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Mathematical Models and Methods in Applied Sciences

On Two-phase Flow in Fractured Media

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A model describing two-phase, incompressible, immiscible flow in fractured media is concerned. A fractured medium is regarded as a porous medium consisting of two superimposed continua, a continuous fracture system and a discontinuous system of medium-sized matrix blocks. Transport of fluids through the medium is primarily within the fracture system. No flow is allowed between blocks, and only matrix-fracture flow is possible. Matrix block system plays the role of a global source distributed over the entire medium. Two-phase flow in a fractured medium is strongly related to phase mobilities and capillary pressures. In this work, four relations for these functions are presented, and existence of weak solutions under each relation will be shown also.

1. Introduction

A dual-porosity model describing two-phase, incompressible, immiscible flow in fractured media is concerned. The phases are the nonwetting "o" (oil) phase and the wetting "w" (water) phase. Within a fractured medium there is an interconnected system of fracture planes dividing the porous medium into a collection of matrix blocks. The fracture planes, while very thin, form paths of high permeability. Most of the fluids reside in matrix blocks, where they move very slow. For model considered here, a fractured medium is regarded as a porous medium consisting of two superimposed continua, a continuous fracture system and a discontinuous system of medium-sized matrix blocks. Fracture system has a lower storativity and higher conductivity than matrix block system. Transport of fluids through the medium is primarily within the fracture system. No flow is allowed between blocks, and fluids that reside in matrix blocks must enter the fractures to move great distance. Essentially, matrix block system plays the role of a global source distributed over the entire medium. As a consequence, two sets of equations are obtained for the flow. One contains macroscopic equations for fracture flow, and the other consists of microscopic equations for flow in matrix blocks. The two sets of equations are coupled through locally defined macroscopic matrix-fracture sources, one for each phase. For more description of flow in the medium, readers are referred to [5, 7, 10, 12, 13] and references therein.

If $\Omega \subset \Re^3$ is a fractured medium, equations for fracture flow [5, 10] are, for

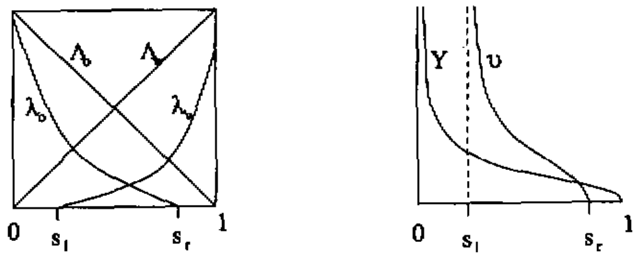

Figure 1: Phase mobilities (left) and capillary pressures (right) of fracture system and matrix blocks.

$x \in \Omega,\ t > 0,$

$$\partial_t S - \nabla_x \cdot (\Lambda_w(S)\nabla_x(P_w - E_w)) = q_w, \tag{1.1}$$

$$-\partial_t S - \nabla_x \cdot (\Lambda_o(S)\nabla_x(P_o - E_o)) = q_o, \tag{1.2}$$

$$\Upsilon(S) = P_o - P_w. \tag{1.3}$$

$S \in [0,1]$ is water saturation; Λ_α $(\alpha = w, o)$ is phase mobility of α-phase, a nonnegative monotone function of S (see Figure 1); P_α denotes pressure; E_α is a function depending on density, gravity, and position; q_α is the matrix-fracture source; and Υ is capillary pressure, a nonnegative decreasing function of S (see Figure 1). Porosity and permeability field have been set 1 for convenience. Incompressibility implies $q_o + q_w = 0$.

Above each point $x \in \Omega$ is suspended topologically a matrix block $\Omega_x \subset \Re^3$. Equations for flow in a matrix block are, for $x \in \Omega,\ y \in \Omega_x,\ t > 0$,

$$\partial_t s - \nabla_y \cdot (\lambda_w(s)\nabla_y p_w) = 0, \tag{1.4}$$

$$-\partial_t s - \nabla_y \cdot (\lambda_o(s)\nabla_y p_o) = 0, \tag{1.5}$$

$$\upsilon(s) = p_o - p_w. \tag{1.6}$$

Each lower case symbol denotes the quantity on Ω_x corresponding to that denoted by an upper case symbol in the fracture system equations. S, P_α, q_α for $\alpha \in \{w, o\}$ in (1.1–1.3) are functions on $\Omega \times [0,T]$, and s, p_α in (1.4–1.6) are on $\Pi_{x\in\Omega}\Omega_x \times [0,T]$. p_α $(\alpha = w, o)$ in (1.4–1.5) only takes derivative with respect to variable y.

The matrix-fracture sources are given by, for $x \in \Omega$, $t > 0$,

$$q_w(x,t) = \frac{-1}{|\Omega_x|}\int_{\Omega_x} \partial_t s(x,y,t)dy = -q_o(x,t), \tag{1.7}$$

where $|\Omega_x|$ is the volume of Ω_x. Boundary $\partial\Omega$ of Ω includes $\partial_1\Omega, \partial_2\Omega$, which satisfying $\partial_1\Omega \cap \partial_2\Omega = \emptyset, \partial\Omega = \partial_1\Omega \cup \partial_2\Omega$. Boundary conditions for fracture system are, for $t > 0$, $\alpha \in \{w, o\}$,

$$P_\alpha = P_{\alpha,b}, \qquad \text{for } x \in \partial_1\Omega, \tag{1.8}$$

$$\Lambda_\alpha(S)\nabla_x(P_\alpha - E_\alpha)\cdot\vec{n} = 0, \qquad \text{for } x \in \partial_2\Omega, \tag{1.9}$$

where $\vec{n}$ is the unit vector outward normal to $\partial\Omega$. Boundary conditions for each matrix block require continuity of pressures, that is, for $t > 0$, $x \in \Omega$, $y \in \partial\Omega_x$, $\alpha \in \{w, o\}$,

$$p_\alpha(x, y, t) = P_\alpha(x, t). \tag{1.10}$$

Initial equilibrium gives

$$S(x, 0) = S_0(x), \qquad \text{for } x \in \Omega, \tag{1.11}$$

$$s(x, y, 0) = s_0(x), \qquad \text{for } x \in \Omega, \quad y \in \Omega_x. \tag{1.12}$$

Two-phase flow in fractured media is strongly related to phase mobilities and capillary pressures [10, 11, 12, 13]. For flow in a bundle of tubes, a mobility curve was measured to be a linear function of phase saturation. In general, phase mobility curves may be determined by being adjusted to history-match field data if all other data are known. Fracture capillary pressure would be near zero for most water saturation values. Matrix mobilities and matrix capillary pressure can be those measured on unfractured media. To maintain gravity/capillary equilibrium, capillary pressure end-points in fracture system and matrix blocks must be set equal [12, 13]. In reality, it is not easy to measure phase mobilities and capillary pressures accurately. Our intention is to look for proper relations for these functions. Some literatures related this problem are listed below. For unfractured media case (that is, $q_w = q_o = 0$), existence of solutions of (1.1–1.3) were studied in [3, 4, 8, 14] and references therein. If one linearizes matrix mobility λ_α ($\alpha = w, o$) or assumes matrix blocks are small, matrix-fracture source q_α is a function of phase saturation. Existence of weak solutions in these cases were considered in [6, 9]. Existence of solutions in a global pressure form of (1.1–1.12) could be found in [7, 17]. In this work, four relations for phase mobilities and capillary pressures are presented. Existence of weak solutions of (1.1–1.12) will be shown for each relation. To reach the goal, a global pressure is introduced to simplify system (1.1–1.12) first. Next, existence of solutions of the simplified system will be shown. Finally, we prove that a subsequence of these solutions converges to a weak solution of (1.1–1.12). Phase mobilities and capillary pressures in [10, 11, 12, 13] satisfy one of the relations here.

Rest of the paper is organized as follows: Notation is recalled and main result is stated in §2. An auxiliary system for (1.1–1.12) is derived and procedure of proof for main result is described in §3. The main result is proved in §4 under the assumption of existence of solution for auxiliary system, which is shown in §5.

2. Notation and Main Result

2.1. *Notation*

Let $\Omega \subset \Re^3$ be open, bounded, and connected with Lipschitz boundary. For every $x \in \Omega$, $\Omega_x \subset \Re^3$ is a bounded region. Identify the product space $\Pi_{x\in\Omega}\Omega_x$ (denoted by Q) as a subset of $\Re^6$. For simplicity, all matrix blocks are assumed to be identical, volume 1, and smooth enough. That is, $\mathcal{Q} = \Omega \times \mathcal{M}$, $|\mathcal{M}| = 1$, and $\mathcal{M} \subset \Re^3$ is assumed to be bounded with Lipschitz boundary $\partial\mathcal{M}$. $\Omega^t \stackrel{\text{def}}{=} \Omega \times [0,t]$ and $\mathcal{Q}^t \stackrel{\text{def}}{=} \mathcal{Q} \times [0,t]$.

$L^r(\mathcal{B})$, $H^m(\mathcal{B})$, $W^{m,r}(\mathcal{B})$, $L^r(\Omega, W^{m,r}(\mathcal{M}))$, $L^r(\Omega, L^r(\partial\mathcal{M}))$, $L^r(0,T;X)$, and $H^m(0,T;X)$ are Sobolev spaces [1] for $r > 1$, $m \in \mathbf{N}$, $\mathcal{B} \subset \mathcal{Q}^T$, and a Banach space X.

$$\begin{cases} \mathcal{W}_0^{m,r}(\Omega) \stackrel{\text{def}}{=} \{f \in W^{m,r}(\Omega) : \ f|_{\partial_1\Omega} = 0\}, \\ \mathcal{V} \stackrel{\text{def}}{=} \mathcal{W}_0^{1,2}(\Omega), \\ \mathcal{W}_y^{1,r}(\mathcal{Q}) \stackrel{\text{def}}{=} \{f \in L^r(\mathcal{Q}) : \nabla_y f \in L^r(\mathcal{Q})\}, \\ \mathcal{U} \stackrel{\text{def}}{=} \mathcal{W}_y^{1,2}(\mathcal{Q}). \end{cases}$$

Note $\mathcal{W}_y^{1,r}(\mathcal{Q}) \subset L^r(\Omega, W^{1,r}(\mathcal{M}))$. Let $\mathcal{T}_x$ be the usual trace map of $W^{1,r}(\mathcal{M})$ into $L^r(\partial\mathcal{M})$. We define the distributed trace $\mathcal{T} : \mathcal{W}_y^{1,r}(\mathcal{Q}) \to L^r(\Omega, L^r(\partial\mathcal{M}))$ by $\mathcal{T}f(x,y) = (\mathcal{T}_x f(x))(y)$.

$$\begin{cases} \mathcal{W}_{y,0}^{1,r}(\mathcal{Q}) \stackrel{\text{def}}{=} \{f \in \mathcal{W}_y^{1,r}(\mathcal{Q}) : \mathcal{T}f = 0\}, \\ \mathcal{U}_0 \stackrel{\text{def}}{=} \mathcal{W}_{y,0}^{1,2}(\mathcal{Q}), \\ \mathcal{W}_1 \stackrel{\text{def}}{=} \mathcal{V} \times \mathcal{V} \times \mathcal{U}_0, \\ \mathcal{W}_2 \stackrel{\text{def}}{=} \mathcal{V} \times \mathcal{V} \times \mathcal{U}_0 \times \mathcal{U}_0, \\ \textit{dual } X \stackrel{\text{def}}{=} \text{dual space of } X, \\ s_l \text{ (resp. } 1 - s_r) \stackrel{\text{def}}{=} \text{residual matrix water (resp. oil) saturation.} \end{cases}$$

