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Ranking of units on the DEA frontier with common weights

Fuh-Hwa Franklin Liu*, Hao Hsuan Peng

Department of Industrial Engineering & Management, National Chiao Tung University, 1001 Ta Shueh Road, Hsin Chu 300, Taiwan, Republic of China

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Abstract

Conventional data envelopment analysis (DEA) assists decision makers in distinguishing between efficient and inefficient decision-making units (DMUs) in a homogeneous group. However, DEA does not provide more information about the efficient DMUs. This research proposes a methodology to determine one common set of weights for the performance indices of only DEA efficient DMUs. Then, these DMUs are ranked according to the efficiency score weighted by the common set of weights. For the decision maker, this ranking is based on the optimization of the group's efficiency.

Keywords: Data envelopment analysis; Common weights analysis; Ranking

1. Introduction

Charnes et al. [1] introduce data envelopment analysis (DEA) to assess the relative efficiency of a homogeneous group of operating decision-making units (DMUs), such as schools, hospitals, or sales outlets. The DMUs usually use a set of resources, referred to as input indices, and transform them into a set of outcomes, referred to as output indices. DEA successfully divides them into two categories: efficient DMUs and inefficient DMUs. The DMUs in the efficient category have identical efficiency scores. However, it is not appropriate to claim that they have the equivalent performance in actual practice. In addition, for the category of inefficient DMUs, the efficiency score is derived from comparisons involving performances of different sets of efficient DMUs. Their performances cannot be compared by comparing them with the range of efficiency scores generated from the different facets.

Andersen et al. [2] evaluate that a DMUs efficiency possibly exceeds the conventional score 1.0, by comparing the DMU being evaluated with a linear combination of other DMUs, while excluding the observations of the DMU being evaluated. They try to discriminate between these efficient DMUs, by using different efficiency scores larger than 1.0. Cook et al. [3] developed prioritization models to rank only the efficient units in DEA. They divide those with equal scores, on the boundary, by imposing the restrictions on the multipliers (weights) in a DEA analysis. Torgersen et al. [4] achieved a complete ranking of efficient DMUs by measuring their importance as a benchmark for inefficient DMUs. Bardhan et al. [5] ranked inefficient DMUs using measure inefficiency dominance (MID) which is based on slack-adjusted DEA models. The measure ranks the inefficient DMUs according to their average proportional inefficiency in all inputs and outputs. Cooper et al. [6] ranked inefficient units according to scalar measures inefficiency proportion (MIP) in DEA, based on the slack variables.

* Correspondence author. Tel./fax: +886 3 573 1739.
E-mail address: fliu@mail.nctu.edu.tw (F.-H.F. Liu).

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Doyle et al. [7], in their research into the ranking of overall DMUs, developed a ranked scale method utilizing the cross-efficiency matrix, by ranking the average efficiency ratios of each unit. The idea of common weights in DEA was first introduced by Cook et al. [8] and Roll et al. [9] in the context of applying DEA to evaluate highway maintenance units. Cook et al. [10,11] gave a subjective ordinal preference ranking by developing common weights through a series of bounded DEA runs, by closing the gap between the upper and lower limits of the weights. Ganley et al. [12] considered the common weights for all the units, by maximizing the sum of efficiency ratios of all the units, in order to rank each unit. They suggest the potential use of the common weights for ranking DMUs. Sinuany-Stern et al. [13] used linear discriminant analysis in order to find a score function which ranks DMUs, given the DEA division into efficient and inefficient sets. Sinuany-Stern et al. [14] developed DR/DEA to provide for given inputs and outputs the best common weights in order to rank all the units on the same scale.

In this paper we aim to search one common set of weights to create the best efficiency score of one group composed of efficient DMUs. Then, we use this common set of weights to evaluate the absolute efficiency of each efficient DMUs in order to rank them. The ranking that adopts the common set of weights generated from our methodology makes sense because a decision maker objectively chooses the common weights for the purpose of maximizing the group efficiency. For instance, the general manager of a bank desires to measure the performance of DEA efficient branches of the bank. He would determine one common set of weights base upon the group performance of the DEA efficient branches. In Section 2, we review the concept of DEA framework. In Section 3, we introduce a two-stage algorithm. The first stage is a linear programming model to search one common set of weights for all efficient DMUs. The second stage is to select an optimal solution while the first stage generates alternative solutions. In Section 4, we take two virtual examples to trial our methodology. The expansion of ranking object from efficient DMUs to all DMUs is completely discussed in Section 5. Finally, Section 6 gives our conclusions.

2. DEA framework

DEA was initially developed as one methodology for assessing the comparative efficiencies of organizational units. The initial problem is usually expressed as: n DMUs to be assessed with m inputs and s outputs indices. For each DMU, say DMU_j, the given values of indices are denoted as $(x_{1j}, x_{2j}, \ldots, x_{mj})$ and $(y_{1j}, y_{2j}, \ldots, y_{sj})$, respectively. Given the data, DEA measures the best practice comparative efficiency of each DMU once and hence needs n optimizations, one for each DMU_j to be evaluated. Let the DMU_j being evaluated on any trial be designated as DMU_o where o ranges over $1, 2, \ldots, n$. We can solve the following fractional programming problem (P1) or linear programming (P2) to obtain objective value (relative efficiency θ_o^*) and one comparative set of weights of inputs ($v_{io}, i = 1, 2, \ldots, m$) and outputs ($u_{ro}, r = 1, 2, \ldots, s$). The symbol ε is a positive Archimedean infinitesimal constant, which is used in order to avoid the appearance of zero weights. This zero case in weights would result in the meaningless of certain indices used in DEA.

