行政院國家科學委員會專題研究計畫成果報告

正交投影和之研究
A study of sum of orthogonal projections

計畫編號：NSC89-2115-M-009-025

執行期間：89年8月01日　至　90年10月31日

主持人：吳培元

執行機構：國立交通大學應用數學系

E-mail: pywu@cc.nctu.edu.tw

本成果報告包括以下應繳交之附件：

- ☑赴國外出差或研習心得報告一份
- ☐赴大陸地區出差或研習心得報告一份
- ☑出席國際學術會議心得報告及發表之論文各一份
- ☐國際合作研究計畫國外研究報告書一份

執行單位：國立交通大學應用數學系

中　華　民　國　90　年　10　月　25　日

中文摘要

在本論文中，我們考慮如何刻劃在希伯特空間上可以表示成有限個正交投影算子之和的算子。當此希伯特空間是有限維時，此問題已由Fillmore完全解決：一個有限維算子是投影算子之和當且僅當其為半正定的，其跡為一大於或等於其秩的整數。在本論文中，我們得到一些必要或充份條件使得一無窮維算子可以這樣子表示。例如，我們證明(一)一個半正定算子之本質範數如大於一，則它必是投影算子之和，且(二)一個嵌射算子如為I+K之形式，其中K為一緊緻算子，且為投影算子之和，則或者K+和K-之跡都為無窮大或者K是一有跡算子且其跡是一大於或等於零之整數。。

SUMS OF ORTHOGONAL PROJECTIONS

MAN-DUEN CHOI and PEI YUAN WU

ABSTRACT

In this paper, we consider the problem of characterizing Hilbert space operators which are expressible as a sum of (finitely many) orthogonal projections. When the underlying space is finite-dimensional, this was completely solved by Fillmore: a finite-dimensional operator is the sum of projections if and only if it is positive, its trace is an integer and the trace is greater than or equal to the rank. In this paper, we obtain necessary/sufficient conditions for infinite-dimensional operators to be expressible as such. For example, we prove that (a) a positive operator with essential norm strictly greater than one is always a sum of projections, and (b) if an injective operator of the form $1+K$, where K is compact, is a sum of projections, then either tr $K_+ =$tr $K_- = \infty$ or K is of trace class with tr K a nonnegative integer.

1991 *Mathematics Subject Classification.* Primary 47B15.

Key words and phrases. Orthogonal projection, essential norm, trace.

The research of the first author was supported by NSERC and the second athor by NSC of Taiwan.

Which bounded linear operator on a complex Hilbert space can be expressed as the sum of finitely many orthogonal projections? (an *orthogonal projection* is an operator P with $P^2 = P = P^*$.) This is the problem we are going to address in this paper. If the underlying space is finite-dimensional, then a complete characterization of such operators was obtained before by Fillmore [2]: a finite-dimensional operator is the sum of projections if and only if it is positive, it has an integral trace and the trace is greater than or equal to the rank. In this paper, we consider this problem for operators on an infinite-dimensional separable space. It turns out that in this situation the necessary/sufficient conditions we obtained for sums of projections are, after some appropriate interpretation, not too much different from the finite-dimensional ones. Although we haven't been able to give a complete characterization, we can reduce the whole problem to the consideration of operators of the form identity + compact.

The organization of this paper is as follows. In Section 1 below, we start by giving a special operator matrix representation for sums of projections (Proposition 1.2). This is used to give a more conceptual proof of the above result of Fillmore (Corollary 1.3). The main result of this section is Theorem 1.5. It says that every positive operator with essential norm strictly greater than one is the sum of projections. This essentially reduces our problem to the consideration of operators of the form identity + compact. We then concentrate on this latter class in Section 2 and derive some necessary conditions for such operators to be sums of projections. It culminates in Theorems 2.3 and 2.5 in which we show that if an injective operator of the form $1 + K$, where K is infinite-rank compact, is a sum of projections, then both the positive and negative parts K_+ and K_- of K have infinte rank and, moreover, the traces of K_+ and K_- are either both infinity or both finite with the difference tr $K_+ -$tr K_- a nonnegative integer. We end this section by conjecturing that the converse is also true. If this is indeed the case, then we have a complete characterization of sums of projections. Finally, in Section 3, we consider some variations of the sum-of-projections problem. They fall into two different categories. One of them involves the characterization of sums of projections which are commuting and /or having some fixed (finite or infinite) rank. The other concerns the characterization of the closure of sums of (two) projections in the norm topology.

Recall that an operator T is *positive* (resp. *strictly positive*), denoted by $T \geq 0$ (resp. $T > 0$) if $\langle Tx, x\rangle \geq 0$ (resp. $\langle Tx, x\rangle > 0$) for any (resp. nonzero) vector x. For Hermitian operators A and B, $A \geq B$ (resp. $A > B$) means that $A - B \geq 0$ (resp. $A - B > 0$). For an operator T on H and $1 \leq n \leq \infty$, $T^{(n)}$ denotes the operator $\underbrace{T \oplus \cdots \oplus T}_{n}$ on $H^{(n)} = \underbrace{H \oplus \cdots \oplus H}_{n}$. The *trace* of T, when defined, is denoted by tr T, and the *range* and *rank* of T are ran T and rank T, respectively. In the following, we will need the Fredholm theory of operators. For this, the reader can consult [1, Chapter XI].

1. REDUCTION

We start by showing that in considering sums of projections we may as well assume that the operator under consideration is injective.

Lemma 1.1. *An operator of the form $T \oplus 0$ is the sum of projections if and only if T itself is.*

Proof. If $T \oplus 0 = \sum_{j=1}^{n} P_j$ is the sum of the projections

$$P_j = \begin{bmatrix} A_j & B_j \\ C_j & D_j \end{bmatrix}, \quad j = 1, \cdots, n,$$

then $\sum_j A_j = T$ and $\sum_j D_j = 0$. Since all the $D_j's$ are positive, the latter equality implies that $D_j = 0$ for all j. Hence $B_j = 0$ and $C_j = 0$ and therefore $T = \sum_j A_j$ is the sum of the projections A_j. ∎

The next result characterizes sums of projections in terms of a certain operator matrix representation.

Proposition 1.2. *Let T be a strictly positive operator. Then T is a sum of projections if and only if $T \oplus 0$ is unitarily equivalent to an operator matrix of the form*

$$\begin{bmatrix} I_1 & & * \\ & \ddots & \\ * & & I_n \end{bmatrix},$$

where 0 and $I_1, \cdots I_n$ denote the zero and identity operators on some spaces.

Proof. If $T = \sum_{j=1}^{n} P_j$ is a sum of projections, then, letting $A = [P_1 \ \cdots P_n]^t$, we have $T = A^*A$. It is well-known that in this case AA^* is unitarily equivalent to $T \oplus 0$

for some zero opeator 0. But

$$AA^* = \begin{bmatrix} P_1 & & * \\ & \ddots & \\ * & & P_n \end{bmatrix}$$

is unitarily equivalent to a matrix of the form

$$\left[\begin{array}{cc|c|cc} I_1 & 0 & \cdots & \multicolumn{2}{c}{*} \\ 0 & 0 & & & \\ \hline \multicolumn{2}{c|}{\vdots} & \ddots & \multicolumn{2}{c}{\vdots} \\ \hline \multicolumn{2}{c|}{*} & \cdots & I_n & 0 \\ & & & 0 & 0 \end{array}\right],$$

which is in term unitarily equivalent to

$$\begin{bmatrix} I_1 & & * \\ & \ddots & \\ * & & I_n \end{bmatrix} \oplus 0,$$

where each I_j acts on a space of dimension rank P_j. Since T is injective, we conclude that T is unitarily equivalent to

$$\begin{bmatrix} I_1 & & * \\ & \ddots & \\ * & & I_n \end{bmatrix}$$

as asserted.

