行政院國家科學委員會專題研究計畫成果報告

預載重剪力變形拱之動力與挫屈行為研究
Dynamic and Buckling Behaviors of Preloaded Arches with Shear Deformation

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中文摘要

本研究首先以變分法推導考慮剪力變形下，預載重拱動力反應之控制方程。然後，以圓拱為例，以解析解探討剪力變形效應對預載重拱自由振動及挫屈之影響。再擴展至變曲率拱之情況，此時之解乃利用級數解及動態勁度法之技巧。

本研究之算例中，將以均勻圓拱及橢圓拱為主。其中探討影響幾何參數為圓拱之 *h/R*（*h* 為拱厚，*R* 為拱半徑）、開口角和邊界條件。橢圓拱則額外探討參數長短軸比之影響。以上之結果均未有文獻討論，相信此結果必可填補學術文獻之不足，並作為工程實務設計之參考。

關鍵詞：預載重；剪力變形效應；拱；動力分析；挫屈

1. Abstract

Variational method is applied to establish the governing equations and the corresponding boundary conditions for the dynamic responses of preloaded arches with shear deformation. An analytical solution for the free vibration of a uniform circular arch subjected to a uniform dead load is developed such that the effects of various initial stress resultant terms from considering the shear deformation in the governing equations on dynamic behavior and buckling load are comprehensively studied. Then, using series solution and stiffness matrix method also develops the free vibration and buckling solutions for preloaded arches with variable curvature and cross-section.

In this research, two types of arches, namely, uniform circular arches and elliptic arches are considered as numerical examples. The parameters studied for the circular arches are h/R (h is the thickness of arch, R is the radius of arch), opening angle, and boundary condition. The additional parameter investigated for elliptic arches is the ratio of the long axis to the short axis. These results are the first known in the literature, and they are reliable references for engineers.

Keywords: preloaded, shear deformation effect, arches, dynamic analysis, buckling.

2. Motive and goal

Curve beam structures have been often used in civil, mechanical, and aerospace engineering applications, for example, arch bridges, springs, and stiffeners in aircraft structures. Research on the vibrations of curved beams began in the 19th century [1], and over 500 references can be found in review articles [2-4]. Most of the research examines the vibrations of unloaded arches and rings, but rather few publications address the vibrations of loaded arches and rings, even though dynamic analyses of loaded arches are frequently needed in many engineering applications.

For simplicity, most studies on the vibrations and stability of loaded circular arches considered cases with inextensional centerline and no shear deformation. For example, Timoshenko and Gere [5] showed closed form solutions for the buckling loads for pin-ended and fixed circular arches with uniformly distributed radial loading. Gjelsvik and Bodner [6] applied an energy method to investigate the stability of a clamped arch subjected to center point loading, while Schreyer and Masur [7] developed an exact solution for an arch with a uniform load. Wasserman [8] developed exact and approximate formulas for determining the lowest natural frequencies and critical loads of arches with flexibly supported ends. Kang *et al*. [9] used the differential quadrature method to determine the critical loads of circular arches.

Centerline extensibility is known to significantly affect the vibrations of rotating thick rings [10]. Chidamparam and Leissa [11] applied the Ritz method to elucidate how centerline extensibility influences the in-plane free vibrations of loaded circular arches. However, they assumed an inextensional centerline in determining the initial axial force distribution along a circular arch that is subjected to a static, distributed vertical load. Matsunaga [12] developed a one-dimensional higher-order theory for arches with initial axial forces and used Fourier series expansion to determine the critical loads of simply-supported circular arches subjected to constant axial forces.

Shear deformation must be considered for thick beams. The aforementioned studies indicate that there is a need to develop the equations governing free vibrations of a loaded circular arch that is shear deformable. This work develops the governing equations using the variational form presented by Washizu [13] for the dynamical problems concerning an elastic body under initial stresses. The developed governing equations include not only the effect of initial axial force but also the effects of other initial stress resultants, such as shear force and moment due to initial loading.

The equations are employed to investigate free vibration and buckling analyses of circular arches and elliptic arches under uniform vertical loading. Developing analytical solutions involves two main steps. First, the static solution for the circular arch under loading is obtained in closed form, while a series solution is developed for the elliptic arch. The solutions for vibration frequencies and buckling loads are then determined using the dynamic stiffness matrix method. A dynamic stiffness matrix is established by a series solution of the governing equations. The proposed solution is applied to elucidate the effects of opening angle and thickness-to-radius ratio on the vibration frequencies and buckling loads of loaded arches. The extent to which the magnitude of a uniformly distributed static load affects vibration frequencies is also considered. The effects of shear deformation on vibration frequencies and buckling loads are demonstrated by comparing the results with the published data obtained by ignoring shear deformation.

