行政院國家科學委員會補助專題研究計畫成果報告

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行政院國家科學委員會專題研究計畫成果報告

球面上預設純曲率之超曲面
Hypersurfaces with prescribed Scalar Curvature in the Sphere

計畫編號：NSC 89-2115-M-009-031
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摘要

假設 N 為(n+1)-維單位球面， R 為定義於 N 上之預設函數。 此報告主要導出： 當 M 為一定義於 n-維單位球面的函數之超曲面時，其對應之非線性方程。 在 n=2 的情形下，即高斯型方程的情形，觀察存在與唯一的可能性。

關鍵詞：純曲率、超曲面、球面

Abstract

Let N be the (n+1)-dimensional unit sphere and R be a function defined on a region of N. Consider M as a graph of a function u defined on a totally geodesic n-sphere, we derive the fully nonlinear partial differential equation for the problem of prescribed scalar curvature R. Then we consider the equation in the case of n=2, and obtain some observations.

Keywords: scalar curvature, hypersurfaces, spheres.

1. Introduction

Let N be a complete (n+1)-dimensional manifold and Ω be a open connected subset of N. Let F be a smooth, symmetric function defined in the n-dimensional Euclidean space. The problem of prescribed curvature is: *Given a smooth function K defined on* Ω, *find a closed hypersurface M contained in* Ω *such that the principal curvatures satisfy the equation F = K on M.*

This is in general a problem for a system of fully nonlinear partial differential equations. For technical reasons it is convenient to consider certain associated scalar elliptic equation. The existence of convex solutions has been studied extensively by various authors. Using the elliptic theory, the problem has been solved in the case when F is the mean curvature (see [BK], [TW] and [HSW]), in the case when F is the Gaussian curvature (see [Ol]), and in the case when F is the general curvature function (see [Ge1], [Ge2] and [CNS]). On the other hand, using the evolutionary approach, the existence of convex solution has been studied by Ecker and Huisken (see [EH]), Gerhardt (see [Ge3] and [Ge4]). Roughly speaking, in the elliptic approach one need find C^0, C^1, C^2 *and* C^3 a priori estimates, and in the evolutionary approach one need find C^0, C^1 and C^2 (and hence $C^{4,\alpha}$) a priori estimates.

In this report, we consider the case when N is the (n+1)-dimensional unit sphere. Let M be the graph of a function u defined on a totally geodesic n-sphere. We establish the following elementary polynomials of degree one and two, and scalar curvature equation:

1. The mean curvature

$$H=\frac{\sqrt{1+u^2}}{\sqrt{1+u^2+|\nabla u|^2}}\sum(\delta_{ij}-\frac{u_iu_j}{1+u^2+|\nabla u|^2})(u_{ij}+u\delta_{ij})$$

2, The square length of the second fundamental form

$$\begin{aligned}S=&\frac{1+u^2}{(1+u^2+|\nabla u|^2)^2}[nu^2(1+u^2+|\nabla u|^2)\\&-2u^2|\nabla u|^2+\frac{u^2|\nabla u|^4}{1+u^2+|\nabla u|^2}\\&+2(1+u^2+|\nabla u|^2)u\Delta u-4uu_iu_iu_{ij}\\&-2u_iu_iu_{kii}u_{kj}+\frac{2u|\nabla u|^2}{1+u^2+|\nabla u|^2}u_iu_iu_{ij}\\&+(1+u^2+|\nabla u|^2)u_{ij}^2\\&+\frac{1}{1+u^2+|\nabla u|^2}u_iu_iu_ku_lu_{ki}u_{lj}].\end{aligned}$$

3. The scalar curvature equation

$$\begin{aligned}0=&\,u_{ij}^2-(\Delta u)^2+2\frac{u_iu_iu_{ij}}{1+u^2+|\nabla u|^2}\Delta u\\&-2\frac{\sum(\sum u_iu_{ij})^2}{1+u^2+|\nabla u|^2}+2(n-2)\frac{uu_iu_iu_{ij}}{1+u^2+|\nabla u|^2}\\&+2\frac{u|\nabla u|^2}{1+u^2+|\nabla u|^2}\Delta u-2(n-1)u\Delta u\\&-\mathrm{n(n-1)}u^2+2(n-1)\frac{u^2|\nabla u|^2}{1+u^2+|\nabla u|^2}\\&+\frac{1+u^2+|\nabla u|^2}{1+u^2}(R-n(n-1)).\end{aligned}$$

In particular, in the case n=2, the scalar curvature equation is just

$$\det(u_{ij}+u\delta_{ij})=(\frac{1+u^2+|\nabla u|^2}{1+u^2})^2(k(u,\mathrm{x})-1).$$

This is a equation of Gaussian curvature which we will give some observations in section 3.