$\Re_0^+ \stackrel{\text{def}}{=} \Re^+ \cup \{0\}$. If $\Upsilon : (0,1] \to \Re_0^+$ (resp. $\upsilon : (s_l, s_r] \to \Re_0^+$) is onto and a strictly decreasing function, let Υ^{-1} (resp. υ^{-1}) be the inverse function of Υ (resp. υ). We define $\mathcal{J} : (0,1] \to (s_l, s_r]$ by $\mathcal{J}(z) \stackrel{\text{def}}{=} \upsilon^{-1}(\Upsilon(z))$, and denote the inverse function of $\mathcal{J}$ by $\mathcal{J}^{-1}$. Let $\mathcal{J}(0.5) \in (s_l, s_r) \subset (0,1)$.

$$\begin{cases} \partial^h f(t) \stackrel{\text{def}}{=} \frac{f(t+h) - f(t)}{h}, \\ P_{c,b} \stackrel{\text{def}}{=} P_{o,b} - P_{w,b}, \\ \Lambda \stackrel{\text{def}}{=} \Lambda_w + \Lambda_o, \\ \lambda \stackrel{\text{def}}{=} \lambda_w + \lambda_o, \\ \mathbf{R}(z) \stackrel{\text{def}}{=} \int_{0.5}^z \frac{\Lambda_w \Lambda_o}{\Lambda} \left|\frac{d\Upsilon}{dS}\right|(\xi) d\xi, \qquad \text{for } z \in (0,1], \\ \mathcal{D}(z) \stackrel{\text{def}}{=} \int_{\mathcal{J}(0.5)}^z \frac{\lambda_w \lambda_o}{\lambda} \left|\frac{dv}{ds}\right|(\xi) d\xi, \quad \text{for } z \in (s_l, s_r]. \end{cases} \tag{2.1}$$

We define $\mathcal{L} : L^r(\Omega) \to L^r(\Omega, L^\infty(\mathcal{M}))$ by $\mathcal{L}f(x,y) = f(x)1_y, x \in \Omega, y \in \mathcal{M}$, where $f(x)1_y$ is constant in $\mathcal{M}$ with value $f(x)$. $f \in L^r(\Omega)$ will be identified with $\mathcal{L}f \in L^r(\Omega, L^\infty(\mathcal{M}))$.

2.2. *Main result*

Taking $(\zeta_w, \zeta_o, \eta_w, \eta_o) \in L^2(0,T;\mathcal{W}_2)$, multiplying (1.1), (1.2), (1.4), (1.5) by ζ_w, ζ_o, η_w, η_o respectively, and integrating these functions over $\mathcal{Q}^T$, one obtains a weak formulation for equations (1.1–1.2) and (1.4–1.5), by (1.8–1.10),

$$\int_{\Omega^T} \partial_t S\, \zeta_w + \int_{\Omega^T} \Lambda_w(S)\nabla_x(P_w - E_w)\nabla_x\zeta_w = -\int_{Q^T} \partial_t s\, \zeta_w, \tag{2.2}$$

$$-\int_{\Omega^T} \partial_t S\, \zeta_o + \int_{\Omega^T} \Lambda_o(S)\nabla_x(P_o - E_o)\nabla_x\zeta_o = \int_{Q^T} \partial_t s\, \zeta_o, \tag{2.3}$$

$$\int_{Q^T} \partial_t s\, \eta_w + \int_{Q^T} \lambda_w(s)\nabla_y p_w \nabla_y \eta_w = 0, \tag{2.4}$$

$$-\int_{Q^T} \partial_t s\, \eta_o + \int_{Q^T} \lambda_o(s)\nabla_y p_o \nabla_y \eta_o = 0. \tag{2.5}$$

Definition 2.1 $\{S, P_w, P_o, s, p_w, p_o\}$ *is a weak solution of the equations* (1.1–1.12) *if there is a number* $r \in (1,2)$ *such that, for* $\alpha \in \{w, o\}$,

1. $P_\alpha - P_{\alpha,b} \in L^r(0,T;\mathcal{W}_0^{1,r}(\Omega)),\ p_\alpha - P_\alpha \in L^r(0,T;\mathcal{W}_{y,0}^{1,r}(Q))$,
2. $\partial_t S + \int_{\mathcal{M}} \partial_t s\, dy \in$ *dual* $L^2(0,T;\mathcal{V}),\ \partial_t s \in$ *dual* $L^2(0,T;\mathcal{U}_0)$,
3. $\Lambda_\alpha \nabla_x P_\alpha \in L^2(\Omega^T),\ \lambda_\alpha \nabla_y p_\alpha \in L^2(Q^T)$,
4. $\Upsilon(S) = P_o - P_w,\ \upsilon(s) = p_o - p_w$,
5. (2.2–2.5) *hold for any* $\zeta_\alpha \in L^2(0,T;\mathcal{V}),\ \eta_\alpha \in L^2(0,T;\mathcal{U}_0)$,
6. $0 < S < 1,\ s_l < s < s_r$,
7. *For* $\zeta \in L^2(0,T;\mathcal{V}) \cap H^1(\Omega^T), \eta \in L^2(0,T;\mathcal{U}) \cap H^1(0,T;L^2(Q)),\ \zeta(T) = \eta(T) = 0$,

$$\int_{\Omega^T} \partial_t S\, \zeta + \int_{Q^T} \partial_t s\, \eta = -\int_{\Omega^T} (S - S_0)\partial_t\zeta - \int_{Q^T} (s - s_0)\partial_t\eta. \tag{2.6}$$

Theorem 2.1 *A weak solution of the equations* (1.1–1.12) *exists if the following conditions hold:*

A1. $\partial_1\Omega \neq \emptyset$,

A2. Λ_w, λ_w *(*resp. Λ_o, λ_o*)* $: [0,1] \to [0,1]$ *are continuous and increasing (*resp. *decreasing),* $\Lambda_w(0) = \Lambda_o(1) = \lambda_w(s_l) = \lambda_o(s_r) = 0,\ \Lambda_w\Lambda_o(z)|_{z\in(0,1)} \neq 0,$ $\lambda_w\lambda_o(z)|_{z\in(s_l,s_r)} \neq 0,\ \inf_{z\in(0,1]}\{\Lambda(z), \lambda(z)\} > 0$,

A3. $\Upsilon : (0,1] \to \Re_0^+$ $(\upsilon : (s_l, s_r] \to \Re_0^+)$ *is onto, decreasing, and a locally Lipschitz continuous function, and* $\inf_{z\in(0,1]} \left|\frac{d\Upsilon}{dS}\right| \times \inf_{z\in(s_l,s_r]} \left|\frac{d\upsilon}{ds}\right| > 0$,

A4. $\partial_t P_{c,b} \in L^1(\Omega^T), E_\alpha \in L^\infty(0,T;W^{1,\infty}(\Omega)), P_{\alpha,b} \in L^2(0,T;H^1(\Omega)), \alpha = w, o$,

A5. $\mathbf{k}_1 \leq \Upsilon^{-1}(P_{c,b}) \leq 1 - \mathbf{k}_1,\ \mathbf{k}_1 \leq S_0(x) \leq 1 - \mathbf{k}_1,\ \upsilon(s_0(x)) = \Upsilon(S_0(x)), x \in \Omega$, *A6.* $\min\{\overline{\lim\limits_{\xi\to 0}}\frac{\Lambda_w(\xi)}{\lambda_w(\mathcal{J}(\xi))}, \overline{\lim\limits_{\xi\to 0}}\frac{\lambda_w(\mathcal{J}(\xi))}{\Lambda_w(\xi)}\} < \infty$, $\min\{\overline{\lim\limits_{\xi\to 1}}\frac{\Lambda_o(\xi)}{\lambda_o(\mathcal{J}(\xi))}, \overline{\lim\limits_{\xi\to 1}}\frac{\lambda_o(\mathcal{J}(\xi))}{\Lambda_o(\xi)}\} < \infty$,

A7. One of the following conditions is satisfied:

$$a)\ \begin{cases} \overline{\lim\limits_{z\to 0}}\frac{\Lambda_w(z)}{|z|^2\left|\frac{d\Upsilon}{dS}(z)\right|} + \overline{\lim\limits_{z\to 1}}\frac{\Lambda_o(z)}{|1-z|^2\left|\frac{d\Upsilon}{dS}(z)\right|} + \sup\limits_{z\in(0,1]}\frac{\frac{dv}{ds}(\mathcal{J}(z))}{\frac{d\Upsilon}{dS}(z)} < \infty, \\ \inf\limits_{z\in(0,1]}\frac{\Lambda_w(z)\Lambda_o(z)}{|z(1-z)|^{\mathbf{k}_2}} \times \inf\limits_{z\in(0,1]}\frac{\lambda_w(\mathcal{J}(z))\lambda_o(\mathcal{J}(z))}{|z(1-z)|^{\mathbf{k}_2}} > 0, \\ \underline{\lim\limits_{z\to 0}}\frac{1}{|z|^{\mathbf{k}_2}|\mathcal{D}(\mathcal{J}(z))|} > 0, \end{cases}$$

$$b)\ \begin{cases} \overline{\lim\limits_{z\to s_l}}\frac{\Lambda_w(\mathcal{J}^{-1}(z))}{|z-s_l|^2\left|\frac{dv}{ds}(z)\right|} + \overline{\lim\limits_{z\to s_r}}\frac{\Lambda_o(\mathcal{J}^{-1}(z))}{|s_r-z|^2\left|\frac{dv}{ds}(z)\right|} + \sup\limits_{z\in(s_l,s_r]}\frac{\frac{d\Upsilon}{dS}(\mathcal{J}^{-1}(z))}{\frac{dv}{ds}(z)} < \infty, \\ \inf\limits_{z\in(s_l,s_r]}\frac{\Lambda_w(\mathcal{J}^{-1}(z))\Lambda_o(\mathcal{J}^{-1}(z))}{|(z-s_l)(s_r-z)|^{\mathbf{k}_2}} \times \inf\limits_{z\in(s_l,s_r]}\frac{\lambda_w(z)\lambda_o(z)}{|(z-s_l)(s_r-z)|^{\mathbf{k}_2}} > 0, \\ \underline{\lim\limits_{z\to s_l}}\frac{1}{|z-s_l|^{\mathbf{k}_2}|\mathcal{D}(z)|} > 0, \end{cases}$$

$$c)\ \begin{cases} \overline{\lim\limits_{z\to 0}}\frac{\Lambda_w(z)}{|z|^2\left|\frac{d\Upsilon}{dS}(z)\right|} + \overline{\lim\limits_{z\to 0}}\frac{\frac{dv}{ds}(\mathcal{J}(z))}{\frac{d\Upsilon}{dS}(z)} < \infty, \\ \overline{\lim\limits_{z\to s_r}}\frac{\Lambda_o(\mathcal{J}^{-1}(z))}{|s_r-z|^2\left|\frac{dv}{ds}(z)\right|} + \overline{\lim\limits_{z\to s_r}}\frac{\frac{d\Upsilon}{dS}(\mathcal{J}^{-1}(z))}{\frac{dv}{ds}(z)} < \infty, \\ \underline{\lim\limits_{z\to 0}}\frac{\Lambda_w(z)}{|z|^{\mathbf{k}_2}} \times \underline{\lim\limits_{z\to 0}}\frac{\lambda_w(\mathcal{J}(z))}{|z|^{\mathbf{k}_2}} > 0, \\ \underline{\lim\limits_{z\to s_r}}\frac{\Lambda_o(\mathcal{J}^{-1}(z))}{|s_r-z|^{\mathbf{k}_2}} \times \underline{\lim\limits_{z\to s_r}}\frac{\lambda_o(z)}{|s_r-z|^{\mathbf{k}_2}} > 0, \\ \underline{\lim\limits_{z\to 0}}\frac{1}{|z|^{\mathbf{k}_2}|\mathcal{D}(\mathcal{J}(z))|} > 0, \end{cases}$$

