in f^k(F_p)$ and $\tilde{p}_{i'} - \tilde{p}_i < \tilde{p}_j - \tilde{p}_i$ for all $j \neq i'$, see Fig. 4. Here $\tilde{p}_j = f^k(p_j)$. This is not possible since no other fixed points of $f^{2^{n+1}}$ exists in between $\tilde{p}_i$ and $\tilde{p}_{i'}$. Hence, either $\tilde{p}_i \in F_2(f^k(p))$ or $f^{2^{n+1}}(\tilde{p}_i)$ is a local maximum point. The other part of the lemma can be similarly obtained. □

Proposition 2.2. *Assume $f : I \to I$ satisfies* (2.2). *Let $1 \leqslant k < 2^{n+1}$.*

(i) *If $f^k(p) = \max f^k(F_p)$, and let $\tilde{p}_i = \min f^k(F_p)$, then there exists*

$$x \in \big[\max\big\{Fix\big(f^{2^{n+1}}\big) \cap [0, \tilde{p}_i)\big\}, \tilde{p}_i\big] := J_1 \tag{2.9}$$

such that $f^{2^{n+1}}(x) = f^k(p)$.

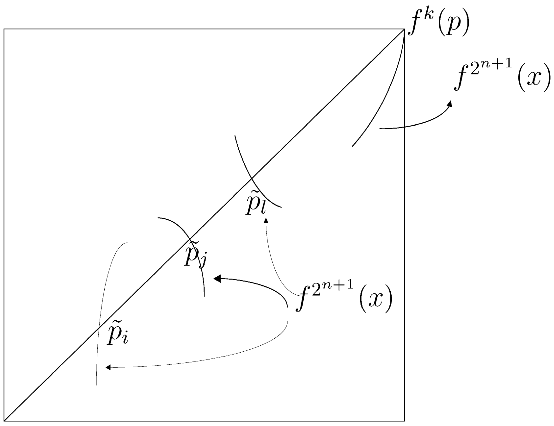


Fig. 3.

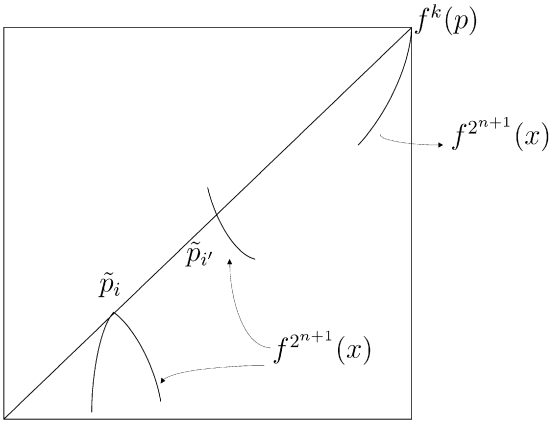


Fig. 4.

(ii) *If* $f^k(p) = \min f^k(F_p)$*, and let* $\tilde{p}_i = \max f^k(F_p)$*, then there exists*

$$x \in \big[\tilde{p}_i, \min\big\{Fix\big(f^{2^{n+1}}\big) \cap (\tilde{p}_i, 1]\big\}\big] := J_2 \tag{2.10}$$

such that $f^{2^{n+1}}(x) = f^k(p)$.

Proof. We only illustrate (i). Let $y = \max(Fix(f^{2^{n+1}}) \cap [0, \tilde{p}_i))$, we note that y is the fixed point of $f^{2^{n+1}}$ that is immediate to the left of $\tilde{p}_i$. The existence of such y is ensured by the assumption (2.2b) and the assertion in Lemma 2.8. It follows from Lemma 2.7(i) that either

$$f^{2^{n+1}-k}(y) < p \tag{2.11a}$$

or

$$f^{2^{n+1}-k}(y) > r(p), \tag{2.11b}$$

where $r(p)$ is defined in (2.5). Now, $f^{2^{n+1}-k}(\tilde{p}_i) = p_i > p$. If (2.11a) holds, then there exists $x \in [y, \tilde{p}_i]$ such that $f^{2^{n+1}-k}(x) = p$. If (2.11b) holds, then there exists $x \in [y, \tilde{p}_i]$ such that $f^{2^{n+1}-k}(x) = r(p)$. Thus, $f^{2^{n+1}}(x) = f^k(r(p))$, and so,

$$f^{2^{n+2}}(x) = f^{2\cdot 2^{n+1}}(x) = f^k\big(f^{2^{n+1}}\big(r(p)\big)\big) = f^k(p). \tag{2.12}$$