Conversely, assume that $T \oplus 0$ is unitarily equivalent to

$$B = \begin{bmatrix} I_1 & & * \\ & \ddots & \\ * & & I_n \end{bmatrix}.$$

Let $C = [C_{ij}]_{i,j=1}^n$ be the positive square root of the positive operator B and $D_j = [C_{1j} \ \cdots \ C_{nj}]^t$ for $j = 1, \cdots, n$. Then $B = C^2 = \sum_{j=1}^{n} D_j D_j^*$. Note that $D_j D_j^*$ is Hermitian and $(D_j D_j^*)^2 = D_j(D_j^* D_j)D_j^* = D_j I_j D_j^* = D_j D_j^*$. Hence $B = \sum_j D_j D_j^*$ is the sum of the projections $D_j D_j^*$. Lemma 1.1 then implies that T is a sum of projections. ∎

Note that in the preceding proposition the sufficiency part is valid even assuming only the positivity of T. The characterization of sums of projections among finite-rank operators can be obtained as an easy corollary, the finite-dimensional case of which is due to Fillmore [2].

Corollary 1.3. *A finite-rank operator T is the sum of projections if and only if* $T \geq 0$, tr T *is an integer and* tr $T \geq$rank T.

Proof. Since every finite-rank operator is the direct sum of a finite-dimensional operator and a zero operator, we may assume that T itself acts on an, say, n-dimensional space. To prove the nontrivial sufficiency part, we may, as in the proof of [2, Theorem 1], subtract some rank-one projections from T and thus assume that tr $T =$ rank $T = n$. By [3, Corollary 2], T is unitarily equivalent to an $n \times n$ matrix with diagonal entries all equal to 1. Propostion 1.2 then implies that T is a sum of projections. ∎

For the remaining part of the paper, we only consider operators on infinite-dimensional separable spaces. We start with the following necessary conditions for sums of projections. They greatly facilitate our search for the exact characterization.

Proposition 1.4. *Let T be a sum of projections.*

(a) *If* $\| T \| < 1$, *then* $T = 0$.

(b) *If* $\| T \|_e < 1$, *then* T *is of finite rank.*

(c) *If* $\| T \| \leq 1$, *then* T *is a projection.*

(d) *If* $\| T \|_e \leq 1$, *then* T *is the sum of a projection and a compact operator.*

Here $\| T \|_e$ denotes the *essential norm* of $T : \| T \|_e = \inf \{ \| T + K \| : K$ compact $\}$.

Proof. Let $T = \sum_{j=1}^{n} P_j$, where the $P_j's$ are projections.

(a) Assume that $P_1 \neq 0$. Since $P_1 \leq T$, for any vector x in ran P_1 we have $\langle P_1 x, x \rangle \leq \langle Tx, x \rangle$. However, $\langle P_1 x, x \rangle = \| P_1 x \|^2 = \| x \|^2$ and $\langle Tx, x \rangle = \| T^{\frac{1}{2}} x \|^2$. From this, we infer that $\| T^{\frac{1}{2}} \| \geq 1$ and hence $\| T \| \geq 1$. This shows that $\| T \| < 1$ implies that $P_1 = 0$. Repeating this argument, we obtain that $P_j = 0$ for all j and therefore $T = 0$.

(b) Passing $T = \sum_j P_j$ to the Calkin algebra, representing the latter as operators on some Hilbert space and following the arguments in (a) yield that T is compact. Now $0 \leq P_j \leq T$ implies that P_j is also compact for every j (cf. [2, p.146]). Hence P_j must be of finite rank. The same is then true for T.

(c) If $\| T \| \leq 1$, then $0 \leq P_1 + P_2 \leq T \leq 1$. It was proved in [2, p.151] that sums of two projections are exactly those which are unitarily equivalent to an operator of the form $A \oplus (2I - A) \oplus 0 \oplus 2I$, where $0 \leq A \leq 1$. From this, we infer that $P_1 + P_2$ is actually itself a projection. Repeating this argument with other projections in the sum $T = \sum_j P_j$ yields that T is a projection.

(d) If $\| T \|_e \leq 1$, then from $0 \leq P_1 + P_2 \leq T$, $\sigma_e(T) \subseteq [0,1]$ and the above structure result of sums of two projections we infer that $\sigma_e(P_1 + P_2) \subseteq \{0, 1\}$. (Here $\sigma_e(A)$ denotes the essential spectrum of an operator A.) Hence $P_1 + P_2$ is the compact perturbation of a projection and, in particular, T is the sum of $n-1$ projections with a compact operator. Repeating this argument, we obtain that T is the sum of a projection and a compact operator. ∎

$\therefore \hat{T} = \hat{P}_1 + \cdots + \hat{P}_n$ & $\|\hat{T}\| = \|T\|_e \leq 1$
(c) $\Rightarrow$ $\hat{T}$ proj $\Rightarrow$ T = proj + compact

In view of Proposition 1.4(b) and Corollary 1.3, to characterize sums of projections we need only consider positive operators with essential norm at least one in the remaining discussions. The case when the essential norm is strictly greater than one is taken care of by the next result, which is the main theorem of this section.

Theorem 1.5. *Any positive operator with essential norm strictly greater than one is the sum of projections.*

This will be proved via the following lemmas.

Lemma 1.6. *If $T = T_1 \oplus \cdots \oplus T_n$ on $H^{(n)}$, where the T_j's are positive operators satisfying $T_1 + \cdots + T_n = nI$, then T is the sum of n projections.*

Proof. Let $P = \frac{1}{n}\left[T_i^{\frac{1}{2}}T_j^{\frac{1}{2}}\right]_{i,j=1}^{n}$ and

$$U_j = \begin{bmatrix} I & & & & 0 \\ & \omega^j I & & & \\ & & \omega^{2j} I & & \\ & & & \ddots & \\ 0 & & & & \omega^{(n-1)j} I \end{bmatrix}, \quad j = 1, \cdots, n,$$

on the space $H^{(n)}$, where ω is the nth primitive root of 1. Then P is a projection and U_j is unitary. An easy computation shows that $\sum_{j=1}^{n} U_j^* P U_j = T_1 \oplus \cdots \oplus T_n = T$, completing the proof. ∎

Lemma 1.7. *If* $0 \leq T \leq \lambda I$ *on* H, *where* λ *is a rational number with* $1 < \lambda < 2$, *then* $T \oplus \lambda I$ *on* $H \oplus H$ *is a sum of projections.*

Proof. Since $T \oplus \lambda I$ is unitarily equivalent to the sum of the two operators

$$T \oplus T^{(\infty)} \oplus (\lambda I - T)^{(\infty)} \oplus (\lambda I)^{(\infty)} \oplus 0^{(\infty)}$$

and

$$0 \oplus (\lambda I - T)^{(\infty)} \oplus T^{(\infty)} \oplus 0^{(\infty)} \oplus (\lambda I)^{(\infty)}$$

on $H \oplus H^{(\infty)} \oplus H^{(\infty)} \oplus H^{(\infty)} \oplus H^{(\infty)}$, to prove our assertion we need only check that $T \oplus (\lambda I - T) \oplus \lambda I$ is a sum of projections. Let $\lambda = \frac{n}{m}$, where n and m are integers satisfying $1 < m < n < 2m$. Then $T \oplus (\lambda I - T) \oplus \lambda I$ is unitarily equivalent to the sum of

$$T \oplus (\lambda I - T) \oplus (\lambda I)^{(2m-n)} \oplus ((\lambda - 1)I)^{(n-m-1)}$$

and

$$0 \oplus 0 \oplus 0^{(2m-n)} \oplus I^{(n-m-1)}$$

on $H \oplus H \oplus H^{(2m-n)} \oplus H^{(n-m-1)}$. Since the first of the latter two operators is a sum of $m+1$ projections by Lemma 1.6 and the second is already a projection, $T \oplus (\lambda I - T) \oplus \lambda I$ is a sum of $m+2$ projections, completing the proof. ∎