$$d)\ \begin{cases} \overline{\lim\limits_{z\to s_l}}\frac{\Lambda_w(\mathcal{J}^{-1}(z))}{|z-s_l|^2\left|\frac{dv}{ds}(z)\right|} + \overline{\lim\limits_{z\to s_l}}\frac{\frac{d\Upsilon}{dS}(\mathcal{J}^{-1}(z))}{\frac{dv}{ds}(z)} < \infty, \\ \overline{\lim\limits_{z\to 1}}\frac{\Lambda_o(z)}{|1-z|^2\left|\frac{d\Upsilon}{dS}(z)\right|} + \overline{\lim\limits_{z\to 1}}\frac{\frac{dv}{ds}(\mathcal{J}(z))}{\frac{d\Upsilon}{dS}(z)} < \infty, \\ \underline{\lim\limits_{z\to s_l}}\frac{\Lambda_w(\mathcal{J}^{-1}(z))}{|z-s_l|^{\mathbf{k}_2}} \times \underline{\lim\limits_{z\to s_l}}\frac{\lambda_w(z)}{|z-s_l|^{\mathbf{k}_2}} > 0, \\ \underline{\lim\limits_{z\to 1}}\frac{\Lambda_o(z)}{|1-z|^{\mathbf{k}_2}} \times \underline{\lim\limits_{z\to 1}}\frac{\lambda_o(\mathcal{J}(z))}{|1-z|^{\mathbf{k}_2}} > 0, \\ \underline{\lim\limits_{z\to s_l}}\frac{1}{|z-s_l|^{\mathbf{k}_2}|\mathcal{D}(z)|} > 0, \end{cases}$$

where $\mathbf{k}_1, \mathbf{k}_2$ *are positive constants. See* (2.1) *for* $\Lambda, \lambda, P_{c,b}, \mathcal{D}$.

Remark 2.1 *1) By A2-3,* $\mathcal{D}$ *is a strictly increasing function on* $(s_l, s_r]$, *so it has a bounded and strictly increasing inverse function* $\mathcal{D}^{-1}$. *Let us extend* $\mathcal{D}^{-1}$ *to* $\Re$ *continuously and linearly with slope* 1. *The new function will not be relabelled.*

2) A7 sets restrictions on phase mobilities and capillary pressures around end-points only. Roughly speaking, A7.a) corresponds to that fracture capillary pressure decreases faster than matrix capillary pressure around end-points. A7.b) is the inverse case of A7.a). By proper combinations of the restrictions in A7.a) and A7.b), we obtain A7.c) and A7.d). A7.c) is the case that fracture capillary pressure drops faster around 0 (resp. *slower around* 1*) than matrix capillary pressure around* s_l (resp. *around* s_r*). A7.d) is the inverse case of A7.c).*

3) If $\frac{\lambda_w\lambda_o}{\lambda}\left|\frac{dv}{ds}\right| \in L^1(s_l, s_r]$ *(assumption in* [7, 17]*),* $\mathcal{D}$ *is bounded. If* $\mathcal{D}$ *is a bounded function on* $(s_l, s_r]$, *then* $A7.a)_3$, $A7.b)_3$, $A7.c)_5$, *and* $A7.d)_5$ *hold obvious.*

4) If $\overline{\lim\limits_{z\to 0}}|z|^{\mathbf{k}_2}|\Upsilon(z)| < \infty$, $A7.a)_3$ *and* $A7.c)_5$ *hold. If* $\overline{\lim\limits_{z\to s_l}}|z-s_l|^{\mathbf{k}_2}|v(z)| < \infty$, *A7.b)$_3$ and A7.d)$_5$ hold. So, if $\mathcal{D}(\mathcal{J}(z))$* (resp. *$\mathcal{D}(z)$) grows slower than $\frac{1}{|z|^{\mathbf{k}_2}}$* (resp. *$\frac{1}{|z-s_l|^{\mathbf{k}_2}}$) as z approaches* 0 (resp. *as z approaches s_l), then A7.a)$_3$ and A7.c)$_5$* (resp. *A7.b)$_3$ and A7.d)$_5$) hold.*

3. Procedure of Proof

Now we derive an auxiliary system for (1.1–1.12), and describe procedure of proof for **Theorem 2.1**. Global pressure [8] is defined as

$$P \stackrel{\text{def}}{=} \frac{1}{2}\Big(P_o + P_w + \int_0^{\Upsilon(S)} \Big(\frac{\Lambda_o}{\Lambda}(\Upsilon^{-1}(\xi)) - \frac{\Lambda_w}{\Lambda}(\Upsilon^{-1}(\xi))\Big) d\xi\Big). \tag{3.1}$$

See (2.1) for Λ. Then $\nabla_x P = \frac{\Lambda_w}{\Lambda}\nabla_x P_w + \frac{\Lambda_o}{\Lambda}\nabla_x P_o$. Let $\zeta_w = \zeta_o = \zeta$ in (2.2–2.3), and add the two equations to obtain

$$\int_{\Omega^T} \Lambda(S)\nabla_x P \nabla_x \zeta - \sum_{\alpha\in\{w,o\}} \int_{\Omega^T} \Lambda_\alpha(S)\nabla_x E_\alpha \nabla_x \zeta = 0. \tag{3.2}$$

If we define

$$\mathcal{G} \stackrel{\text{def}}{=} \mathcal{J}(S), \tag{3.3}$$

(2.2) can be written as

$$\int_{\Omega^T} \partial_t S\ \zeta_w + \int_{\Omega^T} \left(\Lambda_w(S)\nabla_x(P - E_w) - \frac{\Lambda_w(S)\Lambda_o(S)\frac{dv}{ds}(\mathcal{G})}{\Lambda(S)}\nabla_x \mathcal{G}\right)\nabla_x \zeta_w$$
$$= -\int_{Q^T} \partial_t s\ \zeta_w. \tag{3.4}$$

If we repeat the process (3.1–3.4) in each matrix block, (2.4) can be written as

$$\int_{Q^T} \partial_t s\ \eta_w - \int_{Q^T} \frac{\lambda_w(s)\lambda_o(s)\frac{dv}{ds}(s)}{\lambda(s)}\nabla_y s \nabla_y \eta_w = 0. \tag{3.5}$$

Let ε be a small number satisfying

$$0 < \varepsilon < \mathbf{k}_1/4, \tag{3.6}$$

where $\mathbf{k}_1$ is the one in *A5*. Let us extend mobility functions $\Lambda_\alpha, \lambda_\alpha$ ($\alpha = w, o$) constantly and continuously to $\Re$, and find continuous monotone functions $\Lambda_\alpha^\varepsilon, \lambda_\alpha^\varepsilon$ in $\Re$ such that

$$\begin{cases} \varepsilon \le \inf_{z\in\Re}\{\Lambda_\alpha^\varepsilon(z), \lambda_\alpha^\varepsilon(z)\} \le \sup_{z\in\Re}\{\Lambda_\alpha^\varepsilon(z), \lambda_\alpha^\varepsilon(z)\} \le 1, \\ \Lambda_\alpha^\varepsilon(z) = \Lambda_\alpha(z) \text{ and } \lambda_\alpha^\varepsilon(\mathcal{J}(z)) = \lambda_\alpha(\mathcal{J}(z)) \text{ for } z \in [\varepsilon, 1-\varepsilon]. \end{cases} \tag{3.7}$$

Next we define, for $z \in \Re$,

$$\begin{cases} \Lambda^\varepsilon(z) \stackrel{\text{def}}{=} \Lambda_w^\varepsilon(z) + \Lambda_o^\varepsilon(z), \\ \lambda^\varepsilon(z) \stackrel{\text{def}}{=} \lambda_w^\varepsilon(z) + \lambda_o^\varepsilon(z), \\ \widetilde{\Lambda}_\alpha^\varepsilon(z) \stackrel{\text{def}}{=} \Lambda_\alpha(0.5(\frac{z-\varepsilon}{0.5-\varepsilon})), \quad \alpha \in \{w, o\}, \\ \widetilde{\Lambda}^\varepsilon(z) \stackrel{\text{def}}{=} \widetilde{\Lambda}_w^\varepsilon(z) + \widetilde{\Lambda}_o^\varepsilon(z). \end{cases} \tag{3.8}$$

By *A3*, one may find decreasing and Lipschitz functions $\Upsilon^\varepsilon, \upsilon^\varepsilon$ in $\Re$ so that

$$\begin{cases} 0 < \mathbf{k}_3 \le \inf_{z\in\Re} \{|\frac{d\Upsilon^\varepsilon}{dS}|(z), |\frac{d\upsilon^\varepsilon}{ds}|(z)\} \le \sup_{z\in\Re} \{|\frac{d\Upsilon^\varepsilon}{dS}|(z), |\frac{d\upsilon^\varepsilon}{ds}|(z)\} < \infty, \\ \Upsilon^\varepsilon(z) = \Upsilon(z) \text{ and } \upsilon^\varepsilon(\mathcal{J}(z)) = \upsilon(\mathcal{J}(z)) \text{ for } z \in [\varepsilon, 1-\varepsilon], \\ \Upsilon^\varepsilon \text{ (resp. } \upsilon^\varepsilon) \text{ has inverse function } \Upsilon^{\varepsilon,-1} \text{ (resp. } \upsilon^{\varepsilon,-1}) \text{ in } \Re, \\ \mathcal{J}^\varepsilon \left(\stackrel{\text{def}}{=} \upsilon^{\varepsilon,-1}(\Upsilon^\varepsilon)\right) \text{ is linear in } \Re \setminus [\varepsilon, 1-\varepsilon] \text{ and has inverse } \mathcal{J}^{\varepsilon,-1}, \end{cases} \tag{3.9}$$

where $\mathbf{k}_3$ is a constant independent of ε. By *A4-5*, there exist smooth functions $S_0^\varepsilon, s_0^\varepsilon, P_{c,b}^\varepsilon, P_{\alpha,b}^\varepsilon$ $(\alpha = w, o)$ such that

$$\begin{cases} P_{c,b}^\varepsilon = P_{o,b}^\varepsilon - P_{w,b}^\varepsilon, \\ 0 < \frac{\mathbf{k}_1}{2} \le \inf_{(x,t)\in\Omega^T} \{S_0^\varepsilon, \Upsilon^{-1}(P_{c,b}^\varepsilon)\} \le \sup_{(x,t)\in\Omega^T} \{S_0^\varepsilon, \Upsilon^{-1}(P_{c,b}^\varepsilon)\} \le 1 - \frac{\mathbf{k}_1}{2}, \\ s_0^\varepsilon = \mathcal{J}(S_0^\varepsilon), \\ s_0^\varepsilon - \upsilon^{-1}(P_{c,b}^\varepsilon)(x,0) \in \mathcal{V}, \end{cases} \tag{3.10}$$

and, as $\varepsilon \to 0$,

$$\begin{cases} S_0^\varepsilon \to S_0, & \text{in } L^2(\Omega), \\ P_{\alpha,b}^\varepsilon \to P_{\alpha,b}, & \text{in } L^2(0,T;H^1(\Omega)), \\ \partial_t P_{c,b}^\varepsilon \to \partial_t P_{c,b}, & \text{in } L^1(\Omega^T). \end{cases} \tag{3.11}$$