2. The Fully nonlinear PDE

For deriving the equation for the problem of prescribed scalar curvature, we parameterize the standard (n+1)-dimensional unit sphere by (λ ,x) as follows

$$(\lambda,x)\to\frac{1}{\sqrt{1+\lambda^2}}x+\frac{\lambda}{\sqrt{1+\lambda^2}}e,$$

where x is the position vector of the standard n-dimensional unit sphere S^n = {(x, x_{n+1}): x_{n+1}=0}, e = (0,…,0,1) and λ is a real number. Let u be a smooth function defined on the standard n-dimensional unit sphere, and Y be the embedding from the standard n-dimensional unit sphere into the standard (n+1)-dimensional unit sphere given by Y(x)=(u(x), x) via the parameterization of the standard (n+1)-dimensional unit sphere. Let e_1, e_2, …, e_n be an orthonormal frame fields on the n-dimensional unit sphere and ω_1, ω_2, …, ω_n its dual coframes. Taking exterior differentiation, we see that the tangent space of the hypersurface M= graph(u) is spaned by

$$-uu_ix+u_ie+(1+u^2)e_i$$

for i = 1,2,…,n, and the first fundamental form is given by

$$ds^2=\frac{1}{(1+u^2)^2}\sum(u_iu_j+(1+u^2)\delta_{ij})\omega_i\omega_j.$$

And the unit normal vector is

$$N=\frac{1}{\sqrt{1+u^2+|\nabla u|^2}}(-ux+e-\nabla u).$$

Assume that $f_1, f_2, \ldots, f_n$ is an orthonormal frame fields on M, and let $\theta_1, \theta_2, \ldots, \theta_n$ be its dual coframes. We then have

$$f_i = \sum a_{ij}(-uu_j x + u_j e + (1+u^2)e_j)$$

and

$$\bar{}_i = \sum b_{ij}\mu_j,$$

where

$$A = [a_{ij}] = \begin{bmatrix} \frac{1}{\sqrt{(1+u^2)(1+u^2+|\nabla u|^2)}}\frac{\nabla u}{|\nabla u|} \\ \frac{1}{1+u^2}V_2 \\ \cdots \\ \frac{1}{1+u^2}V_n \end{bmatrix},$$

$$B = [b_{ij}] = \begin{bmatrix} \frac{\sqrt{1+u^2+|\nabla u|^2}}{1+u^2}\frac{\nabla u}{|\nabla u|} \\ \frac{1}{\sqrt{1+u^2}}V_2 \\ \cdots \\ \frac{1}{\sqrt{1+u^2}}V_n \end{bmatrix},$$

$\frac{\nabla u}{|\nabla u|}$, V_2, V_n are orthonormal provided $|\nabla u| \neq 0$.

Let $h=[h_{ij}]$, $I=[\delta_{ij}]$ and $U=[u_{ij}]$. It follows from the structure equations that

$$A^t hB = \frac{1}{(1+u^2)(1+u^2+|\nabla u|^2)^{\frac{3}{2}}}[(1+u^2+|\nabla u|^2)I - (\nabla u)^t\nabla u](uI+U).$$

We then have the mean curvature

$$H = tr\, h = \frac{1}{(1+u^2)(1+u^2+|\nabla u|^2)^{\frac{3}{2}}}.$$

$$tr(A^{-1})^t[(1+u^2+|\nabla u|^2)I - (\nabla u)^t\nabla u](uI+U)B^{-1}$$

$$= \frac{\sqrt{1+u^2}}{\sqrt{1+u^2+|\nabla u|^2}}\sum(\text{¢}_{ij} - \frac{u_i u_j}{1+u^2+|\nabla u|^2})(u_{ij} + u\text{¢}_{ij})$$

and the square of the length of the second fundamental form are given by

$$S = tr\, h\, h^t$$

$$= \frac{1}{(1+u^2)^2(1+u^2+|\nabla u|^2)^3} tr(A^{-1})^t\, W B^{-1}\, (A^{-1})^t\, W B^{-1}$$