(P1) *DEA-FP*:

$$
\begin{aligned}
\theta_o^* = \max \quad & \frac{\sum_{r=1}^{s} y_{ro} u_{ro}}{\sum_{i=1}^{m} x_{io} v_{io}} \\
\text{s.t.} \quad & \frac{\sum_{r=1}^{s} y_{rj} u_{ro}}{\sum_{i=1}^{m} x_{ij} v_{io}} \leqslant 1, \quad j = 1, \ldots, n, \\
& u_{ro} \geqslant \varepsilon > 0, \quad r = 1, \ldots, s, \\
& v_{io} \geqslant \varepsilon > 0, \quad i = 1, \ldots, m.
\end{aligned}
$$

(P2) *DEA-LP*:

$$
\begin{aligned}
\theta_o^* = \max \quad & \sum_{r=1}^{s} y_{ro} u_{ro} \\
\text{s.t.} \quad & \sum_{i=1}^{m} x_{io} v_{io} = 1
\end{aligned}
$$

$$-\sum_{i=1}^{m} x_{ij}v_{io} + \sum_{r=1}^{s} y_{rj}u_{ro} \leqslant 0, \quad j = 1, \ldots, n,$$

$$u_{ro} \geqslant \varepsilon > 0, \quad r = 1, \ldots, s,$$

$$v_{io} \geqslant \varepsilon > 0, \quad i = 1, \ldots, m.$$

It is claimed that object DMU_o is comparative efficient, with the efficiency $\theta_o^* = 1.0$, also called an efficient DMU. We defines $E = \{j \mid \theta_j^* = 1.0, \;\; j = 1, 2, \ldots, n\}$ to represent the set of efficient DMUs. It is helpful for decision makers only to focus on the efficient DMUs. However, decision makers always face the problem of how to carry out a further comparison among DMUs on the set E. The following sections provide further analysis from the viewpoint of absolute efficiency, to assist in discriminating amongst the efficient DMUs.

3. Common weights analysis (CWA) methodology

3.1. Development

In conventional DEA models, each DMU in turn maximizes the efficiency score, under the constraint that none of DMUs' efficiency scores is allowed to exceed 1.0. Decision makers always intuitively take the maximal efficiency score 1.0 as the common *benchmark level* for DMUs. We will take advantage of this benchmark level to help us describe concretely the concept about the generation of common weights here. In Fig. 1 the vertical and horizontal axes are set to be the virtual output (weighted sum of s outputs) and virtual input (weighted sum of m inputs), respectively. By the definition of the efficiency score, the common benchmark level is one straight line that passes through the origin, with slope 1.0 in the coordinate. U_r $(r = 1, 2, \ldots, s)$ and V_i $(i = 1, 2, \ldots, m)$ in the weighted sum denote the decision variables of the common weights for the rth output and ith input index, respectively. The notation of a decision variable with superscript symbols "′" represents an arbitrary assigned value. For any two DMUs, DMU_M and DMU_N, if given one set of weights U_r' $(r = 1, 2, \ldots, s)$ and V_i' $(i = 1, 2, \ldots, m)$, then the coordinate of points M' and N' in Fig. 1 are $(\sum_{i=1}^{m} x_{iM} V_i', \sum_{r=1}^{s} y_{rM} U_r')$ and $(\sum_{i=1}^{m} x_{iN} V_i', \sum_{r=1}^{s} y_{rN} U_r')$. The virtual gaps, between points M' and M'^P on the horizontal axes and vertical axes, are denoted as $\Delta_M^{I'}$ and $\Delta_M^{O'}$, respectively. Similarly, for points N' and N'^P, the gaps are $\Delta_N^{I'}$ and $\Delta_N^{O'}$. Therefore, in view of points M' and N', we observe that there exists a total virtual gap $\Delta_M^{I'} + \Delta_M^{O'} + \Delta_N^{I'} + \Delta_N^{O'}$ to the benchmark line. Let the notation of a decision variable with superscript "∗" represent the optimal value of the variable. We want to determine an optimal set of weights U_r^* $(r = 1, 2, \ldots, s)$ and V_i^*, $(i = 1, 2, \ldots, m)$, such that both points M^* and N^* below the benchmark line could be as close to their projection

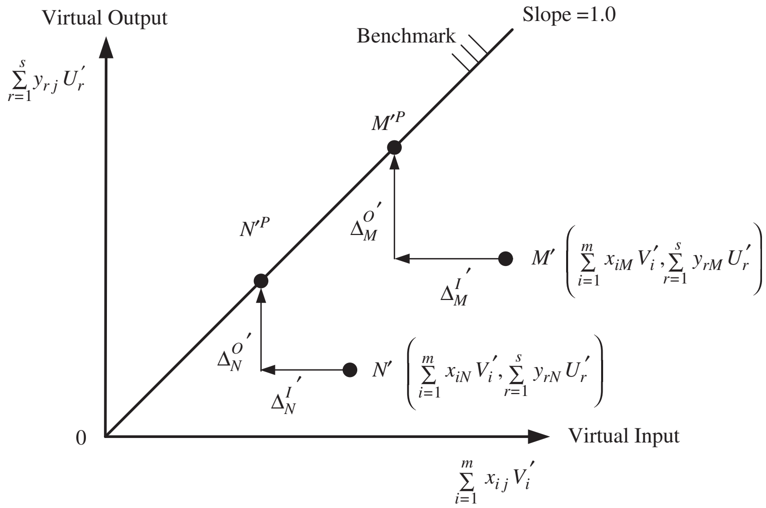

Fig. 1. Gap analysis showing DMU below the virtual benchmark line.