Auxiliary initial and boundary conditions are defined as

$$\begin{cases} \mathcal{G}_0^\varepsilon \stackrel{\text{def}}{=} \mathcal{J}(S_0^\varepsilon), \\ \mathcal{G}_b^\varepsilon \stackrel{\text{def}}{=} \upsilon^{-1}(P_{c,b}^\varepsilon), \\ P_b^\varepsilon \stackrel{\text{def}}{=} \frac{1}{2}\left(P_{o,b}^\varepsilon + P_{w,b}^\varepsilon + \int_0^{P_{c,b}^\varepsilon} \left(\frac{\Lambda_o^\varepsilon}{\Lambda^\varepsilon}(\Upsilon^{\varepsilon,-1}(\xi)) - \frac{\Lambda_w^\varepsilon}{\Lambda^\varepsilon}(\Upsilon^{\varepsilon,-1}(\xi))\right) d\xi\right). \end{cases} \tag{3.12}$$

Auxiliary system of (1.1–1.12) for each ε is to find $\{S^\varepsilon, \mathcal{G}^\varepsilon, P^\varepsilon, s^\varepsilon\}$ so that

$$\partial_t S^\varepsilon + \int_{\mathcal{M}} \partial_t s^\varepsilon dy \in \textit{dual } L^2(0,T;\mathcal{V}), \quad \partial_t s^\varepsilon \in \textit{dual } L^2(0,T;\mathcal{U}_0), \tag{3.13}$$

$$\varepsilon \le S^\varepsilon \le 1-\varepsilon, \quad \mathcal{J}(\varepsilon) \le s^\varepsilon \le \mathcal{J}(1-\varepsilon), \tag{3.14}$$

$$\mathcal{G}^\varepsilon = \mathcal{J}(S^\varepsilon), \quad (\mathcal{G}^\varepsilon - \mathcal{G}_b^\varepsilon,\ P^\varepsilon - P_b^\varepsilon,\ s^\varepsilon - \mathcal{G}^\varepsilon) \in L^2(0,T;\mathcal{W}_1), \tag{3.15}$$

$$\int_{\Omega^T} \partial_t S^\varepsilon \zeta_1 + \int_{\Omega^T} \left(\widetilde{\Lambda}_w^\varepsilon(S^\varepsilon)\nabla_x(P^\varepsilon - E_w) - \frac{\Lambda_w\Lambda_o}{\Lambda}(S^\varepsilon)\nabla_x\Upsilon(S^\varepsilon)\right)\nabla_x\zeta_1 + \int_{Q^T} \partial_t s^\varepsilon \zeta_1 = 0, \tag{3.16}$$

$$\int_{\Omega^T} \widetilde{\Lambda}^\varepsilon(S^\varepsilon)\nabla_x P^\varepsilon \nabla_x \zeta_2 - \sum_{\alpha\in\{w,o\}} \int_{\Omega^T} \widetilde{\Lambda}_\alpha^\varepsilon(S^\varepsilon)\nabla_x E_\alpha \nabla_x \zeta_2 = 0, \tag{3.17}$$

$$\int_{Q^T} \partial_t s^\varepsilon \eta - \int_{Q^T} \frac{\lambda_w\lambda_o}{\lambda}(s^\varepsilon)\nabla_y \upsilon(s^\varepsilon)\nabla_y \eta = 0, \tag{3.18}$$

$$\mathcal{G}^\varepsilon(x,0) = \mathcal{G}_0^\varepsilon, \quad s^\varepsilon(x,y,0) = s_0^\varepsilon, \tag{3.19}$$

for any $(\zeta_1, \zeta_2, \eta) \in L^2(0,T;\mathcal{W}_1)$. See (3.8) for $\widetilde{\Lambda}^\varepsilon, \widetilde{\Lambda}_\alpha^\varepsilon$ $(\alpha = w, o)$. Later the following result will be proved:

Theorem 3.1 *Under A1-5, for each* ε, *there is* $\{S^\varepsilon, \mathcal{G}^\varepsilon, P^\varepsilon, s^\varepsilon, P^\varepsilon_\alpha, p^\varepsilon_\alpha (\alpha = w, o)\}$ *such that* (3.13–3.19) *hold. Moreover,*

$$(P^\varepsilon_w - P^\varepsilon_{w,b},\ P^\varepsilon_o - P^\varepsilon_{o,b},\ p^\varepsilon_w - P^\varepsilon_w,\ p^\varepsilon_o - P^\varepsilon_o) \in L^2(0,T;\mathcal{W}_2), \tag{3.20}$$

$$\Upsilon(S^\varepsilon) = P^\varepsilon_o - P^\varepsilon_w, \quad \upsilon(s^\varepsilon) = p^\varepsilon_o - p^\varepsilon_w, \tag{3.21}$$

$$\int_{\Omega^T} \partial_t S^\varepsilon \zeta_w + \int_{\Omega^T} (\Lambda_w(S^\varepsilon)\nabla_x P^\varepsilon_w - \widetilde{\Lambda}^\varepsilon_w(S^\varepsilon)\nabla_x E_w + (\widetilde{\Lambda}^\varepsilon_w - \Lambda_w)\nabla_x P^\varepsilon)\nabla_x \zeta_w + \int_{Q^T} \partial_t s^\varepsilon \zeta_w = 0, \tag{3.22}$$

$$-\int_{\Omega^T} \partial_t S^\varepsilon \zeta_o + \int_{\Omega^T} (\Lambda_o(S^\varepsilon)\nabla_x P^\varepsilon_o - \widetilde{\Lambda}^\varepsilon_o(S^\varepsilon)\nabla_x E_o + (\widetilde{\Lambda}^\varepsilon_o - \Lambda_o)\nabla_x P^\varepsilon)\nabla_x \zeta_o - \int_{Q^T} \partial_t s^\varepsilon \zeta_o = 0, \tag{3.23}$$

$$\int_{Q^T} \partial_t s^\varepsilon \eta_w + \int_{Q^T} \lambda_w(s^\varepsilon)\nabla_y p^\varepsilon_w \nabla_y \eta_w = 0, \tag{3.24}$$

$$-\int_{Q^T} \partial_t s^\varepsilon \eta_o + \int_{Q^T} \lambda_o(s^\varepsilon)\nabla_y p^\varepsilon_o \nabla_y \eta_o = 0, \tag{3.25}$$

for all $\zeta_\alpha \in L^2(0,T;\mathcal{V}), \eta_\alpha \in L^2(0,T;\mathcal{U}_0)$.

Similar result as **Theorem 3.1** had been considered in [17]. For completion, proof of this theorem will be given in §5. In next section one will see that a subsequence of the solutions of **Theorem 3.1** converges weakly to a solution of (1.1–1.12) as ε approaches 0, which implies **Theorem 2.1**.

4. Existence of A Weak Solution

Objective of this section is to prove **Theorem 2.1 if Theorem 3.1** holds. It is done as follows: First we show $P^\varepsilon_\alpha, p^\varepsilon_\alpha$ for $\alpha \in \{w, o\}$ (solutions of **Theorem 3.1**) are bounded independently of ε (see **Lemmas 4.1, 4.2, 4.3**), next prove $\{S^\varepsilon\}$ has a convergent subsequence in $L^2(\Omega^T)$ (see **Lemmas 4.4, 4.5, 4.6**), then show $\{s^\varepsilon\}$ has a convergent subsequence in $L^2(Q^T)$ (see **Lemmas 4.7, 4.8, 4.9**), and finally conclude the existence of a weak solution of (1.1–1.12). We define

$$\begin{cases} \Theta(z) \overset{\text{def}}{=} \int_0^z (\Upsilon^{-1}(-z) - \Upsilon^{-1}(-\xi))d\xi, & \text{for } z \in (-\infty, 0], \\ \Psi^\varepsilon \overset{\text{def}}{=} -\Upsilon(S^\varepsilon), \\ \theta(z) \overset{\text{def}}{=} \int_0^z (\upsilon^{-1}(-z) - \upsilon^{-1}(-\xi))d\xi, & \text{for } z \in (-\infty, 0], \\ \psi^\varepsilon \overset{\text{def}}{=} -\upsilon(s^\varepsilon), \\ \rho^\varepsilon \overset{\text{def}}{=} \mathcal{J}^{-1}(s^\varepsilon), \\ \Psi^\varepsilon_b \overset{\text{def}}{=} -P^\varepsilon_{c,b}. \end{cases} \tag{4.1}$$

$(4.1)_{2,4,5}$ are well-defined by (3.14). (3.15) implies $\rho^\varepsilon|_{\partial\mathcal{M}} = S^\varepsilon$ in Ω^T. $\Theta(z)$ and $\theta(z)$ are nonnegative functions on $(-\infty, 0]$, and, for any $z_1, z_2 \le 0$,

$$\begin{cases} \Theta(z_1) - \Theta(z_2) \le (\Upsilon^{-1}(-z_1) - \Upsilon^{-1}(-z_2))z_1, \\ \theta(z_1) - \theta(z_2) \le (\upsilon^{-1}(-z_1) - \upsilon^{-1}(-z_2))z_1. \end{cases} \tag{4.2}$$

$\mathcal{X}_{\mathcal{B}}$, $\mathcal{B} \subset Q^T$, is a characteristic function defined as

$$\mathcal{X}_{\mathcal{B}}(z) = \begin{cases} 1, & \text{for } z \in \mathcal{B}, \\ 0, & \text{otherwise.} \end{cases} \tag{4.3}$$

Let us find two nonnegative smooth functions $\mathbf{g}_1$ and $\mathbf{g}_2$ defined on $[0,1]$ such that $\mathbf{g}_1$ (resp. $\mathbf{g}_2$) is decreasing (resp. increasing), $\mathbf{g}_1(0) = \mathbf{g}_2(1) = 1, \mathbf{g}_1(0.6) = \mathbf{g}_2(0.4) = 0$, and $\mathbf{g}_1 + \mathbf{g}_2 > 0$ in $[0,1]$. Let $\widetilde{\mathbf{g}}_1, \widetilde{\mathbf{g}}_2 : (s_l, s_r] \to \Re$ by $\widetilde{\mathbf{g}}_1(\xi) \stackrel{\text{def}}{=} \mathbf{g}_1(\mathcal{J}^{-1}(\xi))$, $\widetilde{\mathbf{g}}_2(\xi) \stackrel{\text{def}}{=} \mathbf{g}_2(\mathcal{J}^{-1}(\xi))$. By *A6*, we define $\mathcal{E} : (0,1] \to \Re$ by

$$\mathcal{E}(z) = \begin{cases} \int_{0.5}^{z} \sqrt{\Lambda_w \Lambda_o} \left|\frac{d\Upsilon}{dS}\right|, & \text{if} \begin{cases} \overline{\lim\limits_{\xi\to 0}} \frac{\Lambda_w(\xi)}{\lambda_w(\mathcal{J}(\xi))} < \infty, \\ \overline{\lim\limits_{\xi\to 1}} \frac{\Lambda_o(\xi)}{\lambda_o(\mathcal{J}(\xi))} < \infty, \end{cases} \\ \int_{\mathcal{J}(0.5)}^{\mathcal{J}(z)} \sqrt{\lambda_w \lambda_o} \left|\frac{dv}{ds}\right|, & \text{if} \begin{cases} \overline{\lim\limits_{\xi\to 0}} \frac{\lambda_w(\mathcal{J}(\xi))}{\Lambda_w(\xi)} < \infty, \\ \overline{\lim\limits_{\xi\to 1}} \frac{\lambda_o(\mathcal{J}(\xi))}{\Lambda_o(\xi)} < \infty, \end{cases} \\ \int_{0.5}^{z} \sqrt{\Lambda_w \Lambda_o} \left|\frac{d\Upsilon}{dS}\right| \mathbf{g}_1 + \int_{\mathcal{J}(0.5)}^{\mathcal{J}(z)} \sqrt{\lambda_w \lambda_o} \left|\frac{dv}{ds}\right| \widetilde{\mathbf{g}}_2, & \text{if} \begin{cases} \overline{\lim\limits_{\xi\to 0}} \frac{\Lambda_w(\xi)}{\lambda_w(\mathcal{J}(\xi))} < \infty, \\ \overline{\lim\limits_{\xi\to 1}} \frac{\lambda_o(\mathcal{J}(\xi))}{\Lambda_o(\xi)} < \infty, \end{cases} \\ \int_{0.5}^{z} \sqrt{\Lambda_w \Lambda_o} \left|\frac{d\Upsilon}{dS}\right| \mathbf{g}_2 + \int_{\mathcal{J}(0.5)}^{\mathcal{J}(z)} \sqrt{\lambda_w \lambda_o} \left|\frac{dv}{ds}\right| \widetilde{\mathbf{g}}_1, & \text{if} \begin{cases} \overline{\lim\limits_{\xi\to 0}} \frac{\lambda_w(\mathcal{J}(\xi))}{\Lambda_w(\xi)} < \infty, \\ \overline{\lim\limits_{\xi\to 1}} \frac{\Lambda_o(\xi)}{\lambda_o(\mathcal{J}(\xi))} < \infty. \end{cases} \end{cases} \tag{4.4}$$