Lemma 1.8. *If T is a positive operator on H and λ is a rational number with $1 < \lambda < 2$, then $T \oplus \lambda I$ on $H \oplus H$ is a sum of projections.*

Proof. We decompose T as $T_1 \oplus \cdots \oplus T_n$, where $(j-1)\lambda I \leq T_j \leq j\lambda I, j = 1, \cdots, n$. Then $T \oplus \lambda I$ is unitarily equivalent to the sum of

$$0 \oplus \cdots \oplus T_j \oplus \cdots \oplus 0 \oplus 0 \oplus \cdots \lambda I \oplus \cdots \oplus 0,\ j = 1, \cdots, n,$$

on $H^{(2n)}$. To complete the proof, we need only show that each $T_j \oplus \lambda I$ is a sum of projections. Since

$$T_j \oplus \lambda I = ((T_j - (j-1)\lambda I) \oplus \lambda I) + (j-1)(\lambda I \oplus 0)$$

and both $(T_j - (j-1)\lambda I) \oplus \lambda I$ and $\lambda I \oplus 0$ are sums of projections by Lemma 1.7, $T_j \oplus \lambda I$ is indeed a sum of projections as asserted. This completes the proof. ∎

We are now ready for the

Proof of Theorem 1.5. Let λ_0 be any point in $\sigma_e(T)$ which is greater than one, and let λ be a rational number with $1 < \lambda < \min(\lambda_0, 2)$. We decompose T as

$T_1 \oplus T_2 \oplus T_3$, where $0 \leq T_1 \leq \lambda I$ and $T_2, T_3 \geq \lambda I$, the latter two on infinite-dimensional spaces. Then $T = (T_1 \oplus \lambda I \oplus (T_3 - \lambda I)) + (0 \oplus (T_2 - \lambda I) \oplus \lambda I)$. Lemma 1.8 implies that these latter two operators are sums of projections. Hence the same is true for T. ∎

2. IDENTITY + COMPACT

In light of the results in Section 1, for the problem of sums of projections we may restrict ourselves to operators which are injective and have essential norm equal to one. The next lemma narrows down further the pool of operators which we need to consider.

Lemma 2.1. *Let T be an injective operator with $\| T \|_e = 1$. If T is a sum of projections, then T is the sum of the identity and a compact operator.*

Proof. By Proposition 1.4 (d), $T = P + K$, where P is a projection and K is compact. On the other hand, since T is injective, we have $\sigma(T) \subseteq \sigma(T - K) = \sigma(P)$ by [4, Corollary of Theorem 3.3]. Hence T itself is a projection. Then $\| T \|_e = 1$ implies that T is of the form identity + compact. ∎

Note that if T is the sum of the identity and a finite-rank operator, then it can be decomposed as $T_1 \oplus I$ on $H_1 \oplus H_2$ with $\dim H_1 < \infty$. The next proposition reduces the characterization of sums of projections among such operators to the finite-dimensional case.

Proposition 2.2. *Let $T = T_1 \oplus I$ on $H_1 \oplus H_2$, where* dim $H_1 < \infty$. *Then T is a sum of projections if and only if T_1 is.*

Proof. We need only prove the necessity part. Let $T = \sum_{j=1}^{n} P_j$ be a sum of projections. We claim that P_iP_j is of finite rank for any $i \neq j$. Indeed, since the subspace $K = H_1 \vee P_iH_1$ is invariant under P_i and its orthogonal complement $K^\perp$, being contained in H_2, is invariant under T, we have the matrix representations

$$T = \begin{bmatrix} 1 & 0 \\ 0 & * \end{bmatrix}, \quad P_i = \begin{bmatrix} Q & 0 \\ 0 & * \end{bmatrix} \quad \text{and} \quad P_j = \begin{bmatrix} R & * \\ * & * \end{bmatrix}$$

on $H = K^\perp \oplus K$. Let Q and R be simultaneously unitarily equivalent to

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

respectively. Since $T \geq P_i + P_j$, we have $1 \geq Q + R$ and hence $1 \geq 1 + A$ or $A \leq 0$. But $P_j \geq 0$ also implies that $A \geq 0$. Thus $A = 0$ and therefore $B = 0$ and $C = 0$. Hence P_iP_j is unitarily equivalent to

$$\left[\begin{array}{cc|c} 1 & 0 & \\ & & 0 \\ 0 & 0 & \\ \hline & 0 & * \end{array}\right] \left[\begin{array}{cc|c} 0 & 0 & \\ & & * \\ 0 & D & \\ \hline & * & * \end{array}\right] = \begin{bmatrix} 0 & * \\ * & * \end{bmatrix}$$

on $K^\perp \oplus K$. Since K is finite-dimensional, this readily implies that rank $(P_iP_j) < \infty$ as claimed.

Let $L = H_1 \vee (\bigvee_{i \neq j} \text{ran } P_iP_j)$ and $M = L \vee (\bigvee_j P_jL)$. Then the subspace M is invariant under all the P_j's. On the decomposition $H = M \oplus M^\perp, T = \sum_j P_j$ may be represented as $T' \oplus 1 = \sum_j P'_j \oplus P''_j$. Hence $T' = \sum_j P'_j$ is a sum of projections on the finite-dimensional M. Since $T' = T_1 \oplus 1$ on $M = H_1 \oplus (M \ominus H_1)$, it is easily seen from Corollary 1.3 that T_1 is also a sum of projections, completing the proof. ∎.

The preceding proof depends heavily on the fact that the summand T_1 acts on a finite-dimensional space. It is unknown whether the same assertion still holds without this assumption.

We now proceed to consider operators of the form identity + infinite-rank compact. The next theorem gives a necessary condition for such operators to be sums of projections.

Theorem 2.3. *If $T = 1 + K$, where K is an infinite-rank compact operator, and is a sum of projections, then both K_+ and K_- have infinite rank.*

Recall that the *positive* and *negative parts* of a Hermitian operator A are by definition $A_+ = \frac{1}{2}(|A| + A)$, and $A_- = \frac{1}{2}(|A| - A)$, respectively, where $|A| = (A^2)^{\frac{1}{2}}$.

Here are some simple facts concerning the positive and negative parts which we will need in the proof of Theorem 2.3.

Lemma 2.4. (a) *If $T = A - B$, where $A, B \geq 0$, then* rank $T_+ \leq$ rank A *and* rank $T_- \leq$ rank B.

(b) *If* A *is a compression of the Hermitian operator* B, *then* rank $A_+ \leq$ rank B_+ *and* rank $A_- \leq$ rank B_-.

(c) *For any Hermitian operators* A *and* B, *the inequalities* rank $(A+B)_+ \leq$ rank A_++rank B_+ *and* rank $(A+B)_- \leq$ rank A_-+ rank B_- *hold.*

Recall that A on $K(\subseteq H)$ is a *compression* of B on H if $A = PB|K$, where P is the projection from H onto K.