Table 1
Simple example to simulate CWA scenario

Index					Assign arbitrary weight $(V_1', V_2') = (25, 1)$ $(U_1', U_2') = (1, 2)$			Assign optimal weight $(V_1^*, V_2^*) = (20.33, 1)$ $(U_1^*, U_2^*) = (1, 3.33)$		
DMU	x_1	x_2	y_1	y_2	$25x_1 + x_2$	$y_1 + 2y_2$	$\Delta_j^{I'} + \Delta_j^{O'}$	$20.33x_1 + x_2$	$y_1 + 3.33y_2$	$\Delta_j^{I^*} + \Delta_j^{O^*}$
A	3	5	6	18	80	42	38	65.90	65.90	0
B	4	3	5	22	103	49	54	84.32	78.26	6.06
C	2	6	14	9	56	32	24	46.66	43.97	2.67
D	3	2	13	15	77	43	34	62.90	62.90	0
Sum							150			8.73

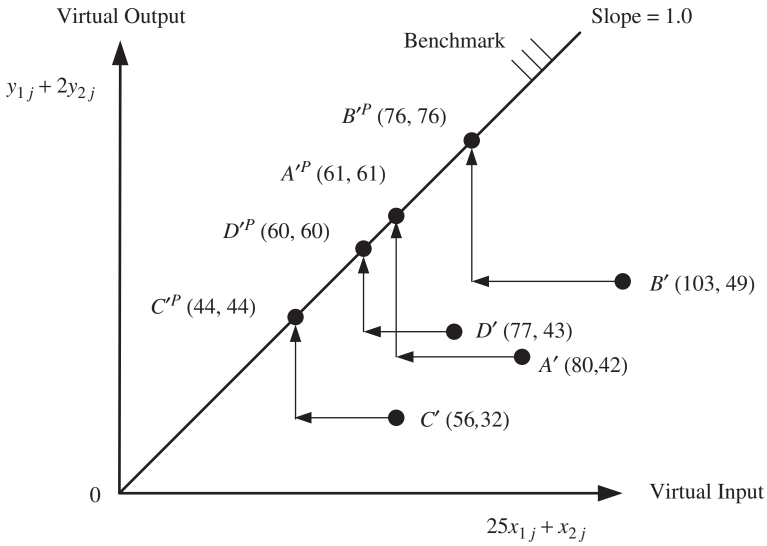

Fig. 2. Coordinates of DMUs weighted by arbitrary common set of weights.

points, M^{*P} and N^{*P} on the benchmark line, as possible. In other words, by adopting the optimal weights, the total virtual gap $\Delta_M^{I^*} + \Delta_M^{O^*} + \Delta_N^{I^*} + \Delta_N^{O^*}$ to the benchmark line is the shortest to both DMUs.

The following numerical example simulates the above scenario. Table 1 depicts the values of DMU_A, DMU_B, DMU_C, and DMU_D on the two input and two output indices. Given an arbitrary weights, $U_r' = (U_1', U_2') = (1, 2)$ and $V_i' = (V_1', V_2') = (25, 1)$, the weighted sum of inputs, weighted sum of outputs, and virtual gap $\Delta_j^{I'} + \Delta_j^{O'}$ for every DMU are recorded. In Fig. 2, points A', B', C' and D' are weighted by arbitrary weights U_r' and V_i', while A'^P, B'^P, C'^P and D'^P are their projection points on the benchmark line. There is a total virtual gap of 150 from the four DMUs to the benchmark line.

Our methodology, presented in the following subsection, generates one optimal set of weights, $U^* = (U_1^*, U_2^*) = (1, 3.33)$ and $V^* = (V_1^*, V_2^*) = (20.33, 1)$. In Fig. 3, A^*, B^*, C^* and D^* are the DMUs weighted by the optimal common set of weights U^* and V^* while A^{*P}, B^{*P}, C^{*P} and D^{*P} are their projection points onto the benchmark line. The minimum total virtual gap will approach 8.73. Obviously, the set of weights is favorable to these efficient DMUs since they are near the benchmark line. (P3) expresses the original model of our methodology. The object function is to minimize the sum of the total virtual gaps of DMUs, in set E, to benchmark line. As for the constraint, the numerator is the weighted sum of outputs plus the vertical virtual gap Δ_j^O and the denominator is the weighted sum of inputs minus the horizontal virtual gap Δ_j^I. The constraint implies that the direction closest to the benchmark line is upwards and

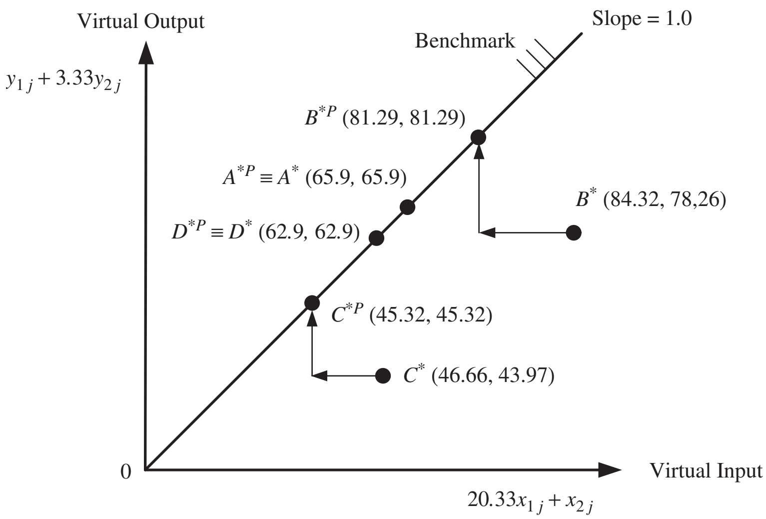

Fig. 3. Coordinates of DMUs weighted by optimal common set of weights.

leftwards at same time. The ratio of the numerator to the denominator equals to 1.0, which means that the projection point on the benchmark line is reached. All the DMUs in set E perform in the same manner. ε is a positive Archimedean infinitesimal constant. We also avoid a case of zero value of indices obtained by choosing the set of zero weights. In our methodology, we assume the benchmark line is located above all DMUs in set E. The optimal common set of weights U_r^* $(r=1,2,\ldots,s)$ and V_i^*, $(i=1,2,\ldots,m)$ to each efficient DMU would be solved and then each efficient DMU could obtain one absolute efficiency score as the standard for comparison. Then, ranking of those efficient DMUs would be completed.