$\mathcal{E}$ in (4.4) may have more than two options. If so, one selects the foremost possible one in (4.4) so that $\mathcal{E}$ is well-defined. $\mathcal{E}$ is a strictly increasing function, so it has a bounded and strictly increasing inverse function $\mathcal{E}^{-1}$. We extend $\mathcal{E}^{-1}$ to $\Re$ so that it is bounded, continuous, and strictly increasing in $\Re$. Let us define

$$\begin{cases} \Phi^\varepsilon \stackrel{\text{def}}{=} \mathcal{E}(S^\varepsilon), \\ \phi^\varepsilon \stackrel{\text{def}}{=} \mathcal{E}(\rho^\varepsilon). \end{cases}$$

Lemma 4.1 *Solutions of* **Theorem 3.1** *satisfy*

$$\sum_{\alpha\in\{w,o\}} \left(\|\sqrt{\Lambda_\alpha(S^\varepsilon)}\nabla_x P_\alpha^\varepsilon\|_{L^2(\Omega^T)} + \|\sqrt{\lambda_\alpha(s^\varepsilon)}\nabla_y p_\alpha^\varepsilon\|_{L^2(Q^T)}\right)$$
$$+\|P^\varepsilon\|_{L^2(0,T;H^1(\Omega))} \le c, \tag{4.5}$$
$$\|\mathbf{R}(S^\varepsilon)\|_{L^2(0,T;H^1(\Omega))} + \|\nabla_y \mathcal{D}(s^\varepsilon)\|_{L^2(Q^T)} + \|\Phi^\varepsilon\|_{L^2(0,T;H^1(\Omega))}$$
$$+\|\phi^\varepsilon\|_{L^2(0,T;\mathcal{U})} \le c, \tag{4.6}$$

where c is a constant independent of ε. See (2.1) *for* $\mathbf{R}, \mathcal{D}$.

Proof: Set $\zeta_2 = P^\varepsilon - P_b^\varepsilon$ in (3.17) to obtain, by *A4* and $(3.12)_3$,

$$\|P^\varepsilon\|_{L^2(0,T;H^1(\Omega))} \le c \text{ (indep. of } \varepsilon). \tag{4.7}$$

By $(4.2)_1$, for all $t, \varpi > 0$,

$$\Theta(\Psi^\varepsilon(t)) - \Theta(\Psi^\varepsilon(t-\varpi)) \le (S^\varepsilon(t) - S^\varepsilon(t-\varpi))\Psi^\varepsilon(t), \tag{4.8}$$

where $\Psi^\varepsilon(t) = \Psi^\varepsilon(0)$ for $-\varpi < t < 0$. Integrate (4.8) over Ω^τ to obtain

$$\frac{1}{\varpi}\int_{\tau-\varpi}^{\tau}\int_\Omega \Theta(\Psi^\varepsilon) \le \int_{\Omega^\tau}(\Psi^\varepsilon - \Psi_b^\varepsilon)\partial^{-\varpi}S^\varepsilon + \int_\Omega \Theta(\Psi^\varepsilon(0)) - \int_0^{\tau-\varpi}\int_\Omega (S^\varepsilon - S^\varepsilon(0))\partial^{\varpi}\Psi_b^\varepsilon + \frac{1}{\varpi}\int_{\tau-\varpi}^{\tau}\int_\Omega (S^\varepsilon - S^\varepsilon(0))\Psi_b^\varepsilon. \tag{4.9}$$

See $(2.1)_1$ for time differentiation. Similarly, by $(4.2)_2$, one obtains

$$\frac{1}{\varpi}\int_{\tau-\varpi}^{\tau}\int_Q \theta(\psi^\varepsilon) \le \int_{Q^\tau}(\psi^\varepsilon - \Psi_b^\varepsilon)\partial^{-\varpi}s^\varepsilon + \int_Q \theta(\psi^\varepsilon(0)) - \int_0^{\tau-\varpi}\int_Q (s^\varepsilon - s^\varepsilon(0))\partial^{\varpi}\Psi_b^\varepsilon + \frac{1}{\varpi}\int_{\tau-\varpi}^{\tau}\int_Q (s^\varepsilon - s^\varepsilon(0))\Psi_b^\varepsilon. \tag{4.10}$$

Summing (4.9) and (4.10) as well as letting $\varpi \to 0$, by boundedness of S^ε and s^ε, we get, for almost all $\tau \in (0, T]$,

$$\int_\Omega \Theta(\Psi^\varepsilon)(\tau) + \int_Q \theta(\psi^\varepsilon)(\tau) \le \int_{\Omega^\tau}(\Psi^\varepsilon - \Psi_b^\varepsilon)\partial_t S^\varepsilon + \int_{Q^\tau}(\psi^\varepsilon - \Psi_b^\varepsilon)\partial_t s^\varepsilon + c(\|\Psi^\varepsilon(0)\|_{L^1(\Omega)}, \|\Psi_b^\varepsilon\|_{L^\infty(0,T;L^1(\Omega))}, \|\partial_t\Psi_b^\varepsilon\|_{L^1(\Omega^T)}). \tag{4.11}$$

Letting $\zeta_\alpha = P_\alpha^\varepsilon - P_{\alpha,b}^\varepsilon, \eta_\alpha = p_\alpha^\varepsilon - P_\alpha^\varepsilon$ for $\alpha \in \{w, o\}$ in (3.22–3.25), one obtains

$$\int_{\Omega^\tau}(\Psi^\varepsilon - \Psi_b^\varepsilon)\partial_t S^\varepsilon + \sum_{\alpha\in\{w,o\}}\int_{\Omega^\tau}\Lambda_\alpha(S^\varepsilon)|\nabla_x P_\alpha^\varepsilon|^2 + \int_{Q^\tau}(\psi^\varepsilon - \Psi_b^\varepsilon)\partial_t s^\varepsilon + \sum_{\alpha\in\{w,o\}}\int_{Q^\tau}\lambda_\alpha(s^\varepsilon)|\nabla_y p_\alpha^\varepsilon|^2 \le c(\|\nabla_x P^\varepsilon, \nabla_x E_\alpha, \nabla_x P_{\alpha,b}^\varepsilon\|_{L^2(\Omega^T)}). \tag{4.12}$$

By (3.11), (4.7), (4.11–4.12), and *A4-5*, we obtain (4.5). Clearly (4.5) implies

$$\int_{\Omega^T}\Lambda_o\Lambda_w(S^\varepsilon)|\nabla_x\Upsilon(S^\varepsilon)|^2 + \int_{Q^T}\lambda_o\lambda_w(s^\varepsilon)|\nabla_y\upsilon(s^\varepsilon)|^2 \le c_0, \tag{4.13}$$

where c_0 is independent of ε. (4.6) is due to *A5*, (4.4), and (4.13). ■

Lemma 4.2 *Suppose* $2 \le \varpi_0 \in \mathbf{N}$ *and* $\frac{2\mathbf{k}_1}{2^{\varpi_0}} \le \min\{\mathcal{J}(\frac{\mathbf{k}_1}{2}) - s_l, s_r - \mathcal{J}(1 - \frac{\mathbf{k}_1}{2})\}$ *where* $\mathbf{k}_1$ *is the one in A5. For any* $\tau(\le T)$, $\varpi(\ge 2+\varpi_0) \in \mathbf{N}$, *and* $\varepsilon\left(< \mu \overset{\text{def}}{=} \frac{\mathbf{k}_1}{2^{\varpi}}\right)$, *solutions of* **Theorem 3.1** *satisfy the following results: If A7.a) holds, then*

$$\sup_{t\le\tau}\left(|\{x\in\Omega : S^\varepsilon(t) \le \mu\}| + |\{(x,y)\in Q : \rho^\varepsilon(t) \le \mu\}|\right) + \sup_{t\le\tau}\left(|\{x\in\Omega : 1-\mu \le S^\varepsilon(t)\}| + |\{(x,y)\in Q : 1-\mu \le \rho^\varepsilon(t)\}|\right) \le \frac{c_0|c_0\tau|^{\varpi-\varpi_0}}{(\varpi-\varpi_0)^{(\varpi-\varpi_0)\mathfrak{k}_w}}, \tag{4.14}$$

if A7.b) holds, then

$$\sup_{t\le\tau}\left(|\{x\in\Omega:\mathcal{G}^\epsilon(t)\le\mu+s_l\}|+|\{(x,y)\in\mathcal{Q}:s^\epsilon(t)\le\mu+s_l\}|\right)$$
$$+\sup_{t\le\tau}\left(|\{x\in\Omega:s_r-\mu\le\mathcal{G}^\epsilon(t)\}|+|\{(x,y)\in\mathcal{Q}:s_r-\mu\le s^\epsilon(t)\}|\right)$$
$$\le\frac{c_0|c_0\tau|^{\varpi-\varpi_0}}{(\varpi-\varpi_0)^{(\varpi-\varpi_0)\mathbf{f}_\varpi}},\tag{4.15}$$

if A7.c) holds, then

$$\sup_{t\le\tau}\left(|\{x\in\Omega:S^\epsilon(t)\le\mu\}|+|\{(x,y)\in\mathcal{Q}:\rho^\epsilon(t)\le\mu\}|\right)$$
$$+\sup_{t\le\tau}\left(|\{x\in\Omega:s_r-\mu\le\mathcal{G}^\epsilon(t)\}|+|\{(x,y)\in\mathcal{Q}:s_r-\mu\le s^\epsilon(t)\}|\right)$$
$$\le\frac{c_0|c_0\tau|^{\varpi-\varpi_0}}{(\varpi-\varpi_0)^{(\varpi-\varpi_0)\mathbf{f}_\varpi}},\tag{4.16}$$

and finally if A7.d) holds, then

$$\sup_{t\le\tau}\left(|\{x\in\Omega:\mathcal{G}^\epsilon(t)\le\mu+s_l\}|+|\{(x,y)\in\mathcal{Q}:s^\epsilon(t)\le\mu+s_l\}|\right)$$
$$+\sup_{t\le\tau}\left(|\{x\in\Omega:1-\mu\le S^\epsilon(t)\}|+|\{(x,y)\in\mathcal{Q}:1-\mu\le\rho^\epsilon(t)\}|\right)$$
$$\le\frac{c_0|c_0\tau|^{\varpi-\varpi_0}}{(\varpi-\varpi_0)^{(\varpi-\varpi_0)\mathbf{f}_\varpi}},\tag{4.17}$$

where $\lim_{\varpi\to\infty}\mathbf{f}_\varpi=1$ *and* c_0 *is a constant independent of* τ,ϖ,ϵ,μ.