We have used (2.4) to justify the last equality of (2.12). It follows from Lemma 2.8 that the graph of $f^{2^{n+1}}$ on $(y, \tilde{p}_i)$ stays above the line $y = x$. Upon using (2.12), we conclude that $f^{2^{n+1}}(x) = f^k(p)$. □

We are now in a position to define the following notations. If $f^k(p) = \max f^k(F_p)$ (respectively $f^k(p) = \min f^k(F_p)$), we denote by

$$r_k(p) \text{ the largest } x \in J_1 \text{ (respectively the smallest } x \in J_2) \text{ so that } f^{2^{n+1}}(x) = f^k(p). \tag{2.13}$$

Here J_1 and J_2 are given in (2.9) and (2.10), respectively. For fixed n, we let $1 \leqslant k \leqslant 2^n$, and set

$$I_k^n(p) = \begin{cases} [r_k(p), f^k(p)] & \text{if } r_k(p) < f^k(p), \\ [f^k(p), r_k(p)] & \text{if } r_k(p) > f^k(p). \end{cases} \tag{2.14}$$

Remark 2.5. We note that for fixed n, $I_k^n = I_{k+2^{n+1}}^n$, for all $1 \leqslant k \leqslant 2^{n+1}$.

Theorem 2.1. *Let $f : I \to I$ be a map satisfying* (2.2)*. For fixed n, $f(I_k^n) \subset I_{k+1}^n$ for all $1 \leqslant k \leqslant 2^{n+1}$.*

Proof. Assume $I_k^n = [f^k(p), r_k(p)]$ and $I_{k+1}^n = [f^{k+1}(p), r_{k+1}(p)]$. The other three possibilities for the pair I_k^n and I_{k+1}^n can be treated similarly. If the assertion of the proposition were false, then there would exist $\tilde{x} \in (f^k(p), r_k(p))$ such that either $f(\tilde{x}) = f^{k+1}(p)$ or $f(\tilde{x}) = r_{k+1}(p)$. If $f(\tilde{x}) = f^{k+1}(p)$, then $f^{2^{n+1}}(\tilde{x}) = f^{2^{n+1}}(f^k(p)) = f^k(p)$, a contradiction. Suppose $f(\tilde{x}) = r_{k+1}(p)$. Then $f^{2^{n+1}+1}(\tilde{x}) = f^{k+1}(p)$, and hence

$$f^{2\cdot 2^{n+1}}(\tilde{x}) = f^{2^{n+1}-1}\big(f^{k+1}(p)\big) = f^{2^{n+1}}\big(f^k(p)\big) = f^k(p),$$

a contradiction. We thus complete the proof of the theorem. □

Remark 2.6.

(1) We will introduce the notation

$$I_1 \to I_2 \quad \text{if } f(I_1) \subset I_2. \tag{2.15a}$$

So far, we have shown that for each $n \in \mathbb{N}$, and each fixed point of A type for f^{2^n}, we may generate a sequence of 2^{n+1} intervals I_k^n, $1 \leqslant k \leqslant 2^{n+1}$, for which

$$I_1^n(p) \to I_2^n(p) \to \cdots \to I_k^n(p) \to I_{k+1}^n(p) \to \cdots \to I_{2^{n+1}}^n(p) \to I_1^n(p). \tag{2.15b}$$

(2) From the construction of $I_1^n(p)$, we see that there must exist a fixed point of A type, say q, for $f^{2^{n+1}}|_{I_1^n(p)}$. Hence, we may inductively generate a sequence $\{I_k^{n+1}(q)\}_{k=1}^{2^{n+2}}$ of intervals. Moreover, each $I_k^n(p)$, $1 \leqslant k \leqslant 2^{n+1}$, contains two intervals, $I_i^{n+1}(q)$ and $I_j^{n+1}(q)$, $1 \leqslant i \neq j \leqslant 2^{n+2}$.

