行政院國家科學委員會專題研究計畫成果報告

非齊次馬可夫鍊的轉換機率

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主持人：彭南夫　國立交通大學統計研究所

共同主持人：

計畫參與人員：

一、摘要：

此篇論文考慮一個具有與時間有關的細微矩陣的非其齊次馬可夫鍊。在一些不同的狀況下，我們可以找到轉換機率。同時我們也找到一些有效的方法，來尋求轉換機率的近似解，也提供各種例子。

關鍵詞：　非齊次馬可夫鍊、轉換機率

Abstract:

Consider a nonhomogeneous Markov chain with time-dependent infinitesimal matrix. The exact transition probabilities under various conditions are found. For general nonhomogeneous Markov chain with continuous infinitesimal matrix, we find an efficient way to approximate transition probabilities. Various examples and some numerical results are also presented.

Ketword: Nonhomogeneous Markov Chains, Transition Probabilities

二、緣由與目的

Nonhomogeneous Markov chains are always more difficult to handle with than homogeneous ones. The solutions to the Kolmogorov forward and backward equations are also hard to obtain. As we shall see later that a nonhomogeneous Poisson process which can be easily dealt with is an exceptional one. The purpose of this article is to find conditions under which the analytic forms of the transition probabilities of nonhomogeneous Markov chains can be obtained. Approximate alternative solutions are also acquired. Various examples and some numerical results are included.

三、成果與討論

Consider a continuous time nonhomogeneous Markov chain $X(t)$ with state space $\{1, 2, \ldots\}$ and time-dependent infinitesimal matrix $Q(t) = \left(q_{ij}(t)\right)$. Let $P_{ij}(s,t) = P(X(t) = j \mid X(s) = i)$ be the transition probability from state i at epoch s to j at epoch t. The well known system of Kolmogorov forward and backward differential equations written in matrix forms are, [1],

$$\text{(1)} \qquad \frac{\partial P(s,t)}{\partial t} = P(s,t)Q(t)$$
$$\frac{\partial P(s,t)}{\partial s} = -Q(s)P(s,t)$$

where $P(s,t) = (P_{ij}(s,t))$ and $\frac{\partial P(s,t)}{\partial t} = \left(\frac{\partial P_{ij}(s,t)}{\partial t}\right)$. For a given t, the solutions to (1) are unique under the condition that $\left|q_{ij}(\tau)\right| < \infty, \forall \tau \le t$, [1]. However, they are difficult to obtain in analytic form. Yet for some cases we do have ways to write them in nice patterns.

Proposition 1

Suppose $Q(x) = \alpha(x)A$, where $\alpha(x)$ is a real valued function and A is a constant matrix. The transition probability matrix becomes $P(s,t)=e^{\int_s^t a(x)dx\,A}$.

Example 1

A nonhomogeneous Poisson process with intensity function $a(x)$ is a typical one. The constant matrix $A=\begin{bmatrix}-1 & 1 & 0 & \cdots \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & \\ \vdots & & & \ddots\end{bmatrix}$.

Hence we have $P(s,t)=e^{\int_s^t a(x)dx\,A}=e^{-\int_s^t a(x)dx}e^{\int_s^t a(x)dx\,(A+I)}$. Due to the special structure of $(A+I)$, it is easy to see that $(e^{\int_s^t a(x)dx\,(A+I)})_{ij}=\left.(\int_s^t a(x)dx)^{j-i}\right/(j-i)!$ which is the core part of the probability function of the Poisson process with mean $\int_s^t a(x)dx$.

Proposition 2

Suppose $Q(x)=\sum a_i(x)A_i$ where $a_i(x)$'s are real valued functions and A_i's are constant matrices satisfying $A_i^2=A_i$ and $A_iA_j=0, \forall i\neq j$. That is, *Q(x)* is diagonalizable with constant eigenspaces but time-dependent eigenvalues. The transition probability matrix becomes $P(s,t)=\sum_{i=1}^{k}e^{\int_s^t a_i(x)dx}A_i$.

Proposition 3

If $C(x)$ satisfies the conditions of Corollary 1, that is, $C(x)=a(x)A$, then $(P_{12}(t),\ldots,P_{1n}(t))=\int_0^t(q_{12}(x),\ldots,q_{1n}(x))e^{\int_x^t a(y)dy\,A}dx$.

If $C(x)$ satisfies the conditions of Corollary 2, that is, $C(x)=\sum a_i(x)A_i$ where A_i 's are idempotent and orthogonal to each other, then $(P_{12}(t),\ldots,P_{1n}(t))=\int_0^t(q_{12}(x),\ldots,q_{1n}(x))\left[\sum_{i=1}^{k}e^{\int_x^t a_i(y)dy}A_i\right]dx$.

Example 2

If $q_{ij}=q_{1j}$ for $i\neq j$, $i>1$ and $j>1$, then $C(x)=diag(-q_{12}+q_{22},\ldots,-q_{1n}+q_{nn})$. Hence $P_{1i}(t)=\int_0^t q_{1i}(x)e^{\int_x^t(-q_{1i}+q_{ii})(y)dy}dx,\quad i\geq 2$. A special case is a 2-state Markov chain whose $C(x)=(-q_{12}+q_{22})(x)=-(q_{12}+q_{21})(x)$, and $P_{12}(t)=\int_0^t q_{12}(x)e^{-\int_x^t(q_{12}+q_{21})(y)dy}dx$. The other three transition probabilities can be obtained by symmetry and complementary.