Proof: *CASE 1:* We claim (4.14). *A7.a)*$_1$ is assumed here. Define $\mathcal{K}_\mu$, $\mathcal{K}_{\varsigma,\mu}$ as

$$\mathcal{K}_\mu(z)\stackrel{\text{def}}{=}\begin{cases}0, & \text{for } 2\mu\le z,\\ z-2\mu, & \text{for } \mu\le z\le 2\mu,\\ -\mu, & \text{for } z\le\mu,\end{cases}$$
$$\mathcal{K}_{\varsigma,\mu}(z)\stackrel{\text{def}}{=}\begin{cases}0, & \text{for } \varsigma(2\mu)\le z,\\ z-\varsigma(2\mu), & \text{for } \varsigma(\mu)\le z\le\varsigma(2\mu),\\ \varsigma(\mu)-\varsigma(2\mu), & \text{for } z\le\varsigma(\mu),\end{cases}$$

where

$$\varsigma(z)\stackrel{\text{def}}{=}\int_{0.5}^{z}\frac{\widetilde{\Lambda}^\epsilon_w}{\widetilde{\Lambda}^\epsilon}(\xi)d\xi,\quad z\in(0,1).\tag{4.18}$$

Define

$$\check{\mathcal{X}}_\mu(z)\stackrel{\text{def}}{=}\begin{cases}1, & \text{for } \mu\le z\le 2\mu,\\ 0, & \text{otherwise.}\end{cases}$$

Then $\check{\mathcal{X}}_\mu(z)=\frac{d}{dz}\mathcal{K}_\mu(z)=\frac{d}{dz}\mathcal{K}_{\varsigma,\mu}(\varsigma(z))$, $\frac{d}{dz}\varsigma(z)=\frac{\widetilde{\Lambda}^\epsilon_w}{\widetilde{\Lambda}^\epsilon}(z)$. By $2\mu\le\frac{\mathbf{k}_1}{2}$ and (3.15),

$$(\zeta_1,\zeta_2,\eta)=(\mathcal{K}_\mu(S^\epsilon),\mathcal{K}_{\varsigma,\mu}(\varsigma(S^\epsilon)),\mathcal{K}_\mu(\rho^\epsilon)-\mathcal{K}_\mu(S^\epsilon))\in L^2(0,T;\mathcal{W}_1).\tag{4.19}$$

Employ (ζ_1,ζ_2,η) of (4.19) in (3.16–3.18) to obtain, by *A4*,

$$\int_{\Omega^\tau}\mathcal{K}_\mu(S^\epsilon)\partial_t S^\epsilon+\int_{\Omega^\tau}\Lambda_w(S^\epsilon)\check{\mathcal{X}}_\mu(S^\epsilon)\nabla_x\Psi^\epsilon\nabla_x S^\epsilon+\int_{\mathcal{Q}^\tau}\mathcal{K}_\mu(\rho^\epsilon)\partial_t s^\epsilon$$
$$\le c_1\int_{\Omega^\tau}\widetilde{\Lambda}^\epsilon_w\check{\mathcal{X}}_\mu(S^\epsilon)|\nabla_x S^\epsilon|,\tag{4.20}$$

where constant c_1 is independent of ε, μ. Suppose

$$\int_{\Omega^\tau} \mathcal{K}_\mu(S^\varepsilon)\partial_t S^\varepsilon + \int_{Q^\tau} \mathcal{K}_\mu(\rho^\varepsilon)\partial_t s^\varepsilon \geq 0, \tag{4.21}$$

(4.20–4.21) imply

$$\int_{\Omega^\tau} \widetilde{\Lambda}^\varepsilon_w \check{\mathcal{X}}_\mu(S^\varepsilon)|\nabla_x S^\varepsilon| \leq \sqrt{\int_{\Omega^\tau} \frac{\widetilde{\Lambda}^\varepsilon_w \check{\mathcal{X}}_\mu}{|\frac{d\Upsilon}{dS}|}(S^\varepsilon)} \sqrt{\int_{\Omega^\tau} \widetilde{\Lambda}^\varepsilon_w \check{\mathcal{X}}_\mu(S^\varepsilon)|\nabla_x \Psi^\varepsilon||\nabla_x S^\varepsilon|}$$
$$\leq c_2 \sqrt{\int_{\Omega^\tau} \frac{\widetilde{\Lambda}^\varepsilon_w \check{\mathcal{X}}_\mu}{|\frac{d\Upsilon}{dS}|}(S^\varepsilon)} \sqrt{\int_{\Omega^\tau} \widetilde{\Lambda}^\varepsilon_w \check{\mathcal{X}}_\mu(S^\varepsilon)|\nabla_x S^\varepsilon|}, \tag{4.22}$$

where constant c_2 is independent of ε, μ. (4.20–4.22) imply

$$\int_{\Omega^\tau} \mathcal{K}_\mu(S^\varepsilon)\partial_t S^\varepsilon + \int_{Q^\tau} \mathcal{K}_\mu(\rho^\varepsilon)\partial_t s^\varepsilon \leq c_3 \int_{\Omega^\tau} \frac{\widetilde{\Lambda}^\varepsilon_w \check{\mathcal{X}}_\mu}{|\frac{d\Upsilon}{dS}|}(S^\varepsilon). \tag{4.23}$$

Define $\mathcal{Z}^\varepsilon_\mu \stackrel{\text{def}}{=} \check{\mathcal{Z}}_\mu(S^\varepsilon) + \widehat{\mathcal{Z}}_\mu(s^\varepsilon)$ where

$$\check{\mathcal{Z}}_\mu(\xi) \stackrel{\text{def}}{=} \int_{2\mu}^{\xi} \mathcal{K}_\mu(z)dz, \qquad \widehat{\mathcal{Z}}_\mu(\xi) \stackrel{\text{def}}{=} \int_{\mathcal{J}(2\mu)}^{\xi} \mathcal{K}_\mu(\mathcal{J}^{-1}(z))dz.$$

(4.23) implies

$$\int_{Q^\tau} \partial_t \mathcal{Z}^\varepsilon_\mu = \int_{\Omega^\tau} \mathcal{K}_\mu(S^\varepsilon)\partial_t S^\varepsilon + \int_{Q^\tau} \mathcal{K}_\mu(\rho^\varepsilon)\partial_t s^\varepsilon \leq c_3 \int_{\Omega^\tau} \frac{\widetilde{\Lambda}^\varepsilon_w \check{\mathcal{X}}_\mu}{|\frac{d\Upsilon}{dS}|}(S^\varepsilon). \tag{4.24}$$

$(3.9)_2$, (4.24), and $A7.a)_1$ yield that, if $0 \leq t_1 \leq t_2 \leq T$,

$$\int_{t_1}^{t_2}\int_Q \partial_t \mathcal{Z}^\varepsilon_\mu \leq c_4 \int_{t_1}^{t_2}\int_Q \mathcal{Z}^\varepsilon_{2\mu}, \tag{4.25}$$

where c_4 is independent of $t_1, t_2, \mu, \varepsilon$. Define

$$\mathcal{F}^\varepsilon(\mu, \tau) \stackrel{\text{def}}{=} \frac{1}{\mu^2} \sup_{t \leq \tau} \int_Q \mathcal{Z}^\varepsilon_\mu(\cdot, t).$$

(4.25) implies that, for $0 \leq t_1 \leq t_2 \leq T$,

$$\mathcal{F}^\varepsilon(\mu, t_2) - \mathcal{F}^\varepsilon(\mu, t_1) \leq c_5(t_2 - t_1)\mathcal{F}^\varepsilon(2\mu, t_2), \tag{4.26}$$

where c_5 is independent of $t_1, t_2, \mu, \varepsilon$. By induction and $(3.10)_2$, one obtains, for $j \in \mathbf{N}$, $jh \leq T$,

$$\mathcal{F}^\varepsilon(\frac{\mathbf{k}_1}{2^\varpi}, jh) \leq (\varpi - \varpi_0 + 1)^{j-1}|c_5 h|^{\varpi - \varpi_0}\mathcal{F}^\varepsilon(\frac{\mathbf{k}_1}{2^{\varpi_0}}, jh). \tag{4.27}$$

If $j = \frac{\varpi - \varpi_0}{\log(\varpi - \varpi_0)}$ and $\tau = jh$ in (4.27), then

$$\mathcal{F}^\varepsilon(\frac{\mathbf{k}_1}{2^\varpi}, \tau) \le \frac{|c_5 \tau|^{\varpi - \varpi_0}}{(\varpi - \varpi_0)^{(\varpi - \varpi_0)\mathbf{f}_\varpi}} \mathcal{F}^\varepsilon(\frac{\mathbf{k}_1}{2^{\varpi_0}}, \tau), \tag{4.28}$$

where $\mathbf{f}_\varpi \to 1$ as $\varpi \to \infty$. Define

$$\begin{cases} \mathcal{B}_1(t) \stackrel{\text{def}}{=} \{x \in \Omega : S^\varepsilon(x,t) \le \mu = \frac{\mathbf{k}_1}{2^\varpi}\}, \\ \mathcal{B}_2(t) \stackrel{\text{def}}{=} \{(x,y) \in \mathcal{Q} : \rho^\varepsilon(x,y,t) \le \mu = \frac{\mathbf{k}_1}{2^\varpi}\}. \end{cases}$$

A7.a)$_1$, (3.11), and (4.28) imply

$$\sup_{t \le \tau} \left(\int \mathcal{X}_{\mathcal{B}_1(t)} + \int \mathcal{X}_{\mathcal{B}_2(t)} \right) \le c_6 \mathcal{F}^\varepsilon(\frac{\mathbf{k}_1}{2^\varpi}, \tau) \le \frac{c_6 |c_5 \tau|^{\varpi - \varpi_0}}{(\varpi - \varpi_0)^{(\varpi - \varpi_0)\mathbf{f}_\varpi}} \mathcal{F}^\varepsilon(\frac{\mathbf{k}_1}{2^{\varpi_0}}, \tau),$$

where constant c_6 is independent of $\tau, \varpi, \varepsilon, \mu$. See (4.3) for $\mathcal{X}_{\mathcal{B}_i}$ $(i = 1,2)$. So proof of the first part of (4.14) is completed.

Proof of the second part of (4.14) is similar to that of the first part, so we just sketch the proof. For comparison with proof of the first part, some notations above will be used again. Define $\mathcal{K}_\mu$, $\mathcal{K}_{\varsigma,\mu}$ as

$$\mathcal{K}_\mu(z) \stackrel{\text{def}}{=} \begin{cases} 0, & \text{for } z \le 1 - 2\mu, \\ z - 1 + 2\mu, & \text{for } 1 - 2\mu \le z \le 1 - \mu, \\ \mu, & \text{for } 1 - \mu \le z, \end{cases}$$

$$\mathcal{K}_{\varsigma,\mu}(z) \stackrel{\text{def}}{=} \begin{cases} 0, & \text{for } z \le \varsigma(1 - 2\mu), \\ z - \varsigma(1 - 2\mu), & \text{for } \varsigma(1 - 2\mu) \le z \le \varsigma(1 - \mu), \\ \varsigma(1-\mu) - \varsigma(1 - 2\mu), & \text{for } \varsigma(1 - \mu) \le z, \end{cases}$$

where ς is the one in (4.18). Define

$$\breve{\mathcal{X}}_\mu(z) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{for } 1 - 2\mu \le z \le 1 - \mu, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\breve{\mathcal{X}}_\mu(z) = \frac{d}{dz}\mathcal{K}_\mu(z) = \frac{d}{dz}\mathcal{K}_{\varsigma,\mu}(\varsigma(z))$. By $2\mu \le \frac{\mathbf{k}_1}{2}$ and (3.15),

$$(\zeta_1, \zeta_2, \eta) = (\mathcal{K}_\mu(S^\varepsilon), \mathcal{K}_{\varsigma,\mu}(\varsigma(S^\varepsilon)), \mathcal{K}_\mu(\rho^\varepsilon) - \mathcal{K}_\mu(S^\varepsilon)) \in L^2(0,T;\mathcal{W}_1). \tag{4.29}$$

Employ (ζ_1, ζ_2, η) of (4.29) in (3.16–3.18) to get

$$\begin{aligned} &\int_{\Omega^\tau} \mathcal{K}_\mu(S^\varepsilon)\partial_t S^\varepsilon + \int_{\Omega^\tau} \Lambda_o(S^\varepsilon)\breve{\mathcal{X}}_\mu(S^\varepsilon)\nabla_x \Psi^\varepsilon \nabla_x S^\varepsilon + \int_{\mathcal{Q}^\tau} \mathcal{K}_\mu(\rho^\varepsilon)\partial_t s^\varepsilon \\ &\quad \le c_1 \int_{\Omega^\tau} \widetilde{\Lambda}_o^\varepsilon \breve{\mathcal{X}}_\mu(S^\varepsilon)|\nabla_x S^\varepsilon|, \end{aligned} \tag{4.30}$$

where constant c_1 is independent of ε, μ. Then following the proof of the first part, one can complete the proof of the second part.