(P3) *CWA-FP*:

$$\begin{aligned} \Delta^* = \min \quad & \sum_{j\in E}(\Delta_j^O+\Delta_j^I) \\ \text{s.t.} \quad & \frac{\sum_{r=1}^{s} y_{rj}U_r+\Delta_j^O}{\sum_{i=1}^{m} x_{ij}V_i-\Delta_j^I}=1, \quad j\in E, \\ & \Delta_j^O,\Delta_j^I \geqslant 0, \quad j\in E, \\ & U_r \geqslant \varepsilon > 0, \quad r=1,\ldots,s, \\ & V_i \geqslant \varepsilon > 0, \quad i=1,\ldots,m. \end{aligned}$$

The ratio form of constraints in (P3) can be rewritten in a linear form, formulated in the constraints of (P4).

(P4) *CWA-LP* 1:

$$\begin{aligned} \Delta^* = \min \quad & \sum_{j\in E}(\Delta_j^O+\Delta_j^I) \\ \text{s.t.} \quad & \sum_{r=1}^{s} y_{rj}U_r-\sum_{i=1}^{m} x_{ij}V_i+(\Delta_j^O+\Delta_j^I)=0, \quad j\in E, \\ & \Delta_j^O,\Delta_j^I \geqslant 0, \quad j\in E, \\ & U_r \geqslant \varepsilon > 0, \quad r=1,\ldots,s, \\ & V_i \geqslant \varepsilon > 0, \quad i=1,\ldots,m. \end{aligned}$$

Then, if we let $\Delta_j^I+\Delta_j^O$ be Δ_j, (P4) is then simplified to the following linear programming (P5).

(P5) *CWA-LP* 2:

$$\begin{aligned}\Delta^* = \min \quad & \sum_{j\in E} \Delta_j \\ \text{s.t.} \quad & \sum_{r=1}^{s} y_{rj} U_r - \sum_{i=1}^{m} x_{ij} V_i + \Delta_j = 0, \quad j \in E, \\ & U_r \geqslant \varepsilon > 0, \quad r = 1, \ldots, s, \\ & V_i \geqslant \varepsilon > 0, \quad i = 1, \ldots, m, \\ & \Delta_j \geqslant 0, \quad j \in E.\end{aligned}$$

(P5) could be rewritten to the equivalent linear programming (P6) by taking out the slack variable Δ_j and aggregating y_r and x_i to be Y_r and X_i, respectively, where $Y_r = \sum_{j\in E} y_{rj}$ and $X_i = \sum_{j\in E} x_{ij}$.

(P6) *CWA-LP* 3:

$$\begin{aligned}-\Delta^* = \max \quad & \sum_{r=1}^{s} Y_r U_r - \sum_{i=1}^{m} X_i V_i \\ \text{s.t.} \quad & \sum_{r=1}^{s} y_{rj} U_r - \sum_{i=1}^{m} x_{ij} V_i \leqslant 0, \quad j \in E, \\ & U_r \geqslant \varepsilon > 0, \quad r = 1, \ldots, s, \\ & V_i \geqslant \varepsilon > 0, \quad i = 1, \ldots, m.\end{aligned}$$

An implicit constraint $\sum_{r=1}^{s} Y_r U_r - \sum_{i=1}^{m} X_i V_i \leqslant 0$ could exist in (P6). This constraint is redundant since it is a linear combination of the first set of constraints. We regard X_i $(i = 1, 2, \ldots, m)$ and Y_r $(r = 1, 2, \ldots, s)$ in (P6) as the input and output indices of one aggregated DMU or group. The goal of (P6) is to maximize the efficiency of the aggregated DMU, under the constraints that the efficiency score of each DMU in set *E* cannot exceed the benchmark level. While the optimal efficiency of the aggregated DMU occurs, one corresponding set of weights is also determined, to be assigned to every DMU in set *E*. The ranking score that adopts the common set of weights generated from (P6) makes sense because the decision maker objectively chooses the common weights for the purpose of maximizing group efficiency.

For instance, the general manager of a bank desires to measure the performance of all branches of the bank. A branch would have a higher performance if the required resources could be reduced and the outputs could be increased. The possible resources could be employees, the number of bank service counters, etc., while the outputs could be multiple business items existing in the bank such as deposit business, loan business, credit card business, etc. In addition, customer satisfaction is an important output. The general manager desires to have a set of weights for these resources and output indices. However, each branch manager may focus on a different business base, a different strategy, or the limited resources. Therefore, it is difficult for the general manager to set the weight of each business item subjectively for the discrimination requirement of branches. The general manager could take advantage of DEA to distinguish the efficient branches from the inefficient ones. While the detailed ranking of efficient branches is necessary, the general manager could determine one common set of weights for the purpose of maximizing the overall efficient branches' efficiency (group efficiency) under the constraints that every efficient branch's highest efficiency score cannot exceed 1.0. Because of only considering the group of efficient branches, the general manager can take those efficient branches as a *virtual* bank. In other words, the general manager can determine one common set of weights for efficient branches, with the purpose of maximizing the virtual bank's efficiency.