(3) We may renumber those nested intervals in a better way. To do this, we begin with $n = 1$. Let $I_1 = I_1^1(p)$ and $I_2 = I_2^1(p)$. Let q be a fixed point of A type for f^2 in I_1. Set $I_{11} = I_1^2(q)$, $I_{21} = I_2^2(q)$, $I_{12} = I_3^2(q)$, and $I_{22} = I_4^2(q)$. Clearly, $I_{11} \cup I_{12} \subset I_1$ and $I_{21} \cup I_{22} \subset I_2$. Moreover,

$$I_{11} \to I_{21} \to I_{12} \to I_{22}.$$

Inductively, for $n = k$, we may generate 2^{k+1} intervals which satisfy the following:

$$I_{\omega^k} \to I_{\alpha(\omega^k)} \tag{2.16a}$$

where $\omega^k = (\omega_1^k, \omega_2^k, \ldots, \omega_k^k)$, and $\omega_i^k = 1$ or 2, $1 \leqslant i \leqslant k$. Given ω^k, let j be the first index such that $\omega_j^k = 1$. Then $\alpha(\omega^k) = ((\alpha(\omega^k))_1, \ldots, (\alpha(\omega^k))_k)$ is defined to be as follows:

$$\big(\alpha(\omega^k)\big)_i = \begin{cases} 3 - \omega_i^k & \text{for } 1 \leqslant i \leqslant j, \\ \omega_i^k & \text{otherwise.} \end{cases} \tag{2.16b}$$

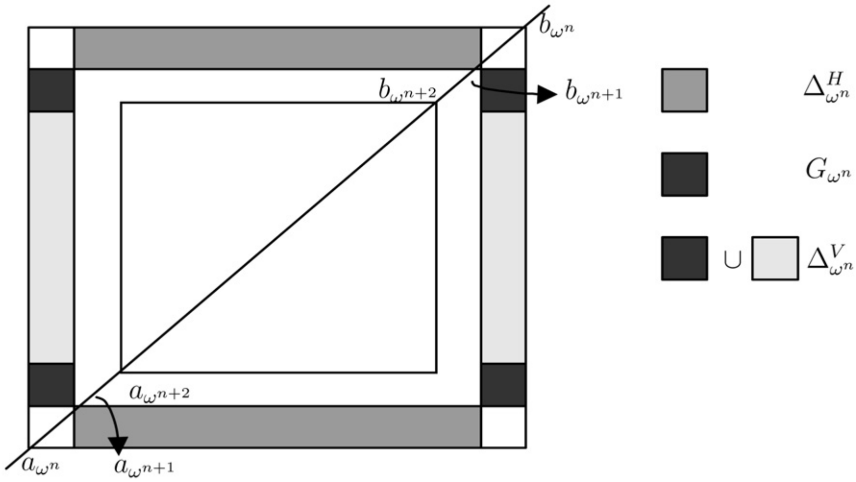


Fig. 5.

Furthermore,

$$I_{\omega^k} \subseteq I_{\omega^{k-1}} \quad \text{if } \omega_i^k = \omega_i^{k-1} \text{ for all } 1 \leqslant i \leqslant k-1. \tag{2.16c}$$

We shall denote by

$$\Sigma_2^n = \{(i_1, i_2, \dots, i_n) \mid i_k = 1 \text{ or } 2 \text{ for } 1 \leqslant k \leqslant n\}. \tag{2.16d}$$

(4) The map α in (2.16b) defined on Σ_2, the space of all (one-side) sequences with symbols in $\{1, 2\}$, is called the dyadic adding machine (see e.g., [3,5]).

For any $\omega = (\omega_1, \omega_2, \dots, \omega_n, \dots) \in \Sigma_2$, let $\omega^n = (\omega_1, \omega_2, \dots, \omega_n)$, we will show that $\lim_{n\to\infty} |I_{\omega^n}| = 0$ where $|I_{\omega^n}|$ denotes the length of I_{ω^n}. To do so, we need to introduce some definitions and notations. Let $I_{\omega^n} = [a_{\omega^n}, b_{\omega^n}]$, we first give the following notations, see Fig. 5. Define Δ_{ω^n}, $\Delta^H_{\omega^n}$, $\Delta^V_{\omega^n}$ and G^{ω^n} as follows:

$$\begin{aligned}
\Delta_{\omega^n} &= \big([a_{\omega^n}, b_{\omega^n}] \times [a_{\omega^n}, b_{\omega^n}]\big) \setminus \big([a_{\omega^{n+1}}, b_{\omega^{n+1}}] \times [a_{\omega^{n+1}}, b_{\omega^{n+1}}]\big),\\
\Delta^H_{\omega^n} &= [a_{\omega^{n+1}}, b_{\omega^{n+1}}] \times \big([a_{\omega^n}, a_{\omega^{n+1}}] \cup [b_{\omega^{n+1}}, b_{\omega^n}]\big),\\
\Delta^V_{\omega^n} &= \big([a_{\omega^n}, a_{\omega^{n+1}}] \cup [b_{\omega^{n+1}}, b_{\omega^n}]\big) \times [a_{\omega^{n+1}}, b_{\omega^{n+1}}],
\end{aligned}$$

and

$$G_{\omega^n} = \big\{[a_{\omega^n}, a_{\omega^{n+1}}] \times [a_{\omega^{n+1}}, a_{\omega^{n+2}}]\big\} \cup \big\{\big([b_{\omega^{n+2}}, b_{\omega^{n+1}}] \cup [b_{\omega^n}, b_{\omega^{n+1}}]\big) \times \big([a_{\omega^{n+1}}, a_{\omega^{n+2}}] \cup [b_{\omega^{n+2}}, b_{\omega^{n+1}}]\big)\big\}.$$

Proposition 2.3. *Let f satisfy* (2.2)*, then the graph of $f^{2^{n+1}}|_{[a_{\omega^n}, b_{\omega^n}]}$ is contained in Δ_{ω^n}.*

Remark 2.7. We note that the graph of $f^{2^{n+1}}|_{[a_{\omega^n}, b_{\omega^n}]}$ has the following 4 possibilities (see Fig. 6).

(a) a_{ω^n} and $b_{\omega^{n+1}}$ are fixed points of $f^{2^{n+1}}$, and $b_{\omega^{n+1}}$ is of A type.
(b) b_{ω^n} and $a_{\omega^{n+1}}$ are fixed points of $f^{2^{n+1}}$, and $a_{\omega^{n+1}}$ is of A type.
(c) a_{ω^n} and $a_{\omega^{n+1}}$ are fixed points of $f^{2^{n+1}}$, and $a_{\omega^{n+1}}$ is of A type.
(d) b_{ω^n} and $b_{\omega^{n+1}}$ are fixed points of $f^{2^{n+1}}$, and $b_{\omega^{n+1}}$ is of A type.

Definition 2.2. Let $p = (x_p, y_p)$ and $q = (x_q, y_q)$ be turning points of $f^{2^n}|_{[a_{\omega^n}, b_{\omega^n}]}$ and $f^{2^{n+1}}|_{[a_{\omega^{n+1}}, b_{\omega^{n+1}}]}$, respectively for some $n \in \mathbb{N}$. We say that p generates q (see Fig. 7) if either of the following conditions (i) and (ii) holds.

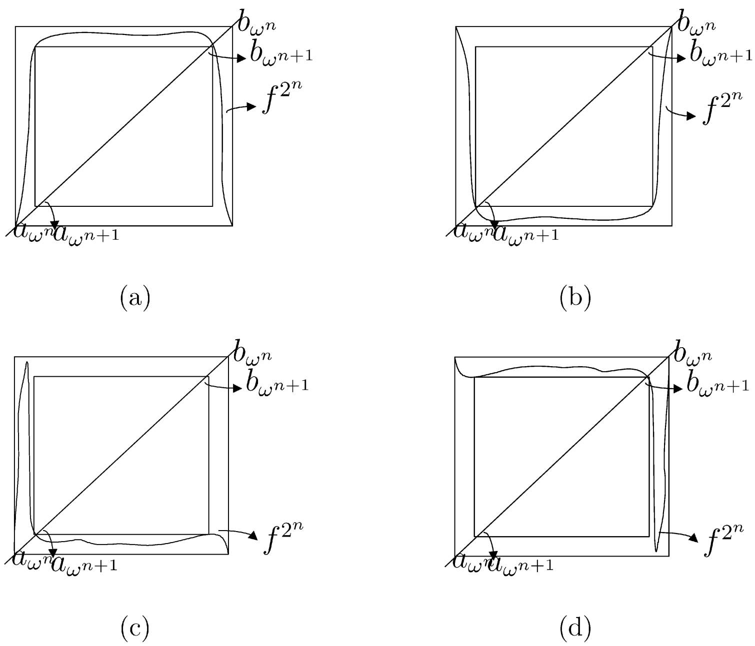


Fig. 6.