CASE 2: We assume *A7.b)*$_1$ and claim (4.15). Proof of this case is similar to that of *CASE 1*. Define $\mathcal{K}_\mu$, $\mathcal{K}_{\varsigma,\mu}$ as

$$\mathcal{K}_\mu(z) \stackrel{\text{def}}{=} \begin{cases} 0, & \text{for } 2\mu + s_l \le z, \\ z - (2\mu + s_l), & \text{for } \mu + s_l \le z \le 2\mu + s_l, \\ -\mu, & \text{for } z \le \mu + s_l, \end{cases}$$

$$\mathcal{K}_{\varsigma,\mu}(z) \stackrel{\text{def}}{=} \begin{cases} 0, & \text{for } \varsigma(2\mu + s_l) \le z, \\ z - \varsigma(2\mu + s_l), & \text{for } \varsigma(\mu + s_l) \le z \le \varsigma(2\mu + s_l), \\ \varsigma(\mu + s_l) - \varsigma(2\mu + s_l), & \text{for } z \le \varsigma(\mu + s_l), \end{cases}$$

where

$$\varsigma(z) \stackrel{\text{def}}{=} \int_{\mathcal{J}(0.5)}^{z} \frac{\widetilde{\Lambda}_w^\varepsilon}{\widetilde{\Lambda}^\varepsilon}(\mathcal{J}^{-1}(\xi))d\xi, \quad z \in (s_l,\ s_r).$$

Let

$$\check{\mathcal{X}}_\mu(z) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{for } \mu + s_l \le z \le 2\mu + s_l, \\ 0, & \text{otherwise.} \end{cases}$$

By $2\mu \le \mathcal{J}(\frac{k_1}{2}) - s_l$ and (3.15),

$$(\zeta_1, \zeta_2, \eta) = (\mathcal{K}_\mu(\mathcal{G}^\varepsilon), \mathcal{K}_{\varsigma,\mu}(\varsigma(\mathcal{G}^\varepsilon)), \mathcal{K}_\mu(s^\varepsilon) - \mathcal{K}_\mu(\mathcal{G}^\varepsilon)) \in L^2(0,T;\mathcal{W}_1). \tag{4.31}$$

Set (ζ_1, ζ_2, η) of (4.31) in (3.16–3.18) to obtain

$$\int_{\Omega^\tau} \mathcal{K}_\mu(\mathcal{G}^\varepsilon)\partial_t S^\varepsilon + \int_{\Omega^\tau} \Lambda_w(S^\varepsilon)\check{\mathcal{X}}_\mu(\mathcal{G}^\varepsilon)\nabla_x \Psi^\varepsilon \nabla_x \mathcal{G}^\varepsilon + \int_{Q^\tau} \mathcal{K}_\mu(s^\varepsilon)\partial_t s^\varepsilon$$
$$\le c_1 \int_{\Omega^\tau} \widetilde{\Lambda}_w^\varepsilon(S^\varepsilon)\check{\mathcal{X}}_\mu(\mathcal{G}^\varepsilon)|\nabla_x \mathcal{G}^\varepsilon|, \tag{4.32}$$

where constant c_1 is independent of ε, μ. As *CASE 1*, (4.32) implies

$$\int_{\Omega^\tau} \mathcal{K}_\mu(\mathcal{G}^\varepsilon)\partial_t S^\varepsilon + \int_{Q^\tau} \mathcal{K}_\mu(s^\varepsilon)\partial_t s^\varepsilon \le c_3 \int_{\Omega^\tau} \frac{\widetilde{\Lambda}_w^\varepsilon(S^\varepsilon)\check{\mathcal{X}}_\mu(\mathcal{G}^\varepsilon)}{|\frac{du}{ds}(\mathcal{G}^\varepsilon)|}. \tag{4.33}$$

Define $\mathcal{Z}_\mu^\varepsilon \stackrel{\text{def}}{=} \check{\mathcal{Z}}_\mu(S^\varepsilon) + \widehat{\mathcal{Z}}_\mu(s^\varepsilon)$ where

$$\check{\mathcal{Z}}_\mu(\xi) \stackrel{\text{def}}{=} \int_{\mathcal{J}^{-1}(2\mu+s_l)}^{\xi} \mathcal{K}_\mu(\mathcal{J}(z))dz, \qquad \widehat{\mathcal{Z}}_\mu(\xi) \stackrel{\text{def}}{=} \int_{2\mu+s_l}^{\xi} \mathcal{K}_\mu(z)dz.$$

$(3.9)_2$, (4.33), and *A7.b)*$_1$ yield that, if $0 \le t_1 \le t_2 \le T$,

$$\int_{t_1}^{t_2}\int_Q \partial_t \mathcal{Z}_\mu^\varepsilon \le c_4 \int_{t_1}^{t_2}\int_Q \mathcal{Z}_{2\mu}^\varepsilon, \tag{4.34}$$

where c_4 is independent of $t_1, t_2, \mu, \varepsilon$. Then following the proof of *CASE 1*, one can show the first part of (4.15). The second part of (4.15) can be shown by a similar argument as the first part of (4.15). By tracing proofs of *CASE 1* and *CASE 2*, one can see that (4.16) and (4.17) hold also. ■

Lemma 4.3 *Suppose* $2 \le \varpi_0 \in \mathbf{N}$ *and* $\frac{2\mathbf{k}_1}{2^{\varpi_0}} \le \min\{\mathcal{J}(\frac{\mathbf{k}_1}{2}) - s_l, s_r - \mathcal{J}(1-\frac{\mathbf{k}_1}{2})\}$ *where* $\mathbf{k}_1$ *is the one in A5. If* $1 < r < 2$ *and* $\varepsilon < \frac{\mathbf{k}_1}{2^{2+\varpi_0}}$, *then*

$$\sum_{\alpha\in\{w,o\}} \left(\|P_\alpha^\varepsilon\|_{L^r(0,T;W^{1,r}(\Omega))} + \|p_\alpha^\varepsilon\|_{L^r(0,T;\mathcal{W}_y^{1,r}(Q))}\right) \le c, \tag{4.35}$$

where c is a constant independent of ε.

Proof: We assume *A1-5* and *A7.a)*$_{1,2}$ hold. Suppose $\frac{\mathbf{k}_1}{2^{\varpi_*+1}} \le \varepsilon < \frac{\mathbf{k}_1}{2^{\varpi_*}} \le \frac{\mathbf{k}_1}{2^{2+\varpi_0}}$. Due to (3.14), we define

$$\begin{cases} \mathcal{B}_{\varpi_0} \overset{\text{def}}{=} \{(x,t)\in\Omega^T: \ \frac{\mathbf{k}_1}{2^{2+\varpi_0}} \le S^\varepsilon\}, \\ \mathcal{B}_{\varpi} \overset{\text{def}}{=} \{(x,t)\in\Omega^T: \ \frac{\mathbf{k}_1}{2^{\varpi+1}} \le S^\varepsilon < \frac{\mathbf{k}_1}{2^{\varpi}}\}, \text{ for } 2+\varpi_0 \le \varpi \le \varpi_* - 1, \\ \mathcal{B}_{\varpi_*} \overset{\text{def}}{=} \{(x,t)\in\Omega^T: \ \frac{\mathbf{k}_1}{2^{\varpi_*+1}} \le \varepsilon \le S^\varepsilon < \frac{\mathbf{k}_1}{2^{\varpi_*}}\}. \end{cases}$$

Lemmas 4.1, 4.2, (3.7), and Hölder inequality imply

$$\begin{aligned} \int_{\Omega^T} |\nabla_x P_w^\varepsilon|^r &\le \left(\int_{\Omega^T} \Lambda_w(S^\varepsilon)|\nabla_x P_w^\varepsilon|^2\right)^{\frac{r}{2}} \left(\int_{\Omega^T} |\Lambda_w(S^\varepsilon)|^{\frac{-r}{2-r}}\right)^{\frac{2-r}{2}} \\ &\le c_1 \left(\int_{\Omega^T} |\Lambda_w(S^\varepsilon)|^{\frac{-r}{2-r}}\right)^{\frac{2-r}{2}} = c_1 \left(\int_{\Omega^T} |\Lambda_w(S^\varepsilon)|^{\frac{-r}{2-r}} \sum_{\varpi=\varpi_0}^{\varpi_*} \mathcal{X}_{\mathcal{B}_\varpi}\right)^{\frac{2-r}{2}} \\ &\le c_2 \text{ (indep. of } \varepsilon). \end{aligned} \tag{4.36}$$

See (4.3) for $\mathcal{X}_{\mathcal{B}}$. By (3.11), $\|P_w^\varepsilon\|_{L^r(0,T;W^{1,r}(\Omega))}$ is bounded independently of ε. By a similar argument, one can show the rest of (4.35). Furthermore, a similar argument will show (4.35) if one of the conditions *A7.b)*, *A7.c)*, and *A7.d)* holds. ∎

Lemma 4.4 *For* $f \in C_0^\infty(\Omega)$ *and sufficiently small* ϖ, *solutions of* **Theorem 3.1** *satisfy*

$$\int_\varpi^T\int_\Omega f(x)(S^\varepsilon(t) - S^\varepsilon(t-\varpi))(\Phi^\varepsilon(t) - \Phi^\varepsilon(t-\varpi)) \le c\varpi\|f\|_{W^{1,\infty}(\Omega)},$$

where c is independent of ε, ϖ.