Proof. (a) Decompose T as $T_1 \oplus (-T_2)$ on $H_1 \oplus H_2$, where T_1 and T_2 are both positive. If

$$A = \begin{bmatrix} A_1 & * \\ * & * \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B_1 & * \\ * & * \end{bmatrix}$$

on $H_1 \oplus H_2$, then we have $T_1 = A_1 - B_1 \leq A_1$. Hence rank T_+ =rank $T_1 \leq$ rank $A_1 \leq$ rank A. Similarly, rank $T_- \leq$rank B.

(b) Let $B = \begin{bmatrix} A & * \\ * & * \end{bmatrix}$, $B_+ = \begin{bmatrix} A_1 & * \\ * & * \end{bmatrix}$ and $B_- = \begin{bmatrix} A_2 & * \\ * & * \end{bmatrix}$ on $H = K \oplus K^\perp$. From $B = B_+ - B_-$, we have $A = A_1 - A_2$. Since $A_1, A_2 \geq 0$, part (a) implies that rank $A_+ \leq$rank $A_1 \leq$rank B_+ and rank $A_- \leq$ rank $A_2 \leq$ rank B_-.

(c) Since $A + B = (A_+ - A_-) + (B_+ - B_-) = (A_+ + B_+) - (A_- + B_-)$ with $A_+ + B_+, A_- + B_- \geq 0$, part (a) implies that rank $(A+B)_+ \leq$ rank $(A_+ + B_+) \leq$ rank A_++rank B_+ and rank $(A+B)_- \leq$ rank $(A_- + B_-) \leq$ rank A_-+ rank B_-. ∎

Proof of Theorem 2.3. Assume that K_+ has finite rank. We will show that this leads to a contradiction. Let $T = \sum_{j=1}^{n} P_j$, where the $P_j's$ are projections, let $Q = 1 - P_1$ and let $Q + K = T_1 \oplus 0$ on $H_1 \oplus H_2$ with T_1 injective. If T_1 is of finite rank, then so is $Q+K$, which implies that Q is compact. For a projectino, this is equivalent to Q being finite-rank, and thus K is finite-rank, contradicting our assumption. Hence T_1 must be of infinite rank. On the other hand, since $Q + K = \sum_{j=2}^{n} P_j, T_1$ is, by Lemma 1.1, also a sum of $n-1$ projections. We also have $\| T_1 \|_e = \| Q + K \|_e = \| Q \|_e \leq 1$. Since T_1 has infinite rank, Proposition 1.4 (b) implies that $\| T_1 \|_e = 1$. Thus $T_1 = 1 + K_1$ for some compact operator K_1 by Lemma 2.1. We next show that K_1 also has infinite rank.

Let $Q = \begin{bmatrix} Q_1 & * \\ * & Q_2 \end{bmatrix}$ and $K = \begin{bmatrix} L_1 & * \\ * & L_2 \end{bmatrix}$ on $H_1 \oplus H_2$. From $Q + K = T_1 \oplus 0$, we obtain $Q_1 + L_1 = 1 + K_1$ and $Q_2 + L_2 = 0$. Since $Q = \begin{bmatrix} Q_1 & * \\ * & Q_2 \end{bmatrix}$ is a projection, from its structure theory we may assume that $Q_1 = 0 \oplus 1 \oplus (1 - X)$ and $Q_2 = 0 \oplus 1 \oplus X$ for some operator X with $0 < X < 1$, where the zero and identity operators may act on different spaces (cf. [8, Theorem 2]). Then $K_1 = Q_1 + L_1 - 1 = ((-1) \oplus 0 \oplus (-X)) + L_1 \equiv Y + L_1$. Since the operator $-X$, being a direct summand of $-Q_2 = L_2$, can be considered as a compression of K, Lemma 2.4(b) implies that

$$\text{rank } \ Y_+ = \text{rank } (-X)_+ \leq \text{rank } K_+ < \infty$$

and hence

$$\text{rank } K_{1+} \leq \text{rank } Y_+ + \text{rank } L_{1+} \leq 2 \text{ rank } K_+ < \infty$$

by Lemma 2.4 (c) and (b). If K_1 has finite rank, then

$$\begin{aligned}
\text{rank } L_1 &= \text{rank } (L_{1+} - L_{1-}) \\
&\leq \text{rank } L_{1+} + \text{rank } L_{1-} \\
&\leq \text{rank } L_{1+} + \text{rank}(L_{1-} + Y_-) \\
&= \text{rank } L_{1+} + \text{rank } (L_{1+} - L_1 + Y_+ - Y) \\
&\leq 2 \text{ rank } L_{1+} + \text{rank } (L_1 + Y) + \text{rank } Y_+ \\
&\leq 2 \text{ rank } K_+ + \text{rank } K_1 + \text{rank } Y_+ < \infty,
\end{aligned}$$

where the last inequality follows from Lemma 2.4(b). Let

$$K_+ = \begin{bmatrix} K_1' & * \\ * & K_2' \end{bmatrix} \quad \text{and} \quad K_- = \begin{bmatrix} K_1'' & * \\ * & K_2'' \end{bmatrix}$$

on $H_1 \oplus H_2$. From $K = K_+ - K_-$, we have $L_1 = K_1' - K_1''$. Hence

(1) $\quad \text{rank } K_1'' \leq \text{rank } L_1 + \text{rank } K_1' \leq \text{rank } L_1 + \text{rank } K_+ < \infty.$

Analogously, we have

$$\begin{aligned}
\text{rank } L_2 &\leq \text{rank } L_{2+} + \text{rank } L_{2-} \\
&\leq \text{rank } K_+ + \text{rank } (-Q_2)_- \\
&= \text{rank } K_+ + \text{rank}(0 \oplus (-1) \oplus (-X))_- \\
&= \text{rank } K_+ + \dim M + \text{rank } (-X)_-,
\end{aligned}$$

where M is the space on which the identity summand in $Q_2 = 0 \oplus 1 \oplus X$ acts. Note that dim $M < \infty$ since $Q_2 = -L_2$ is compact. On the other hand, we also have

$$\text{rank } X \leq \text{rank } Y = \text{rank } (K_1 - L_1) \leq \text{rank } K_1 + \text{rank } L_1 < \infty$$

and hence rank $(-X)_- < \infty$. It follows that rank $L_2 < \infty$ and therefore

$$(2) \qquad \begin{aligned} \text{rank } K_2'' &= \text{rank}(L_2 - K_2') \leq \text{rank } L_2 + \text{rank } K_2' \\ &\leq \text{rank } L_2 + \text{rank } K_+ < \infty. \end{aligned}$$

Since $K_- = \begin{bmatrix} K_1'' & * \\ * & K_2'' \end{bmatrix}$ is positive, (1) and (2) together imply that rank $K_- < \infty$ and thus K has finits rank, contradicting our assumption. This shows that K_1 has infinite rank.

So for we have proved that $T_1 = 1 + K_1$ is the sum of $n - 1$ projections with K_1 infinite-rank compact satisfying rank $K_{1+} < \infty$. We can repeat the above arguments with T_1 replacing T and proceed by induction to obtain a projection $T_{n-1} = 1 + K_{n-1}$ with K_{n-1} infinite-rank compact and rank $K_{n-1+} < \infty$. In particular, $K_{n-1} = 1 - T_{n-1}$ is a projection which is infinite-rank compact. This is impossible. Hence we must have rank $K_+ = \infty$. Similarly, rank $K_- = \infty$. ∎

The final result in this section gives information on the trace for sums of projections of the form identity + compact.