3. Contents of the research

3.1 Methodology

The equations governing the free vibration of a loaded arch and the associated boundary conditions are developed according to the following variational principle given by Washizu [13] for the dynamic problem of an elastic body with equilibrium initial stresses, $\sigma_{ij}^{(0)}$, :

$$\delta \int_{t_1}^{t_2} \left\{ T - U - \iiint_V \frac{1}{2} \sigma_{ij}^{(0)} \varepsilon_{ij}^{(H)} dV \right\} dt = 0 , \qquad (1)$$

where T and U are the kinetic and strain energies, given by

$$T = \iiint_V \frac{1}{2} \rho (\dot{\bar{v}}^2 + \dot{\bar{w}}^2) dV , \qquad (2)$$

$$U = \iiint_V \frac{1}{2} \sigma_{ij} \varepsilon_{ij}^{(L)} dV , \qquad (3)$$

ρ is the material density, and the dots denote the derivative with respect to time. In the strain components ε_{ij} , the superscript "L" represents infinitesimal strain parts, while the superscript "H" denotes high order terms. The term with $\sigma_{ij}^{(0)}$ represents the additional strain energy contributed by the initial static stresses.

As in Timoshenko first-order beam theory, the in-plane displacement components of an arch can be assumed to be

$$\bar{v}(r,\theta,t) = v(\theta,t) - z\psi(\theta,t) \qquad (4a)$$

$$\bar{w}(r,\theta,t) = w(\theta,t) , \qquad (4b)$$

where v and w represent the tangential and radial displacements of the centroidal axis, respectively, and ψ is the angle of rotation of the centroidal axis due to bending only. The resulting non-zero strain component, expressed in terms of displacement functions, are

$$\varepsilon_{\theta\theta} = \varepsilon_{\theta\theta}^{(L)} + \varepsilon_{\theta\theta}^{(H)} , \qquad (5a)$$

$$\varepsilon_{r\theta} = \varepsilon_{r\theta}^{(L)} + \varepsilon_{r\theta}^{(H)} , \qquad (5b)$$

$$\varepsilon_{\theta\theta}^{(L)} = \frac{1}{r} \left(\frac{\partial v}{\partial \theta} - \frac{z \partial \psi}{\partial \theta} + w \right) , \qquad (5c)$$

$$\varepsilon_{\theta\theta}^{(H)} = \frac{1}{2r^2} \left[\left(\frac{\partial v}{\partial \theta} - \frac{z \partial \psi}{\partial \theta} + w \right)^2 + \left(\frac{\partial w}{\partial \theta} - v + z\psi \right)^2 \right], \qquad (5d)$$

$$\varepsilon_{r\theta}^{(L)} = \frac{1}{2} \left[\frac{1}{r} \left(\frac{\partial w}{\partial \theta} - v + z\psi \right) - \psi \right] , \qquad (5e)$$

$$\varepsilon_{r\theta}^{(H)} = \frac{1}{2r} \left(\frac{\partial v}{\partial \theta} + w - \frac{z \partial \psi}{\partial \theta} \right) (-\psi) . \qquad (5f)$$

Introduce the following definition of stress resultants:

$$(N, M) = \int_A \sigma_{\theta\theta} (1, z) \, dA , \qquad (6a)$$

$$Q = \int_A \sigma_{r\theta} dA , \qquad (6b)$$

$$(N^{(0)}, M^{(0)}, P^{(0)}) = \int_A \sigma_{\theta\theta}^{(0)} (1, z, z^2) \, dA , \qquad (6c)$$

$$(Q^{(0)}, T^{(0)}) = \int_A \sigma_{r\theta}^{(0)} (1, z) \, dA . \qquad (6d)$$

The relationships between the stress resultants and the displacement components for an arch with h/R sufficiently less than unity are

$$N = \frac{EA}{R} \left(\frac{\partial v}{\partial \theta} + w \right) , \; M = \frac{EI}{R} \frac{\partial \psi}{\partial \theta} , \quad \text{and}$$

$$Q = \kappa GA \left(\frac{1}{R} \frac{\partial w}{\partial \theta} - \frac{v}{R} - \psi \right) , \qquad (7)$$

where E and G are elastic and shear moduli, respectively; A and I are the area and moment of inertia of the cross section, respectively; and κ is the correction factor for the shear force and equals 0.85 for a rectangular cross-section.