In comparison with a non-radial DEA model in the CRS case, the difference is that (P6) is used to search one common set of weights, in order to evaluate the absolute efficiency score. Furthermore, (P6) can be used to discriminate between the DEA efficient DMUs which resulted from the DEA model in the CRS case. In order to obtain more information, we transform (P6) to its dual form (P7).

(P7) *CWA-DLP* 1:

$$
\begin{aligned}
\max \quad & \varepsilon\left(\sum_{r=1}^{s} P_r + \sum_{i=1}^{m} Q_i\right) \\
\text{s.t.} \quad & \\
& \sum_{j\in E} y_{rj}\pi_j - P_r = Y_r, \quad r = 1, \ldots, s, \\
& \sum_{j\in E} x_{ij}\pi_j + Q_i = X_i, \quad i = 1, \ldots, m, \\
& \pi_j \geqslant 0, \quad j \in E, \\
& P_r \geqslant 0, \quad r = 1, \ldots, s, \\
& Q_i \geqslant 0, \quad i = 1, \ldots, m.
\end{aligned}
$$

Similarly, (P7) can be used to compare with the Phase II extension of a traditional CCR model (max-slack model) while the parameter θ_o^* is equal to 1.0, as the depicted (P8). The major difference between (P7) and (P8) is that P_r and Q_i are, respectively, the total *shortfalls* and *excesses* of all efficient DMUs relative to the benchmark line, corresponding to the output index r and input index i.

(P8) *Phase II extension of CCR model*

$$
\begin{aligned}
\max \quad & \varepsilon\left(\sum_{r=1}^{s} s_r^+ + \sum_{i=1}^{m} s_r^-\right) \\
\text{s.t.} \quad & \\
& \sum_{j\in E} y_{rj}\lambda_j - s_r^+ = y_{ro}, \quad r = 1, \ldots, s, \\
& \sum_{j\in E} x_{ij}\lambda_j + s_i = x_{io}, \quad i = 1, \ldots, m, \\
& \lambda_j \geqslant 0, \quad j \in E, \\
& s_r^+ \geqslant 0, \quad r = 1, \ldots, s, \\
& s_i^- \geqslant 0, \quad i = 1, \ldots, m.
\end{aligned}
$$

The variable value π_j^* in (P7) is the *shadow price* of the linear programming (P6). Then, the variations of criterion Eq. (1) will result in the variation of constraint Eq. (2). That is, if the right-hand side of the *j*th constraint increases 1 unit, then the criterion Eq. (2) will get the variation π_j^*.

$$
\sum_{r=1}^{s} y_{rj}U_r - \sum_{i=1}^{m} x_{ij}V_i \leqslant 0 + 1, \tag{1}
$$

$$
\left(\sum_{r=1}^{s}\left(\sum_{j\in E} y_{rj}\right)U_r - \sum_{i=1}^{m}\left(\sum_{j\in E} x_{ij}\right)V_i\right) + \pi_j^*(0+1). \tag{2}
$$

π_j^* represents the total virtual gap scale to the benchmark line that can be reduced while we release the upper bound of efficiency 1.0 for DMUs. If there are multiple DMUs on the benchmark line, π_j^* will give valuable information to indicate which one most influences the total virtual gap. It is useful for determining the priority of DMUs on the benchmark line. In the following subsections, we analyze further the ranking rules of those efficient DMUs.

3.2. CWA-efficient and CWA ranking rule

In this section, we will introduce the definitions of the CWA-efficient and CWA ranking rules. First, the CWA-efficiency score of DMU_j is defined as Eq. (3).

$$\zeta_j^* = \frac{\sum_{r=1}^{s} y_{rj} U_r^*}{\sum_{i=1}^{m} x_{ij} V_i^*}, \quad j \in E. \tag{3}$$

By the value of CWA-efficiency, we can distinguish the DMUs into two separable classes, DMUs on the benchmark and those below the benchmark.

Definition 1. DMU_j is CWA-efficient (on the benchmark) if $\Delta_j^* = 0$ or $\zeta_j^* = 1.0$. Otherwise, DMU_j is CWA-inefficient (below the benchmark).

The following three definitions is necessary to distinguish whether the DMUs are on or below the benchmark line.

Definition 2. The performance of DMU_j is better than DMU_i if $\zeta_j^* > \zeta_i^*$.

Definition 3. If $\zeta_j^* = \zeta_i^* < 1$, i.e. they are both CWA-inefficient (below benchmark line), then the performance of DMU_j is better than DMU_i if $\Delta_j^* < \Delta_i^*$.

Definition 4. If $\zeta_j^* = \zeta_i^* = 1$, i.e. they are both CWA-efficient (on benchmark line), then the performance of DMU_j is better than DMU_i if $\pi_j^* > \pi_i^*$.

Each DMUs CWA-efficiency score is limited to no greater than 1.0, so there is no DMU upon the benchmark line. Furthermore, we can even ensure that there is at least one DMU that joins the assessment located on the benchmark line.

Theorem 1. *There is at least one DMU that joins the assessment located on the benchmark line*.