Theorem 2.5. *Let $T = 1 + K$, where K is infinite-rank compact. If T is injective* *and is a sum of projections, then either* tr $K_+ =$tr $K_- = \infty$ *or* K *is of trace class with* tr K *a nonnegative integer.*

For the proof, we need the following lemma. It is the trace analogue of Lemma 2.4 and its proof is also similar to that of the latter, which we omit.

Lemma 2.6. (a) *If* $T = A - B$, *where* $A, B \geq 0$, *then* tr $T_+ \leq$ tr A *and* tr $T_- \leq$ tr B.

(b) *If* A *is a compression of the Hermitian* B, *then* tr $A_+ \leq$ tr B_+ *and* tr $A_- \leq$ tr B_-.

(c) *For any Hermitian operators* A *and* B, *the inequalities* tr $(A+B)_+ \leq$ tr $A_+ +$ tr B_+ *and* tr $(A+B)_- \leq$ tr $A_- +$ tr B_- *hold.*

Proof of Theorem 2.5. As in the proof of Theorem 2.3, let $T = \sum_{j=1}^{n} P_j$, where P_j's are projections, $Q = 1 - P_1$ and $Q + K = T_1 \oplus 0$ on $H_1 \oplus H_2$ with T_1 injective. Then T_1 has infinite rank and equals $1 + K_1$ for some compact operator K_1 as before. Let $Q = \begin{bmatrix} Q_1 & * \\ * & Q_2 \end{bmatrix}$ and $K = \begin{bmatrix} L_1 & * \\ * & L_2 \end{bmatrix}$ on $H_1 \oplus H_2$, and let $Q_1 = 0 \oplus 1 \oplus (1 - X)$ on $M_1 \oplus N_1 \oplus H$ and $Q_2 = 0 \oplus 1 \oplus X$ on $M_2 \oplus N_2 \oplus H$, where $0 < X < 1$. Then $Q_1 + L_1 = 1 + K_1$ and $Q_2 + L_2 = 0$. Assuming that tr $K_+ < \infty$, we will prove that K is of trace class and tr K is a nonnegative integer.

Since $K_1 = (Q_1 - 1) + L_1 = ((-1) \oplus 0 \oplus (-X)) + L_1$, we obtain

(3) $\quad$ tr $K_1 = -\dim M_1 -$ tr $X +$ tr L_1.

Note that the right-hand side of (3) makes sense since dim $M_1 < \infty$ (because $(-1) \oplus 0 \oplus (-X) = Q_1 - 1 = K_1 - L_1$ is compact), $0 <$tr $X \leq \infty$ (because $X > 0$) and tr L_1 is either finite or $-\infty$ (because tr $L_{1+} \leq$tr $K_+ < \infty$ by Lemma 2.6(b)). Also note that in the decomposition $Q_2 = 0 \oplus 1 \oplus X$, the summand 1 on N_2 does not appear. Indeed, if it does, then $L_2 = -Q_2 = 0 \oplus (-1) \oplus (-X)$ has -1 as an eigenvalue. This implies that $1 + L_2$ has eigenvalue 0, which in turn results in the noninjectivity of the positive operator $T = \begin{bmatrix} 1 + L_1 & * \\ * & 1 + L_2 \end{bmatrix}$, contradicting our assumption. Hence

$$(4) \qquad \text{tr } L_2 = -\text{tr } Q_2 = -\text{tr } X.$$

Moreover, from $K = \begin{bmatrix} L_1 & * \\ * & L_2 \end{bmatrix}$ we also have

$$(5) \qquad \text{tr } K = \text{tr } L_1 + \text{tr } L_2.$$

It follows from (3),(4) and (5) that tr $K_1 = -$dim M_1+tr L_2+ tr L_1 =tr $K-$dim M_1 or

$$(6) \qquad \text{tr } K = \text{tr } K_1 + \dim M_1.$$

On the other hand, since $K_1 = ((-1) \oplus 0 \oplus (-X)) + L_1$ and $X > 0$, Lemma 2.6 (c) and (b) imply that tr $K_{1+} \leq$tr $((-1) \oplus 0 \oplus (-X))_+$+tr L_{1+} =tr $L_{1+} \leq$tr $K_+ < \infty$. Hence the injective $T_1 = 1 + K_1$ is the sum of $n-1$ projections with K_1 compact satisfying tr $K_{1+} < \infty$. Note that if K_1 has finite rank, then express K_1 as $A \oplus 0$, where A acts on an m-dimensional space, and apply Proposition 2.2 to infer that $1 + A$ is a sum of projections. Hence tr $(1 + A)$ is an integer and tr $(1 + A) \geq$ rank $(1 + A)$. Note that $1 + A$ is injective since $1 + K_1$ is. Therefore, tr K_1 =tr A =tr $(1 + A) - m$ is an integer and is greater than or equal to rank $(1 + A) - m = 0$. By (6), tr K is also a nonnegative integer. Thus we may assume that K_1 has infinite rank. We then repeat the preceding arguments with T_1 replacing T and proceed by induction to obtain am injective projection $T_{n-1} = 1 + K_{n-1}$, where K_{n-1} is compact with tr K $-$ tr K_{n-1} a nonnegative integer by (6). Since $T_{n-1} = 1$ or $K_{n-1} = 0$, we conclude that tr K is a nonnegative integer as asserted. Analogous arguments apply in case tr $K_- < \infty$. This completes the proof. ∎

Theorems 2.3 and 2.5 together give some necessary conditions on the rank and trace in order that an operator of the form identity + infinite-rank compact be a sum of projections. Are these conditions sufficient? We conjecture that they are.

Conjecture 2.7. *If K is a Hermitian compact perator with* $K \geq -1$, rank K_+ =rank $K_- = \infty$ *and* tr K_+ =tr $K_- \leq \infty$, *then* $1 + K$ *is a sum of projections.*

Note that if this is indeed true, then so is the following assertion, which together with other results in this paper will yield a complete characterization for sums of projections: if K is a Hermitian trace-class operator with $K \geq -1$, rank K_+ =rank $K_- = \infty$ and tr K a nonnegative integer, then $1 + K$ is a sum of projections. Indeed, let K be represented as the diagonal operator diag $(d_1, d_2, \cdots)$ and tr $K = n \geq 0$. Since rank $K_+ = \infty$, there are infinitely many strictly positive d_j's. We may assume that $d_1, \cdots, d_n > 0$. Let K' =diag $(d_1 - 1, \cdots, d_n - 1, d_{n+1}, \cdots)$. Then K' is of trace class with $K' \geq -1$, rank K'_+ =rank $K'_- = \infty$ and tr K'_+ =tr $K'_- < \infty$. It follows from Conjecture 2.7, if indeed true, that $1 + K'$ is a sum of projections. Therefore $1 + K = (1 + K')$+diag $(1, \cdots 1, 0, 0, \cdots)$ is also a sum of projections as asserted.

3. MISCELLANIES

In this section, we consider some variations of the sum-of-projections problem. They are of two different types. One type involves sums of projections with some additional properties. For example, we may require that the projections be commuting to each other and /or having some fixed (finite or infinite) rank. Another tpe concerns operators which can be approximated by sums of (two) projections in the norm topology. We start with the case of sums of commuting projections.

Proposition 3.1. (a) *T is the sum of commuting projections if and only if T is positive and $\sigma(T)$, the spectrum of T, consists of finitely many nonnegative integers.*

(b) *T is the sum of commuting infinite-rank projections if and only if T is positive, has infinite rank and $\sigma(T)$ consists of finitely many nonnegative integers.*

(c) *Let $k \geq 1$ be a fixed integer. Then T is the sum of commuting rank-k projections if and only if T has finite rank, $\sigma(T)$ consists of finitely many nonnegative integers, k divides* tr T *and* $0 \leq T \leq (\text{tr } T/k)I$.