By performing the variation as indicated in eqn (1), the governing equations for the free vibrations of a statically loaded arch with the displacement field specified by eqns (4) are obtained and expressed as

$$\frac{N'}{R} + \frac{Q}{R} + \frac{1}{R} \left(\frac{N^{(0)}}{R} v' + \frac{M^{(0)}}{R} \psi' + \frac{N^{(0)} w}{R} - Q^{(0)} \psi \right)' + \frac{N^{(0)}}{R^2} w' - \frac{1}{R^2} (N^{(0)} v + M^{(0)} \psi) = \rho A \ddot{v} , \qquad (8a)$$

$$\frac{Q'}{R} - \frac{N}{R} + \frac{1}{R} \left(\frac{N^{(0)}}{R} w' - \frac{N^{(0)} v}{R} - \frac{M^{(0)} \psi}{R} \right)' - \frac{N^{(0)}}{R^2} v'$$

$$-\frac{M^{(0)}}{R^2}\psi' + \frac{Q^{(0)}\psi}{R} - \frac{N^{(0)}w}{R^2} = \rho A\ddot{w}, \qquad (8b)$$

$$\frac{M'}{R} + Q + \frac{1}{R}\left(\frac{M^{(0)}}{R}v' + \frac{M^{(0)}w}{R} + \frac{P^{(0)}}{R}\psi'\right)' + \frac{M^{(0)}}{R^2}w'$$
$$+\frac{Q^{(0)}}{R}(v'+w) - \frac{M^{(0)}v}{R^2} - \frac{P^{(0)}\psi}{R^2} + \frac{\psi}{R}(T^{(0)})' = \rho I\ddot{\psi}; \qquad (8c)$$

and the associated boundary conditions are

$$v = 0 \quad \text{or}$$
$$-N - \frac{N^{(0)}}{R}v' - \frac{M^{(0)}}{R}\psi' - \frac{N^{(0)}w}{R} + Q^{(0)}\psi = 0, \qquad (9a)$$
$$w = 0 \quad \text{or}$$
$$-Q - \frac{N^{(0)}}{R}w' + \frac{N^{(0)}v}{R} + \frac{M^{(0)}\psi}{R} = 0, \qquad (9b)$$
$$\psi = 0 \quad \text{or}$$
$$-M - \frac{M^{(0)}}{R}v' - \frac{M^{(0)}w}{R} - \frac{P^{(0)}}{R}\psi' - T^{(0)}\psi = 0, \qquad (9c)$$

where the primes denote derivatives with respect to θ.

Substituting the relations between stress resultants and displacement components into the governing equations yields a set of three differential equations with variable coefficients in terms of the displacement functions. The solution of the governing equations is obtained in two main steps. The static solution is first found for determining the distribution of initial stress resultants caused by static loading. Then, the solutions for vibration frequencies and buckling loads are obtained by solving the proposed governing equations. In this study, analytic solutions are developed by incorporating the concept of a finite element method into a series solution, constructed using the Frobenius method. That is, the solutions are established by decomposing the considered arch into numerous curve elements. The stiffness matrix for each element is developed by using the corresponding series solution.

The above formulation is established in frequency domain. Consequently, it can also be directly applied to study the stationary random responses of preloaded arch subjected to multiple earthquake input [14].

3.2 Applications

To demonstrate the validity of the proposed solution, a convergence study on the vibration frequencies for a unloaded circular arch was carried out. The convergent results show excellent agreement with the exact solution given by Tseng *et al*. [15].

Figures 1 and 2 exhibit the variation of λ ($=\omega R^2\sqrt{\rho A/EI}$) with β ($=\gamma R^3/EI$, γ is the intensity of loading) for clamped circular arches with h/R=0.1 and θ_0 =40° and 80°, respectively, considering various combinations of static stress resultants, namely $N^{(0)}$, $Q^{(0)}$, $M^{(0)}$, $P^{(0)}$, and $T^{(0)}$ in eqns (8a)-(8c). In the legend of these figures, the stress resultants inside parentheses are those considered in eqns (8a)-(8c) to obtain the results. (ALL) labels the results obtained by considering all the stress resultants in eqns (8a)-(8c). Only the results for the first symmetric mode are shown. Traditionally, the static axial force $N^{(0)}$ is thought of as the most important factor that influences the vibration behavior of a preloaded beam. Figure 1 reveals considerable differences between the results obtained by considering all the static stress resultants in eqns. (8a)-(8c) and those obtained by considering $N^{(0)}$ only, especially in the region of small λ, whereas only slight differences are observed in Fig. 2. Notably, Fig. 1 clearly shows that, for constant β, the results obtained by considering $N^{(0)}$ and $Q^{(0)}$ may not always agree with those obtained by considering all initial stress resultants more closely than those obtained by considering only $N^{(0)}$. The unimportance of $T^{(0)}$ is noted from the slight differences between the results obtained by considering all initial stress resultants and those obtained by considering $N^{(0)}$, $Q^{(0)}$, $M^{(0)}$ and $P^{(0)}$ (see Fig. 1).