(i) $x_p = x_q$ and $y_q = f^{2^{n+1}}(x_p)$.
(ii) $f^{2^n}(x_q) = x_p$ and $y_p = y_q$.

We will also call p and q the nth generation and the $(n+1)$th generation turning points, respectively.

Clearly, for each fixed $\omega \in \Sigma_2$, $\{a_{\omega^n}\}$ is an increasing sequence and $\{b_{\omega^n}\}$ is a decreasing sequence. Moreover, $a_{\omega^n} < b_{\omega^m}$ for any $n, m \in \mathbb{N}$. Let

$$\lim_{n\to\infty} a_n = \alpha \quad \text{and} \quad \lim_{n\to\infty} b_n = \beta, \tag{2.17}$$

and so $\alpha \leqslant \beta$. We wish to show that $\alpha = \beta$. Suppose not, i.e., $\alpha < \beta$. Then we need the following terminology. Let

$p = (x_p, y_p)$ be a turning point of f^{2^n} with $x_p \notin (\alpha, \beta)$, and $q = (x_q, y_q)$ be a turning point of $f^{2^{n+1}}$ with $x_q \in (\alpha, \beta)$. If p generates q, then p is said to be a "generating turning point" (see Fig. 7(c)). (2.18)

Remark 2.8.

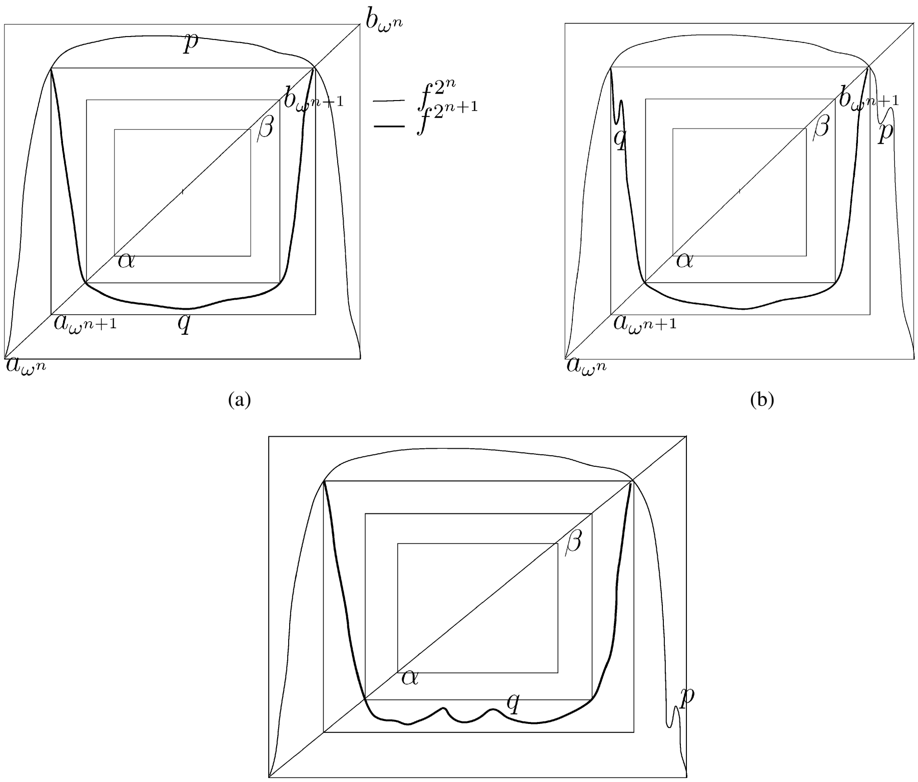
(i) If p is a generating turning point, then p generates some q.
(ii) If p is a generating turning point, then $p \in G_{\omega^n} \subset \Delta^V_{\omega^n}$.
(iii) If p generates q and q generates r, then we say that p, q and r belong to the same "family tree."