Proof. (a) Let $T = \sum_{j=1}^{n} P_j$, where the $P_j's$ are commuting projections. Then the abelian C*-algebra A generated by the $P_j's$ and the identity operator is $*-$isomorphic to the C*$-$algebra $C(X)$ of continuous functions on some compact Hausdorff space X. Under this isomorphism, the spectrum of any operator in A is equal to the range of the corresponding function in $C(X)$. Hence $\sigma(T)$ consists of numbers of the form $x_1 + \cdots + x_n$, where each x_j is 0 or 1, which are all nonnegative integers.

Conversely, if T satisfies the given conditions, then we may assume that it is of the form $\sum_{j=1}^{n} \oplus k_j I_j$, where the $k_j's$ are integers satisfying $k_1 \geq \cdots \geq k_n \geq 0$. Then the expression

$$T = \sum_{j=1}^{n} (k_j - k_{j+1}) (\underbrace{I_1 \oplus \cdots \oplus I_j \oplus 0 \oplus \cdots \oplus 0}_{n})$$

with $k_{n+1} = 0$ expresses T as the sum of k_1 many commuting projections.

(b) If $T = \sum_{j=1}^{n} P_j$, where the $P_j's$ are commuting infinite-rank projections, then, in particular, $0 \leq P_1 \leq T$. This implies easily that ran $P_1 \subseteq \overline{\text{ran} T}$ or rank $P_1 \leq$rank T and thus rank $T = \infty$.

Conversely, if T satisfies the given conditions, then we may assume that $T = \sum_{j=1}^{n} \oplus k_j I_j$ on $\sum_{j=1}^{n} \oplus H_j$, where $k_1 \geq \cdots \geq k_n \geq 0$ and for some $n_0, 1 \leq n_0 \leq n$, we have $k_{n_0} > 0$, dim $H_j < \infty$ for $1 \leq j < n_0$ and dim $H_{n_0} = \infty$. Let m be an integer such that $m \geq \left(\sum_{j \neq n_0} k_j \right) / k_{n_0}$, and "split" H_{n_0} into the direct sum of m copies of infinite-dimensional subspaces. Then T is unitarily equivalent to

$$\left(\sum_{j\neq n_0} \oplus k_j I_j\right) \oplus \left(\sum_{i=1}^{m} \oplus k_{n_0} I_{n_0}\right).$$

Obviously, this latter operator can be written as the sum of mk_{n_0} many commuting projections of the form

$$(0 \oplus \cdots \oplus I_j \oplus \cdots \oplus 0) \oplus (0 \oplus \cdots \oplus I_{n_0} \oplus \cdots \oplus 0),$$

each of which has infinite rank.

(c) If $T = \sum_{j=1}^{n} P_j$, where the $P_j's$ are commuting rank$-k$ projections, then obviously T has finite rank and $\sigma(T)$ consists of finitely many nonnegative integers by (a). We also have tr $T = \sum_j$tr $P_j = \sum_j$ rank $P_j = nk$ and $\| T \| \leq \sum_j \| P_j \| = n =$tr T/k. The latter condition implies that $T \leq$(tr $T/k)I$.

The converse is proved by induction on $n =$tr T/k. If $n = 1$, then $0 \leq T \leq 1$, tr $T = k$ and $\sigma(T)$ consists of integers, which implies that T itself is a rank$-k$ projection. For the general case, we may assume that $T =$diag $(t_1, \cdots, t_m)$ on $\mathbf{C}^m$, where the $t_j's$ are integers satisfying $n \geq t_1 \geq \cdots \geq t_m \geq 0$. Note that in this case we have $t_k \geq 1$. Indeed, if otherwise $t_k = \cdots = t_m = 0$, then $nk =$tr $T = \sum_{j=1}^{k=1} t_j \leq (k-1)n$ which is impossible. Let $T_1 =$diag $(t_1 - 1, \cdots, t_k - 1, t_{k+1}, \cdots, t_m)$. Then $T_1 \geq 0$, the eigenvalues of T_1 are nonnegative integers and tr $T_1 =$tr $T - k = (n-1)k$. Note that we also have $T_1 \leq (n-1)I$. Indeed, if otherwise, then since $t_j - 1 \leq n - 1$ for $j = 1, \cdots, k$ we have $t_{k+1} \geq n$, which implies that $nk =$tr $T \geq \sum_{j=1}^{k+1} t_j \geq (k+1)n$, a contradiction. Thus the induction hypothesis can be applied to T_1 so that T_1 is a sum of $n-1$ commuting rank$-k$ projections. Via simultaneous diagonalization, we may assume that these projections are all represented as diagonal matrices. Thus $T = T_1 + \text{diag}\,(1, \cdots, 1, 0, \cdots, 0)$ is the sum of n commuting rank$-k$ projections. This completes the proof. ∎

Dropping the requirement that the projections be commuting, we have the following analogue of Proposition 3.1 (b).

Proposition 3.2. *T is the sum of infinite-rank projections if and only if T itself has infinite rank and is the sum of projections.*

Proof. We only prove the sufficiency part. Assume that $T = \sum_{j=1}^{n} P_j$ is a sum of projections, where P_j is of finite rank for $j = 1, \cdots, m$ $(1 \leq m < n)$ and of infinite rank otherwise. Let K be the finite-dimensional subspace $\bigvee_{j=1}^{m}(\text{ran } P_j \vee P_{m+1}(\text{ran } P_j))$. Then K is invariant for $P_1, \cdots, P_{m+1}$. Let $P_j = Q_j \oplus 0, j = 1, \cdots, m$, and $P_{m+1} = Q_{m+1} \oplus R$ on the decomposition $K \oplus K^{\perp}$. Then $T' \equiv \sum_{j=1}^{m+1} P_j = \left(\sum_{j=1}^{m+1} Q_j\right) \oplus R$. Since the infinite-rank R is unitarily equivalent to $I^{(m+1)} \oplus 0$, where the identity operator acts on an infinite-dimensional space, T' is unitarily equivalent to the sum of the operators

$$Q_j \oplus (0 \oplus \cdots \oplus \underset{jth}{I} \oplus \cdots \oplus 0) \oplus 0, \quad j = 1, \cdots, m+1,$$

each of which is an infinite-rank projection. It follows that T is a sum of infinite-rank projections. ∎

It would be interesting to have a noncommutative analogue of Proposition 3.1(c), that is, a characterization of sums of rank-k projections for each $k \geq 1$.

We next turn to problems concerning operators approximatable by sums of (two) projections. Note that in the finite-dimensional case, the set of sums of (two) projections is itself closed. These are easy consequences of the results of Fillmore [2]. We start with sums of two projections. Recall that an operator can be expressed as such if and only if it is unitarily equivalent to an operator of the form $0 \oplus 1 \oplus 2I \oplus A \oplus (2I - A)$, where $0 < A < 1$ (cf. [2, p.151]). The next result gives a characterization of operators (on an infinite-dimensional space) which can be approximated by such operators in norm.

Proposition 3.3. *The norm closure of the set of operators which can be written as a sum of tw projections consists of those which are unitarily equivalent to an operator of the form* $0 \oplus 1 \oplus 2I \oplus A \oplus (2I - B)$, *where* $0 < A, B < 1, \sigma(A) = \sigma(B)$ *and the multiplicities of each isolated eigenvalue of* A *and* B *are equal.*

Proof. If $T = A \oplus (2I - B)$, where A and B satisfy the stated properties, then by [5, Theorem 1] there is a sequence of unitary operators $\{U_n\}$ such that $U_n^* A U_n$ converges to B in norm. Let $T_n = A \oplus (2I - U_n^* A U_n)$. Then each T_n is a sum of two projections by [2, p.151] and T_n converges to T in norm. This shows that T is in the asserted closure.