$$= \frac{1+u^2}{(1+u^2+|\nabla u|^2)^2}[nu^2(1+u^2+|\nabla u|^2)$$

$$-2u^2|\nabla u|^2 + \frac{u^2|\nabla u|^4}{1+u^2+|\nabla u|^2}$$

$$+2(1+u^2+|\nabla u|^2)u\Delta u - 4uu_i u_i u_{ij}$$

$$-2u_i u_i u_{kii} u_{kj} + \frac{2u|\nabla u|^2}{1+u^2+|\nabla u|^2}u_i u_i u_{ij}$$

$$+(1+u^2+|\nabla u|^2)u_{ij}^{\ 2}$$

$$+\frac{1}{1+u^2+|\nabla u|^2}u_i u_i u_k u_l u_{ki} u_{lj}],$$

where $W = (1+u^2+|\nabla u|^2)I - (\nabla u)^t\nabla u](uI+U)$.

Finally we have the scalar curvature

$$R = n(n-1) + H^2 - S$$

$$= n(n-1) + \frac{1+u^2}{1+u^2+|\nabla u|^2}[-u_{ij}^{\ 2} + (\Delta u)^2$$

$$-2\frac{u_i u_i u_{ij}}{1+u^2+|\nabla u|^2}\Delta u$$

$$+2\frac{\sum(\sum u_i u_{ij})^2}{1+u^2+|\nabla u|^2} - 2(n-2)\frac{uu_i u_i u_{ij}}{1+u^2+|\nabla u|^2}$$

$$-2\frac{u|\nabla u|^2}{1+u^2+|\nabla u|^2}\Delta u + 2(n-1)u\Delta u$$

$$+ \mathrm{n(n-1)}u^2 - 2(n-1)\frac{u^2|\nabla u|^2}{1+u^2+|\nabla u|^2}].$$

We then have the scalar curvature equation. In particular, in the case n = 2, we have

$$\det(u_{ij} + u\text{¢}_{ij}) = (\frac{1+u^2+|\nabla u|^2}{1+u^2})^2(k(u, \mathrm{x}) - 1).$$

where K is the Gaussian curvature at (u, x).

At the points where $|\nabla u| = 0$, the above formulas for H, S and R still work.

3. Some Observations

The scalar curvature equation given in section 2 is very complicated. We now consider the case of n=2 which is a equation of Gaussian curvature. In general, for applying the maximum principle and the elliptic theory, one like to make some restrictions on the solutions for the equation when consider the equations of Gaussian curvature,. e.g., convex solutions or positive solutions (see [Ol] and [Ge3]). However, here we only consider some general observations without any constraints.

Since the equation was raised from a geometric setting, there must have natural geometric restrictions if the problem admits a solution. The following condition follows from the Gauss-Bonnet Theorem.

Let K be a function defined on S^3 . Suppose that K(λ, x) is essentially not less than 1+λ^2, for all (λ, x) in S^3. We claim the equation has no solutions.

Let u be a solution of the equation

$$\det[u_{ij}+u¢_{ij}]=\left(\frac{1+u^2+|\nabla u|^2}{1+u^2}\right)^2(k-1).$$

According to the Gauss - Bonnet Theorem,

$$\int \mathrm{K}dv = 4®,$$

where dv is the volume element of the graph u.

Since

$$dv=\frac{(1+u^2+|\nabla u|^2)^{\frac{1}{2}}}{(1+u^2)^{\frac{3}{2}}}dv_0,$$

where dv_0 is the volume element of S^2.

It follows that

$$4® \geq \int \frac{\mathrm{K}}{1+u^2}dv_0 > \int 1 dv_0 = 4®,$$

a contradition.

Thus for the existence of solutions there has a necessary condition: K(λ, x) ≦ 1+λ^2 somewhere. This necessary condition makes sense. One just notices that if u is the constant function u = λ then K=1+λ^2 . In particular, we have the following observation: if K = K(λ), K(λ) ≦ 1+λ^2 for some $\lambda=\lambda_1$ and K(λ) ≧ 1+λ^2 for some $\lambda=\lambda_2$ then there has a solution. There are existence results which have analogous conditions (see [Ol], [TW] and [HSW]), even these equations are not in the same type. We may expect that if K(λ, x) ≦ 1+λ^2 for some $\lambda=\lambda_1$ for all x , K(λ, x) ≧ 1+λ^2 for some $\lambda=\lambda_2$ for all x, and K is monotonic in λ then there has a solution.