Because of the limitation of pages for this report, more results for loaded circular arches are given in [16]. The analyses for loaded elliptic arches are also shown in [16] in detail.

4. Discussion

The present results are compared to those published, obtained by neglecting shear deformation. The comparison reveals that under tensile static loading, shear deformation significantly affects vibration behavior only for thick (h/R=0.1) and shallow (say, $\theta_0 \le 40^o$) arches. The shear deformation importantly affects thick arches, even with large opening angles, under compressive static loading. The vibration frequencies and the lowest buckling load of preloaded arches, according to shear deformation theory, are not always smaller than those obtained by neglecting shear deformation.

Traditionally, $N^{(0)}$ is considered primarily to affect vibration frequencies and buckling loads of loaded arches. This study demonstrates that the vibration frequencies and buckling loads obtained by considering only $N^{(0)}$ in the proposed equations may differ considerably from those obtained by considering all initial stress resultants, especially for thick and shallow arches.

5. Comment and conclusion

We have achieved the goals of the project given in the proposal. Based on the results in this work, a paper has been submitted for publishing in *International Journal of Solid and Structure*. Furthermore, some of the results have also been reported in *the 6th Structural Engineering Conference*, held in Ping-Tung, Taiwan 2002 [17].

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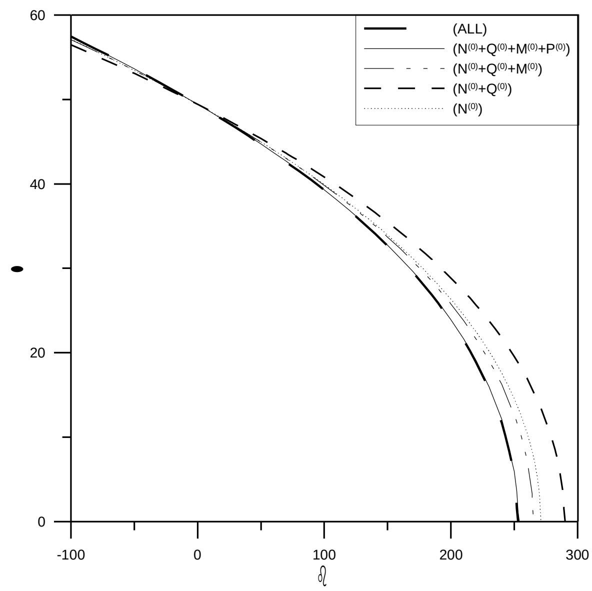

Fig. 1: Variation of λ with β for an arch with h/R=0.1 and $\theta_0 = 40^o$ (for 1st symmetric mode)

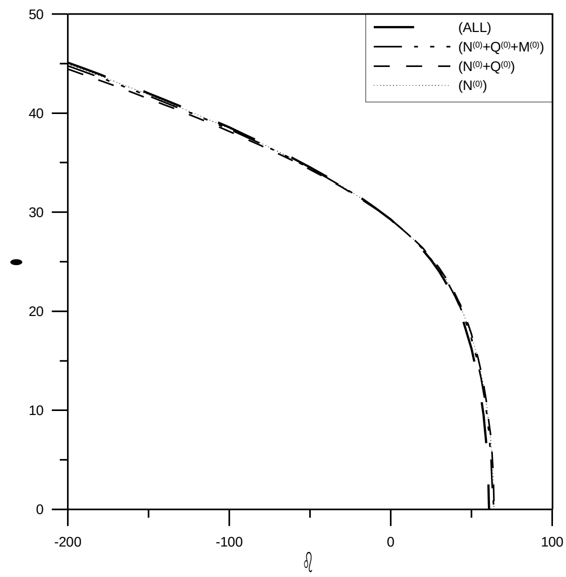

Fig. 2: Variation of λ with β for an arch with h/R=0.1 and $\theta_0 = 80^o$ (for 1st symmetric mode)