Conversely, assume that $\{T_n\}$ is a sequence of sums of two projections which converges to an operator T in norm. Then $C_n \equiv (T_n - 1)_+$ (resp. $D_n \equiv (T_n - 1)_-$) converges to $C \equiv (T-1)_+$ (resp. $D \equiv (T-1)_-$) in norm. Since $0 \le T \le 2$, we have $0 \le C, D \le 1$. We first show that $\sigma(C) \cup \{0,1\} = \sigma(D) \cup \{0,1\}$. Indeed, from the structure of sums of two projections, we have $\sigma(C_n) \cup \{0,1\} = \sigma(D_n) \cup \{0,1\}$ for all n. Since the function which maps an operator to its spectrum is continuous when restricted to the normal ones (cf. [9, Problem 105]), we obtain, as n approaches infinity, $\sigma(C) \cup \{0,1\} = \sigma(D) \cup \{0,1\}$ as asserted. Next, let $\lambda, 0 < \lambda < 1$, be any isolated eigenvalue of C. We will show that λ, as an (isolated) eigenvalue of D, has the same multiplicity as for C. Let $0 < \varepsilon_1 < \varepsilon_2 < \min\{\lambda, 1-\lambda\}$ be such that $\sigma(C) \subseteq [0, \lambda - \varepsilon_2) \cup (\lambda - \varepsilon_1, \lambda + \varepsilon_1) \cup (\lambda + \varepsilon_2, 1] \equiv \Omega$, and let $f : [0,1] \to [0,1]$ be a continuous function such that

$$f(t) = \begin{cases} 1 & \text{if } t \in (\lambda - \varepsilon_1, \lambda + \varepsilon_1) \\ 0 & \text{if } t \in [0, \lambda - \varepsilon_2) \cup (\lambda + \varepsilon_2, 1]. \end{cases}$$

Since $\sigma(C_n)$ converges to $\sigma(C)$ as n approaches infinity, there exists an N such that $\sigma(C_n) \subseteq \Omega$ for all $n \ge N$. From $C_n \to C$ in norm, we obtain $f(C_n) \to f(C)$ in norm (cf. [9, Problem 126]). Since $P_n \equiv f(C_n)$ and $P \equiv f(C)$ are projections, we infer that rank P_n =rank P for all large n (cf. [9, Problem 57]). Similarly, we have rank Q_n =rank Q for all large n, where $Q_n = f(D_n)$ and $Q = f(D)$ are projections. Since rank P_n =rank Q_n for all n by the structure of sums of two projections, we obtain rank P =rank Q or dim $\{x : Cx = \lambda x\}$ =dim $\{y : Dy = \lambda y\}$. The assertions in the statement of our proposition then follows immediately. ∎

Our final result is a characterization of the norm closure of the set of sums of projections.

Theorem 3.4. *The norm closure of the set of sums of projections consists of all positive operators which either have essential norm greater than or equal to one or have finite rank and are a sum of projections.*

Proof. If T is a positive operator with $\| T \|_e \geq 1$, then, for every $n \geq 1, T + \frac{1}{n}I$ is a positive operator with $\| T + \frac{1}{n}I \|_e > 1$ and hence is a sum of projections by Theorem 1.5. Hence T, as a norm limit of the sequence $\left\{T + \frac{1}{n}I\right\}$, is in the asserted closure.

Conversely, let $T_n \to T$ in norm, where each T_n is a sum of projections. Assume that $\| T \|_e < 1$. We will show that T must be of finite rank and is a sum of projections. Since $\| T_n \|_e$ converges to $\| T \|_e$, we may assume that $\| T_n \|_e < 1$ for all n. Proposition 1.4(b) implies that T_n is of finite rank and thus T is compact. Assume first that $\| T \| < 1$. As $\| T_n \|$ converges to $\| T \|$, we may assume that $\| T_n \| < 1$ for all n. Thus Proposition 1.4(a) implies that $T_n = 0$ and hence $T = 0$. For the remaining part of the proof, we assume that $\| T \| \geq 1$. Let k_n =rank T_n for $n \geq 1$. Since rank $T \leq \lim\inf_{n\to\infty}$ rank T_n (cf. [7, Appendix]), if the quantity on the right-hand side is infinity, then there exists a subsequence $\{T_{n_j}\}$ such that rank $T_{n_j} \to \infty$. For convenience, we will assume that k_n =rank $T_n \to \infty$. For each n, let $\lambda_n^{(j)}$ (resp. $\lambda^{(j)}$) denote the jth largest eigenvalue (counting multiplicity) of T_n (resp. T), and let d_n (resp. d) be the number of $\lambda_n^{(j)\prime}s$ (resp. $\lambda^{(j)\prime}s$) which are greater than or equal to 1. It is known that $\lambda_n^{(j)} \to \lambda^{(j)}$ as $n \to \infty$ for each j (cf. [6, Theorem I.4.2]) and hence $d_n \to d$. Let $\delta = \min\{\frac{1}{2}, 1 - \lambda^{(d+1)}\} > 0$, and let n be so large that $\| T_n \| \le \| T \| + 1, k_n > \frac{2}{\delta} d \| T \|, d_n = d$ and $\lambda_n^{(d+1)} \le \lambda^{(d+1)} + \frac{\delta}{3}$. Then

$$\begin{aligned} k_n = \text{rank } T_n \le \text{tr } T_n &= \sum_{j=1}^{k_n} \lambda_n^{(j)} \\ &\le d \| T_n \| + (k_n - d)\lambda_n^{(d+1)} \\ &\le d(\| T \| + 1) + (k_n - d)\left(\lambda^{(d+1)} + \frac{\delta}{3}\right) \\ &\le d(\| T \| + 1) + (k_n - d)\left(1 - \frac{2}{3}\delta\right). \end{aligned}$$

Hence

$$d \| T \| + \frac{2}{3}\delta d \ge \frac{2}{3}\delta k_n > \frac{4}{3} d \| T \|$$

or $d \| T \| < 2\delta d$. Since $\| T \| \ge 1$, we have $d \ge 1$. Therefore, $\| T \| < 2\delta \le 1$, contradicting our assumption. This shows that $\liminf_{n\to\infty} \text{rank } T_n < \infty$ and, in particular, rank T is finite. Passing to a subsequence, we may assume that rank T_n is a constant, say, k for all n. We have rank $T \le k$ and $\lambda_n^{(j)} \to \lambda^{(j)}$ as $n \to \infty$ for each $j, 1 \le j \le k$. Thus tr $T_n = \sum_{j=1}^{k} \lambda_n^{(j)} \to \sum_{j=1}^{k} \lambda^{(j)} =$tr T. Since tr T_n is an integer and tr $T_n \ge$rank $T_n = k$ by Corollary 1.3, we deduce that tr T is also an integer and tr T=tr $T_n \ge k \ge$ rank T. By Corollary 1.3 again, T is a sum of projections, completing the proof. ∎

To conclude this paper, we remark that on an infinite-dimensional space, the closure of the set of sums of projections in the weak operator topology (WOT) consists of all positive operators. Indeed, if $T \ge 0$, then, letting $n \ge \| T \|$, we have $0 \le \frac{1}{n} T \le 1$.

[9, Problem 224] implies that there are projections P_j such that $P_j \to \frac{1}{n}T$ in the WOT. Hence $nP_j \to T$ in the WOT, which shows that T is in the WOT-closure of sums of projections. The same proof also shows that for any $n \geq 1$, the WOT-closure of sums of n projections equals the set $\{T : 0 \leq T \leq nI\}$.