Example 3

The situation is more complicated for a 3-state Markov chain rather than a 2-state one. Explicit forms of transition probabilities can be obtained if conditions are imposed. Suppose $q_{12}=q_{32}$, we obtain $P_{12}(t)=\int_0^t q_{12}(x)e^{-\int_x^t(q_{12}+q_{21}+q_{23})(y)dy}dx$, $P_{32}(t)=\int_0^t q_{32}(x)e^{-\int_x^t(q_{32}+q_{21}+q_{23})(y)dy}dx$ and. The first one is obtained by noting that *C(x)* is upper triangular. The second one is obtained by interchanging the roles of state 1 and state 3. The third one is obtained by assuming $q_{21}=q_{31}$ first, then multiplying the vector $(0,1,1)'$ from the right to both sides of (4), and finally interchanging the roles of state 1 and state 2. The way we used is originated from lumping the state 1 and state 3.

Example 4

For $t\leq 5$, we let

$$Q(t)=\begin{bmatrix} Q_{11} & \sin\frac{t}{5} & \cos\frac{t}{5} & e^{-\frac{t}{5}} \\ 1+\sin\frac{t}{5} & Q_{22} & \cos\frac{t}{5} & e^{-\frac{t}{5}} \\ 1+\cos\frac{t}{5} & \sin\frac{t}{5} & Q_{33} & e^{-\frac{t}{5}} \\ e^{+\frac{t}{5}} & \sin\frac{t}{5} & \cos\frac{t}{5} & Q_{44} \end{bmatrix}$$ which satisfies the condition $Q(t)(1,1,1,1)'=(0,0,0,0)'$. The $Q(t)$ is chosen to make C(x) diagonal with $C_{11}(x)=-(1+2\sin\frac{x}{5}+\cos\frac{x}{5}+e^{-\frac{x}{5}})$, $C_{22}(x)=-(1+2\cos\frac{x}{5}+\sin\frac{x}{5}+e^{-\frac{x}{5}})$ and $C_{33}(x)=-(\sin\frac{x}{5}+\cos\frac{x}{5}+e^{-\frac{x}{5}}+e^{+\frac{x}{5}})$. Direct computations show the transition probabilities.

It is understood that we are not able to obtain analytic solutions to (1) in most cases. Approximate solutions are naturally an alternative. To find an efficient way, we assume that all $q_{ij}(x)$'s are continuous functions. Moreover, letting Δt be small, we have

(5) $P(s,s+\Delta t)=I+\int_s^{s+\Delta t}Q(x)P(x,s+\Delta t)dx$

$\approx I+Q(s)P(s,s)\Delta t=I+Q(s)\Delta t$.

Furthermore, the equation (1) gives $P(s,t+\Delta t)-P(s,t)=P(s,t)Q(t)\Delta t$. Thus,

(6) $P(s,t+\Delta t)=P(s,t)[I+Q(t)\Delta t]$.

Combining (5) and (6), and letting $\Delta t={(t-s)}/{n}$, $t_i=s+i\Delta t$, $i=0,\ldots,n$, we have

(7)

$$P(s,t)\approx\prod_{i=0}^{n-1}[I+Q(t_i)\Delta t]=\prod_{i=0}^{n-1}\left[I+\frac{Q(t_i)(t-s)}{n}\right].$$

If Q is constant, then $P(s,t)\approx\left[I+\frac{Q(t-s)}{n}\right]^n$.

Choosing $n=2^m$ for some positive integer m, we need to operate only $\log_2 n=m$ times of matrix multiplication. Compared to the external uniformization method proposed in [2], which the inverse of a matrix is required and therefore sometimes inefficient, our method is easier but less accurate. However, it can be easily compensated by taking large n technically yet $\log_2 n$ practically.

With regard to (7), we see that Q is assumed to be constant in each interval $(i\Delta t,\ (i+1)\Delta t)$ and we approximate $P(i\Delta t,\ (i+1)\Delta t)$ by $\left[I+\frac{Q(t_i)(t-s)}{n}\right]$. In fact, $P(i\Delta t,\ (i+1)\Delta t)$ is $e^{\left[\frac{Q(t_i)(t-s)}{n}\right]}$ under that assumption. Hence, we modify (7) as

(8) $P(s,t)\approx\prod_{i=0}^{n-1}\left[I+\frac{Q(t_i)(t-s)}{nm}\right]^m$.

Again, we choose $m=2^k$ so that $\log_2 m=k$ matrix multiplications are needed in each interval. Also, it is usually true that computing $Q(t_i)$ contains heavier load than matrix multiplication does. Thus, we prefer to use (8) for moderate n and large m rather than use (7) for large n directly.

Example 4 (continued)

Another purpose of choosing such Q in Example 4 is to comprise miscellaneous functions in it, increasing or decreasing. Let t =5 and $m=2^5$. We target on computing P(0,5). The exact solutions obtained in Example 4, computed numerically though, can reach to extremely high precision and thus are regarded as standards. Computer results show that n=1358 is needed to reach the accuracy of 10^{-5} when (7) is used, while only n=90 is required by using (8) to reach the same precision.

四、計畫成果自評

This approach should be helpful to finding transition probabilities of nonhomogeneous Markov chain with time-dependent infinitesimal matrix.

五、參考文獻

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