Proof. We will use the proof of contradiction to explain the existence of above theorem. Assume that there is no DMU on benchmark, so we can obtain the optimal criterion and the corresponding optimal value U_r^*, V_i^* and Δ_j^* where $\Delta_j^* > 0$ in P(4) for all $j \in E$ (formulated in Eq. (4)). That is, each DMUs efficiency is less than 1 (formulated in Eq. (5)).

$$\frac{\sum_{r=1}^{s} y_{rj} U_r^* + \Delta_j^*}{\sum_{i=1}^{m} x_{ij} V_i^*} = 1, \quad j \in E, \tag{4}$$

$$\frac{\sum_{r=1}^{s} y_{rj} U_r^*}{\sum_{i=1}^{m} x_{ij} V_i^*} < 1, \quad j \in E, \tag{5}$$

$$\frac{\sum_{r=1}^{s} k_j (y_{rj} U_r^*)}{\sum_{i=1}^{m} x_{ij} V_i^*} = 1, \quad j \in E. \tag{6}$$

We can set the constant k_j $(k_j > 1)$ such that the efficiency is equal to 1 for every $j \in E$ (formulated in Eq. (6)). Let K be the minimum of set $\{k_j, \text{ for } j \in E\}$, then we can obtain another feasible common set of weights KU_r^* and V_i^* accompanies the smaller Δ_j^* (at least one equals to 0) for all $j \in E$ in (P5). The case will result in smaller criterion and contradicts the fact that the current criterion has been minimized. Hence, there is at least one DMU locates on benchmark line. □

3.3. Virtual gap analysis

The virtual gaps between virtual input and output indices for each CWA-inefficient DMU could be further decomposed into the real gap of each performance index. We can further analyze this by translating the model (P6) to the equivalent model (P9). As (P9) showed, P_r and Q_i can be partitioned as $P_r = \sum_{(j\in E)} p_{rj}$ and $Q_i = \sum_{(j\in E)} q_{ij}$ and p_{rj} and q_{ij} are the shortfall at the output index r and excess at input index i of DMU_j, respectively, to the benchmark. So $p_{rj} = P_r\lambda_j$ and $q_{ij} = Q_i\lambda_j$ with convex combinations of multipliers $\lambda_j \geqslant 0$ ($j \in E$) and $\sum_{(j\in E)} \lambda_j = 1$.

(P9) *CWA-DLP* 2:

$$\max \quad \sum_{j\in E} \varepsilon\left(\sum_{r=1}^{s} p_{rj} + \sum_{i=1}^{m} q_{ij}\right)$$

s.t.

$$\sum_{j\in E} y_{rj}\pi_j = \sum_{j\in E}(y_{rj} + p_{rj}), \quad r = 1,\ldots,s,$$

$$\sum_{j\in E} x_{ij}\pi_j = \sum_{j\in E}\left(x_{ij} - q_{ij}\right), \quad i = 1,\ldots,m,$$

$$\pi_j \geqslant 0, \quad j \in E,$$

$$p_{rj} \geqslant 0, \quad r = 1,\ldots,s, \quad j \in E,$$

$$q_{ij} \geqslant 0, \quad i = 1,\ldots,m, \quad j \in E.$$

The shortfall p^*_{rj} and excess q^*_{ij} of (P9) could be obtained indirectly by the following theorem.

Theorem 2. *The shortfall p^*_{rj} and excess q^*_{ij} of CWA-inefficient* DMU_j *to benchmark corresponding to the output index r and input index i are $P^*_r\ (\varDelta^*_j/\varDelta^*)$ and $Q^*_i\ (\varDelta^*_j/\varDelta^*)$.*

Proof. Since p^*_{rj} and q^*_{ij} are shortfall and excess of CWA-inefficient DMU_j to the benchmark, we have Eq. (7) holds because of the Definition 1.

$$\frac{\sum_{r=1}^{s}(y_{rj} + p^*_{rj})U^*_r}{\sum_{i=1}^{m}(x_{ij} - q^*_{ij})V^*_i} = 1, \tag{7}$$

$$\frac{\sum_{r=1}^{s}\left(y_{rj} + \frac{P^*_r\varDelta^*_j}{\varDelta^*}\right)U^*_r}{\sum_{i=1}^{m}\left(x_{ij} - \frac{Q^*_i\varDelta^*_j}{\varDelta^*}\right)V^*_i} = 1. \tag{8}$$

To prove Eq. (8) is a truth, we first decompose the numerator and denominator to obtain Eqs. (9) and (10), respectively.

$$\sum_{r=1}^{s} y_{rj}U^*_r + \sum_{r=1}^{s}\frac{P^*_r\varDelta^*_j}{\varDelta^*}U^*_r, \tag{9}$$

$$\sum_{i=1}^{m} x_{ij}V^*_i - \sum_{i=1}^{m}\frac{Q^*_i\varDelta^*_j}{\varDelta^*}V^*_i. \tag{10}$$

Subtract Eq. (10) from Eq. (9) resulted Eq. (11).

$$\sum_{r=1}^{s} y_{rj}U^*_r - \sum_{i=1}^{m} x_{ij}V^*_i + \frac{\varDelta^*_j}{\varDelta^*}\left(\sum_{r=1}^{s} P^*_rU^*_r + \sum_{i=1}^{m} Q^*_iV^*_i\right). \tag{11}$$

Since the lower bound of U^*_r and V^*_i is ε, according to *Complementary Slackness Theorem*, the following relationship holds.

$$\sum_{r=1}^{s} P^*_rU^*_r + \sum_{i=1}^{m} Q^*_iV^*_i = \varepsilon\left(\sum_{r=1}^{s} P^*_r + \sum_{i=1}^{m} Q^*_i\right) = \varDelta^*. \tag{12}$$

Therefore, the formula inside the parenthesis in Eq. (11) could be substituted by the right-hand side in Eq. (12). Obviously, Eq. (11) easily translates to Eq. (13).