The following proposition shows that the a priori C^1 estimate of u follows from the a priori C^0 estimate of u.

Prop.(C^1-estimate) Let K be a function defined on S^3 , K(λ, x)≠1, for all (λ, x) in S^3 . If u is a bounded solution of

$$\det[u_{ij}+u¢_{ij}]=\left(\frac{1+u^2+|\nabla u|^2}{1+u^2}\right)^2(k-1)$$

on S^2 . Then ∇u $_\infty$ ≦ u $_\infty$.

Proof. Let $\nu=u^2+|\nabla u|^2$. Assume that ν attains its maximum value at x_0 . Then we have

$\nu_i = 0$ at x_0 for all i = 1,2. It follows that

$uu_i+\sum u_j u_{ji}=0$ and hence

$\sum (u_{ij}+u¢_{ij})u_j = 0$ for all i = 1,2.

Since

$$\det[u_{ij}+u¢_{ij}]=\left(\frac{1+u^2+|\nabla u|^2}{1+u^2}\right)^2(k-1)\neq 0,$$

$u_i = 0$ for all i = 1, 2. Thus

$\max(u^2+|\nabla u|^2)=u^2(x_0)\leq \max(u^2)$,

$\max(|\nabla u|^2)\leq \max(u^2)$.

The multiplicity of the solutions depends on the behavior of the function K. One can find that u = c(x,a) is a solution when K=1 for all real number c and all a in S^2 since K is invariant under O(3)-actions and the dilation of λ. In this case we observe that there has no uniformly bounds for the C^0 -norm. This is different to the case of Euclidean space; in the case of Euclidean space, positive solutions has a uniformly bounds for the C^0 –norm for certain class of K (see [Ol]) . The equation in the Euclidean space is almost similar to our equation **except a minus sign in the term of u in the determinant.** From this, we learn that for finding a C^0 –estimate it is necessary to restrict solutions in some class of solutions. A priori C^2 and C^3 estimates were rather complicated even in the elliptic case (see [Ol] and [Ge1]).

4. Final Comments

The problem of prescribed curvature is an interesting problem, the existence of solutions has been studied extensively by various authors. For technical reasons, all known results are using either elliptic or parabolic approach. There are still many nonconvex solutions that were not found. The main problem will be: how to find a method which can show the existence of solutions when the equation is not elliptic or parabolic. In this report we establish the equation of prescribed scalar curvature, a equation related to 2-mean curvatures. The equation is well worth studying as a equation of homogeneous degree two.

5. References

[BK] I. Bakelman and B. Kantor, Existence of spherically homeomorphic hypersurfaces in Euclidean space with prescribed mean curvature, Geometry and Topology, Leningrad, 1(1974), 3-10.

[CNS] L. Caffarelli, L. Nirenberg and J. Spruck, The Dirichlet problem for nonlinear second order elliptic equations, **III**: Functions of the eigenvalues of the Hessian, Acta Math. 155(1985) 261-301.

[EH] K. Ecker and H. Huisken, Parabolic methods for the construction of space slices of prescribed mean curvature in comological spacetimes, Comm. Math. Phys. 135(1991), 595-613.

[Ge1] C. Gerhardt, Closed Weingarten hypersurfaces in Riemannian manifolds, J. Differential Geometry, 43(1996), 612-641.

[Ge2] C. Gerhardt, Closed Weingarten hypersurfaces in space forms, Geometric analysis and the calculus of variations, Internat. Press, Cambridge, 1996, 71-97.

[Ge3] C. Gerhardt, Hypersurfaces of prescribed curvature in Lorentzian manifolds, Indiana Univ. Math. J. 49(2000), 1125-1153.

[Ge4] C. Gerhardt, Hypersurfaces of prescribed mean curvature in Lorentzian manifolds, Math. Z. 224(2000), 83-97.

[HSW] Y. J. Hsu, S. J. Shiau and T. H. Wang, Graphs with prescribed mean curvature in the spheres, preprint.

[Ol] V. I. Oliker, Hypersurfaces in R^{n+1} with prescribed Gaussian curvature and related equations of Monge-Ampere type, Comm. Partial Differential Equations 9(1984), 807-838.

[TW] A. E. Trieibergs and S. W. Wei, Embedded hypersurfaces with prescribed mean curvature, J. Differential Geometry 18(1983) 513-521.