Set

$$GT_{\omega^n} = \text{the set of generating turning points of the } n\text{th generation.} \tag{2.19}$$

We are now in a position to prove the following proposition.

Proposition 2.4. *Let f satisfy* (2.2) *and suppose that $\alpha \neq \beta$. If $p \in GT_{\omega^n}$ and $q \in GT_{\omega^m}$ for some $n, m \in \mathbb{N}$, then p and q belong to the same family tree if and only if $n = m$.*



(a) (b) (c)

Fig. 7. (a) p generates q. (b) p generates q. (c) p is a generating turning point.

Proof. Using Remark 2.8(ii), Fig. 5, Definition 2.2(i), (ii), and (2.18) we see immediately that the assertion of the proposition holds true. □

Theorem 2.2. *Let f satisfy* (2.2). *For any $\omega = (\omega_1, \omega_2, \ldots, \omega_n, \ldots) \in \Sigma_2$, let $\omega^n = (\omega_1, \omega_2, \ldots, \omega_n)$, we have that* $\lim_{n\to\infty} |I_{\omega^n}| = 0$.

Proof. Since $I_{\omega^n} \supset I_{\omega^{n+1}}$, if $\lim_{n\to\infty} |I_{\omega^n}| \neq 0$, then $\lim_{n\to\infty} a_n = \alpha \neq \beta = \lim_{n\to\infty} b_n$. Set $I_\infty = [\alpha, \beta]$. We denote by C_n the set of turning points of f^{2^n} in (α, β). The cardinality of C_n is denoted by α_n. If $\alpha_n \leqslant M$, for all n, then $V_{I_\infty}(f^{2^n}) < \infty$ as $n \to \infty$. This contradicts to $\mathcal{H}_1$ condition. Suppose there exists a strictly increasing sequence $\{\alpha_{n_k}\}$ such that $\lim_{k\to\infty} \alpha_{n_k} = \infty$. Then there exists a subsequence $\{n_{k_j}\}$ of $\{n_k\}$ for which the cardinality of $GT_{\omega^{n_{k_j}}}$ goes to infinity as $j \to \infty$. For each $p_{n_{k_j}} \in GT_{\omega^{n_{k_j}}}$, there is $q_{n_{k_1}}(j) \in GT_{\omega^{n_{k_1}}}$ for which $p_{n_{k_j}}$ and $q_{n_{k_1}}(j)$ belong to the same family tree. It then follows from Proposition 2.4 that $q_{n_{k_1}}(j)$ are distinct for all j. This contradicts to the assumption that f has a finitely many number of turning points. □

We are now ready to prove our Main Theorem.

Proof of the Main Theorem. Suppose f is restrictive. From Lemma 2.1, Remark 2.3 and Proposition 2.1, we then may assume, without loss of generality, that f satisfies (2.2). Using Theorems 2.1 and 2.2, we see that for any $\epsilon > 0$, we may choose n sufficiently large that $I_k^n(p) < \epsilon$ for all $1 \leqslant k \leqslant 2^n$ and that $|f^m(I_1^n(p))| < \epsilon$ for all m. Hence, f does not have sensitive dependence on initial conditions, a contradiction to Theorem 1.3. We thus complete the proof of the theorem. □

References

[1] G. Chen, T. Huang, Y. Huang, Chaotic behavior of interval maps and total variations of iterates, Internat. J. Bifur. Chaos 14 (2004) 2161–2186.
[2] W. de Melo, S. van Strien, One Dimensional Dynamics, Springer-Verlag, New York, 1991.
[3] R. Devany, An Introduction To Chaotic Dynamical Systems, Addison–Wesley, New York, 1989.
[4] A. Katok, B. Hasselblatt, Introduction to the Modern Theory of Dynamical Systems, Cambridge Univ. Press, 1995.
[5] M. Misiurewicz, W. Szlenk, Entropy of piecewise monotone mappings, Studia Math. 67 (1980) 45–63.
[6] C. Robinson, Dynamical Systems, Stability, Symbolic Dynamics, and Chaos, CRC Press, London, 1993.