$$\sum_{r=1}^{s} y_{rj} U_r^* - \sum_{i=1}^{m} x_{ij} V_i^* + \Delta_j^*. \tag{13}$$

Eq. (13) is equal to zero by the fulfillment of constraints in (P5). Hence, Eq. (8) comes into existence and the theorem is proved. □

3.4. Selection of the alternative optimal common sets of weights

It is worth noting that (P5) sometimes encounters the existence of alternative weights; moreover, different weights can result in different rankings of efficient DMUs. It is necessary for the decision maker to select the applicable one from these efficient DMUs. We propose one approach to assist decision makers in dealing with the issue of alternative rankings. Lets it is observed that one set of output indices with two sets of different index value has the same weighted sum. While the same weighted sum exists for both sets of different index value by adopting different sets of weights, Obata et al. [15] propose that it is preferable for output indices to adopt the smaller range of weight. In other words, a larger range of output index values is desirable to the decision maker. He explains that the benefit of the weighted sum is from the indices value itself, rather than from the individual weights. Similarly, it is preferable to use the larger range of weights for input indices. That is, a smaller range of input index values is desirable to the decision maker. The following procedure is suggested as a way to search the appropriate range of the existing alternative set of weights, using the L_1-norm.

Stage 1: Solve (P5) and obtain the optimal value Δ^*.

Stage 2: Solve the following linear programming (P10) to obtain one optimal common set of weights.

(P10) *Optimal weight analysis*:

$$\begin{aligned}
\min \quad & \sum_{r=1}^{s} U_r - \sum_{i=1}^{m} V_i \\
\text{s.t.} \quad & \sum_{r=1}^{s} y_{rj} U_r - \sum_{i=1}^{m} x_{ij} V_i + \Delta_j = 0, \quad j \in E, \\
& \sum_{j \in E} \Delta_j = \Delta^*, \\
& U_r \geqslant \varepsilon > 0, \quad r = 1, \ldots, s, \\
& V_i \geqslant \varepsilon > 0, \quad i = 1, \ldots, m, \\
& \Delta_j \geqslant 0, \quad j \in E.
\end{aligned}$$

In stage 1, we first have to look for the minimization of the total virtual gap. Then select one appropriate weight in stage 2, under the optimal status of (P1). Thus, we keep the optimal criteria value of (P5) as one constraint in the linear programming (P10) and then take the minimization of the sum of output weights and maximization of the sum of input weights as the criterion.

4. Dealing with the data sets with special properties

Two data sets are introduced here, as an illustration. In the first data set, the scales of DMUs are in large ranges. In the second data set, the number of performance indices is much larger than the number of DMUs. These two examples usually produce a similarly large number of efficient DMUs. The proposed CWA methodology is able to deal with the data sets with special properties.

Table 2
Example 1 with large scale ranges

DMU_j	x_{1j}	x_{2j}	y_{1j}	θ_j^*	CCR-Slack (s_1^{-*})
C1	470 000	700 000	200 000	1	0
C2	4800	7000	2000	1	100
C3	49	70	20	1	0
C4	5	7	2	1	0
C5	510	700	200	1	40
C6	52 000	70 000	20 000	1	5000
C7	530 000	700 000	200 000	1	60 000

Table 3
Corresponding outcomes of example 1 assessed by CWA

DMU_j	Δ_j^*	ζ_j^*	Rank	p_{1j}^*	$(p_{1j}^*/x_{1j})*100$
C1	0	1.000	1	0	0
C2	100	0.992	2	100	2.1
C3	2	0.983	3	2	4.1
C4	0.3	0.975	4	0.3	6.0
C5	40	0.967	5	40	7.8
C6	5000	0.959	6	5000	9.6
C7	60 000	0.951	7	60 000	11.3

4.1. Data with large-scale ranges

Table 2 gives the simulated data set of seven DMUs with two inputs and one output. The figures for DMU_{C1}, DMU_{C2}, DMU_{C3}, DMU_{C5}, DMU_{C6} and DMU_{C7} are shown to be many times larger than DMU_{C4}, with efficiency scores = 1.0. Slacks appeared in the index x_1. First, we take advantage of the input (output)-oriented CCR model to determine the efficiency of each DMU. For the sake of scale issues and the existence of slacks, it may not be possible to rank the seven DMUs, but, intuitively, their rankings are DMU_{C1}, DMU_{C2}, DMU_{C3}, DMU_{C4}, DMU_{C5}, DMU_{C6} and DMU_{C7}.

Employing the proposed CWA methodology, we obtained the optimal common set of weights $(V_1^*, V_2^*, U_1^*) = (1, 1, 5.85)$. The rankings, according to their CWA efficiency scores ζ, are consistent with the intuitive rankings. As shown in Table 3, DMU_{C1} is the only one that locates on the benchmark line. The last two columns show the gap to the benchmark on index x_1 in current value and percentage. It shows that CWA is able to provide complete ranking and gap in performance indices information for all DMUs.

4.2. Data with large number of indices

DEA models usually have no more than $n/2$ indices when assessing n DMUs. Otherwise, the number of efficient DMUs becomes unreasonably large. It means that the discriminating power of DEA is reduced. The example uses seven DMUs, with three inputs and three outputs. The last column in Table 4 shows that the seven DMUs are efficient by radial efficiency 1 obtained by CCR-Input-oriented model.