ACKNOWLEDGEMENT. The main results in Section 2 were obtained in 1987-88 while the second author was visiting University of Toronto arranged by the first author. He is grateful to the latter for this research experience.

REFERENCES

1. J.B. Conway, *A course in functional analysis*, 2nd ed., Springer-Verlag, New York, 1990.

2. P.A. Fillmore, *On sums of projections*, J. Func. Anal., 4(1969), 146-152.

3. P.A. Fillmore, *On similarity and the diagonal of a matrix*, Amer. Math. Monthly, 76(1969), 167-169.

4. P.A. Fillmore, J.G. Stampfli and J.P. Williams, *On the essential numerical range, the essential spectrum, and a problem of Halmos*, Acta Sci. Math. (Szeged), 33(1972), 179-192.

5. R. Gellar and L. Page, *Limits f unitarily equivalent normal operators*, Duke Math. J., 41 (1974), 319-322.

6. I.C. Gohberg and M.G. Krein, *Introduction to the theory of linear nonselfadjoint operators*, Amer. Math. Soc., Providence, 1969.

7. P.R. Halmos, *Irreducible operators*, Michigan Math. J., 15(1968), 215-223.

8. P.R. Halmos, *Two subspaces*, Trans. Amer. Math. Soc., 144(1969), 381-389.

9. P.R. Halmos, *A Hilbert space problem book*, 2nd ed., Springer-Verlag, New York, 1982.

MAN-DUEN CHOI
Department of Mathematics
University of Toronto
Toronto, Ontario M5S 1A1
Canada

E-mail:choi@math.toronto.edu

PEI YUAN WU
Department of Applied Mathematics
National Chiao Tung University
Hsinchu, Taiwan
Republic of China

E-mail:pywu@cc.nctu.edu.tw

89R096
NSC89-2115-M-009-025
正交投影和之研究

參加「第三屆亞洲數學會議」報告

報告人：吳培元
（國立交通大學應用數學系）

「亞洲數學會議」（Asian Mathematical Conference）是由「東南亞數學會」（Southeast Asian Mathematical Society）各會員國輪流主持，其目的是促進東南亞各國間數學研究成果的交流。第一屆是在1990年由香港地區主辦，第二屆在1995年由泰國主辦，今年第三屆由菲律賓的 University of the Philippines - Diliman 主辦，在十月二十三日至二十七日，於該校區所在的 Quezon City 及校園內進行。預定第四屆將在2004年由新加坡主辦。

此次與會的台灣數學家共有四人，除本人外，其他三位（阮希石，鄭日新及李國偉）均來自中央研究院。本人係在十月二十二日搭乘剛復航的長榮班机抵達馬尼拉機場，和阮、鄭二人共同乘坐由大會當局安排的車子，約大半個鐘頭後，到達預訂的旅館安頓下來。

正式的會議由十月二十三日（星期一）開始，當天的議程在校園外 Quezon City 的 Social Security System Building 內的大廳中進行，開幕典禮中並由菲大合唱團及土風舞社學生表演歌舞，其後即開始一整天的演講活動，上、下午共五場 plenary lectures。首場由現在 Yale Univ.，曾獲得 Fields Medal 的 E. Zelmanov 講 "What makes a group finite?" 主要是介紹 restricted Burnside Problem 的最新研究進展，其後四位分別

由中國大陸、韓國、日本和新加坡的學者作報告，每人一個小時。第二天的議程移到菲大校園內，分別在數學系的建築和鄰近的國家地質研究所內三個場地同步進行，總共有二十四場 invited lectures，每場四十分鐘。本人的演講 "Poncelet property for numerical ranges" 被安排在下午兩點開始，因其內容上溯到十七、八世紀投影幾何的一些古典結果，對於一般非本行的數學家應還可以接受，故雖沒有同行的研究者，但演講後仍有多人前來表達高度的興趣。八月二十五日的議程仍移回 SSS Building 內舉行。上午是三場 invited lectures，包括了復旦大學的李大潛和中研院的阮希石的演講。下午則由會議主辦當局安排了馬尼拉市的半日旅遊，參觀了 Intramuros 內的 Casa Manila Museum 和 Fort Santiago。只是馬尼拉市區內交通擁塞，花了太多時間在來回程的路上。八月二十六的會議仍移回菲大校園內舉行，大部份仍是 invited lectures，到下午時分也安排了不少場次每人二十分鐘的 short communications。當天晚上除了加菜聚餐外，另在數學系建築內搭起了舞台座椅，仍由菲大合唱團及舞蹈社學生演出合唱歌曲及菲國舞蹈，包括了其國舞的竹竿舞。八月二十七日是會議的最後一天，安排有一整天大量的 short communications。我因將搭乘當天中午的飛機回台灣，故就沒有前往會場而由旅館搭乘會議當局安排的車子直接到機場搭機回台，結束了前後六天的旅程。

本次會議顯示亞洲數學的水準較之過往已有顯著的提昇，但離歐美等地的水準仍有相當差距。包括了台灣、中國大陸、新加坡、馬來西亞等地的華人表現也很傑出，另一個較佳的集團則是日本人。

我本人參與此項會議因研究同樣課題的學者很少，故在研究專業上並沒有具体與人討論的机會。只利用機會聽了很多他人的演講，了解一下各地數學發展的現況，同時也和各地學者介紹「台灣數學期刊」，推廣其在國際上的能見度，並把手边帶去的三本該期刊分贈給各相關人士，作了一些學術交流的業績。

89R096
NSC89-2115-M-009-025
正交投影和之研究

至美加地區兩校訪問心得報告

撰寫人：吳培元　　2001年7月25日

暑假期間，經國科會國外差旅費的資助，前往美國 Williamsburg, Virginia 的 College of William and Mary 及加拿大 Toronto, Ontario 的 University of Toronto 兩校數學系訪問，就本年度的研究計畫「正交投影和之研究」內容作進一步的探討。

我是在七月九日自台北出發，經 Newark 機場轉機，夜宿一夜後，於七月十日一早八點半到達 Richmond, Virginia 國際機場，由 College of William and Mary 數學系教授 Charles R. Johnson 接機，旋即到該系安頓展開工作。Johnson 教授係該校講座教授，在十餘年前應聘到該系，建立了一個在矩陣分析方面堅強的研究群，有五、六位成員之多。他本人也曾在去年和今年來台灣參加國際會議及到各校訪問。我在該系共訪問四整天，其間與 Johnson 本人及 Ilya Spitkovsky、李志光等三人多所交談，交換研究心得，尤其是數值域研究方面，他們提供了不少參考資料，對以後研究工作的推進有很大的幫助。

我在七月十四日結束在 William & Mary 的訪問，搭機往加拿大的 Toronto，於當天下午時分住進 Univ. of Toronto 的客房 Hart House。此行的接待者是該校數學系教授蔡文端 (Man-Duen Choi)，他目前也擔任「台灣數學期刊」編輯之一，過去也曾來台灣訪問過，和我有長期的合作關係。經其安排，本人在七月十六日下午在該系作一次演講，講題是 "Numerical ranges of finite matrices"，其後的

幾天訪問期間，我和蔡教授兩人每天從早到晚坐在辦公室內，就正交投影和之問題苦思，想將其中部份未解決的問題徹底完成，結果三天以來稍有進展，但還是未能全部完成。

在七月二十日，我搭機自Toronto回到台北，結束了這一次前後十三天的旅程，後續的研究工作仍要繼續進行。