Table 5 gives the detailed ranking information assessed by adopting CWA. We still find that there are five DMUs still on benchmark line. If we release the upper bound of the efficiency score 1.0 for these DMUs, then π_j^* leads to a reduction in scale in the total virtual gap towards the benchmark line. Obviously, a CWA-efficient DMU with a larger π_j^* is the better one. The total virtual gap can be reduced to a maximum 3.225, compared to the other DMUs on the benchmark line ,while we release the upper bound of efficiency score to over 1.0. Therefore, after comparing with π_j^*, we are able to determine the final ranking of CWA-efficient DMUs to be DMU_{D3}, DMU_{D6}, DMU_{D1}, DMU_{D4}, DMU_{D5}, DMU_{D7}, and DMU_{D2}.

Table 4
Example 2 with the number of indices is much larger than DMUs

DMU	x_{1j}	x_{2j}	x_{3j}	y_{1j}	y_{2j}	y_{3j}	Efficiency (θ_j^*)
D1	1621	436	205	174	497	22	1
D2	2718	314	221	172	497	22	1
D3	1523	345	215	160	443	22	1
D4	5514	1314	553	487	1925	63	1
D5	1941	507	309	220	521	36	1
D6	1496	321	339	109	699	38	1
D7	932	158	200	37	431	19	1

Table 5
Corresponding outcomes of example 2 assessed by CWA

DMU	Δ_j^*	π_j^*	ζ_j^*	Rank
D3	0	3.225	1.000	1
D6	0	1.772	1.000	2
D1	0	1.118	1.000	3
D4	0	0.922	1.000	4
D5	0	0.028	1.000	5
D7	304.864	0.000	0.847	6
D2	925.362	0.000	0.778	7

Table 6
Example 3 including the initial DMUs in example 2

DMU	x_{1j}	x_{2j}	x_{3j}	y_{1j}	y_{2j}	y_{3j}
D1	1621	436	205	174	497	22
D2	2718	314	221	172	497	22
D3	1523	345	215	160	443	22
D4	5514	1314	553	487	1925	63
D5	1941	507	309	220	521	36
D6	1496	321	339	109	699	38
D7	932	158	200	37	431	19
D8	2013	1037	412	198	471	32
D9	1891	976	399	191	491	22
D10	2277	891	418	241	379	28
D11	1995	693	349	167	412	31

5. Expansion and discussion

In this section, we extend the object of CWA ranking from DMUs in set E to $E \cup E^C$ in (P3) where E^C represents the set of inefficient DMUs. It is unfortunate that a paradoxical case exists, that some DMUs in E^C are better than DMUs in E. However, the phenomenon is acceptable and explainable without violating the original concept of DEA. In fact, each DMU in E^C would have a particular reference set that is composed of parts of DMUs in E. One should not assume that one certain DMU in E is better than all DMUs in E^C.

We verify the inference mentioned above by practicing one complete example including DMUs in set $E \cup E^C$. As listed in Table 6, Example 2 with 7 DMUs in set E is extended to Example 3 with 11 DMUs in set $E\cup E^C$. Using DEA model (P2) and CWA methodology, models (P5) and (P10), the results are depicted in Table 7. In view of CWA, we observe DMU_{D2} of set E is ranked 11, and is worse than DMU_{D8}, DMU_{D9}, DMU_{D10}, and DMU_{D11} of set E^C. Although DMU_{D12} belongs to set E, it is not an element of the reference set to any DMU_{D8}, DMU_{D9}, DMU_{D10}, and DMU_{D11}. In other words, individual DMU_{D8}, DMU_{D9}, DMU_{D10}, and DMU_{D11} really are not dominated by DMU_{D2}.

Table 7
The reference set, DEA and CWA efficiency score of 11 DMUs in example 3

DMU	Reference set	DEA efficiency (θ_j^*)	CWA efficiency (ζ_j^*)	Rank
D1	D1	1	1	1
D2	D2	1	0.69	11
D3	D3	1	0.99	4
D4	D4	1	0.97	5
D5	D5	1	1	3
D6	D6	1	1	2
D7	D7	1	0.82	6
D8	D1, D5	0.87	0.72	7
D9	D1, D5	0.91	0.73	8
D10	D5	0.93	0.74	9
D11	D5, D6	0.79	0.71	10

Table 8
The DEA efficiency score revaluated only to 5 debatable DMUs in Table 7

DMU	DEA efficiency (θ_j^*)
D2	1
D8	1
D9	1
D10	1
D11	1

Therefore, in view of DEA, one should not conclude that DMU_{D2} in set E is better than DMU_{D8}, DMU_{D9}, DMU_{D10}, and DMU_{D11} in set E^C. In addition, using DEA model (P2) to measure the relative efficiency of only these 5 DMUs, at this time one would observe that they belong to the equivalent set E, just depicted in Table 8. Therefore, the CWA ranking seems also to be workable in set $E \cup E^C$ without violating the original concept of DEA.

CWA ranking reflects two consequences. The first is that it is primarily used in ranking the DMUs in set E. The second is that when it is used in ranking the DMUs in set $E \cup E^C$, one could still obtain a reasonable conclusion without conflicting with the DEA's initial classification.

6. Conclusion

This paper researches one common set of weights that is the most favorable for determining the absolute efficiency for DMUs in set E at the same time. The practical application of this methodology is aimed at the ranking of a group of DMUs without advanced priority in them, such as DMUs in set E. New intuitional ranking rules, obtained from absolute efficiency, could help decision makers understand the performance of DMUs. The CWA methodology is also workable in two assessments where the number of indices is much larger than the number of DMUs, and the scales of DMUs are in large ranges. Two examples given in this paper show good ranking results.

The methodology proposed in this paper is initially adopted in the ranking of DMUs in set E. This paper also discussed the possibility of expansion in ranking objects from set E to set $E \cup E^C$. We have obtained a positive conclusion and illustrate how CWA ranking could not conflict with the original DEA classification.

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