Proof: Let $f \in C_0^\infty(\Omega)$. One can see

$$\begin{aligned} \zeta_1(x,t) &\overset{\text{def}}{=} f(x)\int_{max(t,\varpi)}^{min(t+\varpi,T)} \varpi\, \partial^{-\varpi}\Phi^\varepsilon(x,\tau)d\tau \ \in L^2(0,T;\mathcal{V}), \\ \eta(x,y,t) &\overset{\text{def}}{=} f(x)\int_{max(t,\varpi)}^{min(t+\varpi,T)} \varpi\, \partial^{-\varpi}(\phi^\varepsilon - \Phi^\varepsilon)(x,y,\tau)d\tau \ \in L^2(0,T;\mathcal{U}_0). \end{aligned}$$

See $(2.1)_1$ for time differentiation. Employ ζ_1 and η above in (3.16) and (3.18) respectively to get, by Fubini's theorem and **Lemma 4.1**,

$$\int_\varpi^T\int_\Omega f(x)\varpi^2\partial^{-\varpi}S^\varepsilon\ \partial^{-\varpi}\Phi^\varepsilon(x,\tau) + \int_\varpi^T\int_Q f(x)\varpi^2\partial^{-\varpi}s^\varepsilon\ \partial^{-\varpi}\phi^\varepsilon(x,y,\tau)$$

$$= \int_{\Omega^T} \partial_t S^\varepsilon(x,t)\zeta_1 + \int_{Q^T} \partial_t s^\varepsilon(x,y,t)(\eta+\zeta_1)$$
$$= -\int_{\Omega^T} \left(\widetilde{\Lambda}^\varepsilon_w \nabla_x (P^\varepsilon - E_w) - \frac{\Lambda_w\Lambda_o}{\Lambda}\nabla_x \Upsilon(S^\varepsilon)\right)\nabla_x\zeta_1 + \int_{Q^T}\frac{\lambda_w\lambda_o}{\lambda}\nabla_y \upsilon(s^\varepsilon)\nabla_y\eta$$
$$\leq c\varpi \|f\|_{W^{1,\infty}(\Omega)}, \tag{4.37}$$

where c is independent of ε, ϖ. So proof is completed. ∎

Let $\mathbf{m}\in \mathbf{N}$, $\delta = \frac{T}{\mathbf{m}}$, $\mathcal{I}_{i,\delta} = [(i-1)\delta, i\delta)$. We define $\mathcal{A}^\delta : L^1([0,T]) \to L^1([0,T])$ by

$$\mathcal{A}^\delta(\zeta)(t) \stackrel{\text{def}}{=} \frac{1}{\delta}\int_{\mathcal{I}_{i,\delta}} \zeta(\tau)d\tau, \qquad \text{for } t\in \mathcal{I}_{i,\delta}. \tag{4.38}$$

Lemma 4.5 *As* $\delta\to 0$, $\|\Phi^\varepsilon - \mathcal{A}^\delta(\Phi^\varepsilon)\|_{L^2(\Omega^T)}$ *converges to* 0 *uniformly in* ε.
Proof: Let $0\neq f\in C_0^\infty(\Omega)$. Define

$$\mathcal{B}(\varepsilon,\varpi,\mathbf{n}) \stackrel{\text{def}}{=} \Big\{ t\in(\varpi,T):\ \|\Phi^\varepsilon\|_{H^1(\Omega)}(t) + \|\Phi^\varepsilon\|_{H^1(\Omega)}(t-\varpi)$$
$$+\frac{1}{\varpi}\int_\Omega \frac{\varpi^2 f(x)}{\|f\|_{W^{1,\infty}(\Omega)}}\partial^{-\varpi}S^\varepsilon(x,t)\partial^{-\varpi}\Phi^\varepsilon(x,t)dx > \mathbf{n}\Big\}. \tag{4.39}$$

By **Lemmas 4.1, 4.4** and (4.39), $\int_{\mathcal{B}(\varepsilon,\varpi,\mathbf{n})} \mathbf{n}\, dt \leq c$, where c is independent of ε,ϖ. So

$$|\mathcal{B}(\varepsilon,\varpi,\mathbf{n})| \leq c/\mathbf{n}, \qquad \text{for all } \varepsilon,\varpi. \tag{4.40}$$

Next we claim: *If* $\mathbf{n}$ *is fixed, then, as* $\varpi\to 0$,

$$\|\Phi^\varepsilon(\cdot,t) - \Phi^\varepsilon(\cdot,t-\varpi)\|_{L^2(\Omega)} \to 0, \qquad \textit{uniformly in } \varepsilon \textit{ and } t, \tag{4.41}$$

where $t\in(\varpi,T)\setminus \mathcal{B}(\varepsilon,\varpi,\mathbf{n})$.

Proof of claim: If not, there is a constant $c_1>0$ and a sequence $\{t_\varpi,\varepsilon_\varpi\}$ such that, as $\varpi\to 0$,

$$\begin{cases} t_\varpi \in (\varpi,T)\setminus\mathcal{B}(\varepsilon_\varpi,\varpi,\mathbf{n}),\\ \|\Phi^{\varepsilon_\varpi}\|_{H^1(\Omega)}(t_\varpi) + \|\Phi^{\varepsilon_\varpi}\|_{H^1(\Omega)}(t_\varpi-\varpi) \leq \mathbf{n},\\ \int_\Omega \frac{\varpi^2 f(x)}{\|f\|_{W^{1,\infty}(\Omega)}}\partial^{-\varpi}S^{\varepsilon_\varpi}(x,t_\varpi)\partial^{-\varpi}\Phi^{\varepsilon_\varpi}(x,t_\varpi)dx \leq \mathbf{n}\varpi,\\ \|\Phi^{\varepsilon_\varpi}(t_\varpi) - \Phi^{\varepsilon_\varpi}(t_\varpi-\varpi)\|_{L^2(\Omega)} \geq c_1. \end{cases} \tag{4.42}$$

By $(4.42)_2$ and compactness principle, there is a subsequence (not be relabelled) of $\{\Phi^{\varepsilon_\varpi}(t_\varpi), \Phi^{\varepsilon_\varpi}(t_\varpi-\varpi)\}$ converging to $\{g_1,g_2\}$ strongly in $L^2(\Omega)$ and pointwise almost everywhere. By $(4.42)_4$,

$$\|g_1-g_2\|_{L^2(\Omega)} \geq c_1. \tag{4.43}$$

Since $\mathcal{E}^{-1}$ is bounded on $\Re$, by $(4.42)_3$,

$$\int_\Omega (\mathcal{E}^{-1}(g_1) - \mathcal{E}^{-1}(g_2))(g_1-g_2)\frac{f(x)}{\|f\|_{W^{1,\infty}(\Omega)}}dx$$
$$= \lim_{\varpi\to 0}\int_\Omega \frac{\varpi^2 f(x)}{\|f\|_{W^{1,\infty}(\Omega)}}\partial^{-\varpi}S^{\varepsilon_\varpi}(x,t_\varpi)\partial^{-\varpi}\Phi^{\varepsilon_\varpi}(x,t_\varpi)dx = 0. \tag{4.44}$$

Because $\mathcal{E}^{-1}$ is strictly increasing on $\Re$ and because f can be any nonnegative smooth function, (4.44) implies $g_1 = g_2$ almost everywhere, which contradicts to (4.43). So the claim is true.

(4.40–4.41) imply, as $\varpi \to 0$,

$$\int_{\varpi}^{T} \|\Phi^\varepsilon(\cdot,t) - \Phi^\varepsilon(\cdot,t-\varpi)\|^2_{L^2(\Omega)} dt \to 0, \quad \text{uniformly in } \varepsilon. \tag{4.45}$$

By (4.38) and (4.45), if $\delta = \frac{T}{\mathbf{m}}$, then

$$\begin{aligned}\int_0^T \|\Phi^\varepsilon - \mathcal{A}^\delta(\Phi^\varepsilon)\|^2_{L^2(\Omega)} dt &= \sum_{i=1}^{\mathbf{m}} \int_{\mathcal{I}_{i,\delta}} \left\| \frac{1}{\delta} \int_{\mathcal{I}_{i,\delta}} (\Phi^\varepsilon(x,t) - \Phi^\varepsilon(x,\tau)) d\tau \right\|^2_{L^2(\Omega)} dt \\ &\le \sum_{i=1}^{\mathbf{m}} \int_{\mathcal{I}_{i,\delta}} \frac{1}{\delta} \int_{t-i\delta}^{t-(i-1)\delta} \|\Phi^\varepsilon(\cdot,t) - \Phi^\varepsilon(\cdot,t-\varpi)\|^2_{L^2(\Omega)} d\varpi dt \\ &\le \frac{2}{\delta} \int_0^\delta \int_\varpi^T \|\Phi^\varepsilon(\cdot,t) - \Phi^\varepsilon(\cdot,t-\varpi)\|^2_{L^2(\Omega)} dt d\varpi. \end{aligned} \tag{4.46}$$

Right hand side of (4.46) converges to 0 uniformly in ε as $\delta \to 0$. So the lemma follows. ■

Lemma 4.6 *There is a convergent subsequence of* $\{S^\varepsilon, \mathcal{G}^\varepsilon\}$ *in* $L^2(\Omega^T)$.

Proof: By **Lemma 4.1**, $\|\Phi^\varepsilon\|_{L^2(0,T;H^1(\Omega))} \le c_1$, which is independent of ε. So for all δ,

$$\|\mathcal{A}^\delta(\Phi^\varepsilon)\|_{L^2(0,T;H^1(\Omega))} \le c_2 \text{ (indep. of } \varepsilon). \tag{4.47}$$

By **Lemma 4.5**, (4.47), and diagonal process, one can find a subsequence of $\{\Phi^\varepsilon\}$ converging to Φ in $L^2(\Omega^T)$ strongly and pointwise almost everywhere. By boundedness and continuity of $\mathcal{E}^{-1}$ as well as convergence of $\{\Phi^\varepsilon\}$ in $L^2(\Omega^T)$, it is not difficult to find a convergent subsequence for $\{S^\varepsilon\}$. Convergence of $\{\mathcal{G}^\varepsilon\}$ is due to the convergence of $\{S^\varepsilon\}$ and boundedness of $\mathcal{J}$. ■

For convenience, it is assumed that S^ε converges to S in $L^2(\Omega^T)$ and pointwise almost everywhere.

Lemma 4.7 $0 < S < 1$.

Proof: Suppose *A7.a)*$_1$ holds. By **Theorem 3.1** and **Lemma 4.6**, $0 \le S \le 1$. We claim $\mathbf{b} \stackrel{\text{def}}{=} |\{(x,t) \in \Omega^T : S = 0\}| = 0$. If not, by Egoroff's theorem [15] and **Lemma 4.6**, there is a set $\mathcal{B} \subset \Omega^T$ such that (i) $|\mathcal{B}| < \frac{\mathbf{b}}{3} \ne 0$ and (ii) S^ε converges uniformly to S in $\Omega^T \setminus \mathcal{B}$.

Take ϖ_0, ϖ_1 large enough so that

$$\begin{cases} 2 < \varpi_0 < \varpi_1 - 2, \\ \frac{2\mathbf{k}_1}{2^{\varpi_0}} \le \min\left\{\mathcal{J}(\frac{\mathbf{k}_1}{2}) - s_l, s_r - \mathcal{J}(1-\frac{\mathbf{k}_1}{2})\right\}, \\ \frac{|c_0 T|^{\varpi_1 - \varpi_0 + 1}}{(\varpi_1 - \varpi_0)^{(\varpi_1 - \varpi_0)\ell_{\varpi_1}}} \le \frac{\mathbf{b}}{3}, \end{cases} \tag{4.48}$$

where $\mathbf{k}_1$ is the one in *A5* and $c_0, \mathbf{f}_\varpi$ are those in **Lemma 4.2**. By **Lemma 4.2** and (4.48), for all $\varepsilon < \mu \stackrel{\text{def}}{=} \frac{\mathbf{k}_1}{2^{\varpi_1}}$,

$$\left|\{(x,t) \in \Omega^T : S^\varepsilon \leq \mu\}\right| \leq \frac{|c_0 T|^{\varpi_1 - \varpi_0 + 1}}{(\varpi_1 - \varpi_0)^{(\varpi_1 - \varpi_0)\mathbf{f}_{\varpi_1}}} \leq \frac{\mathbf{b}}{3}. \tag{4.49}$$

Since S^ε converges uniformly to S in $\Omega^T \setminus \mathcal{B}$, there is a $\varepsilon_0 \leq \mu\, (= \frac{\mathbf{k}_1}{2^{\varpi_1}})$ such that, for any $\varepsilon < \varepsilon_0$,

$$|S^\varepsilon - S|(x,t) \leq \mu, \quad \text{for } (x,t) \in \Omega^T \setminus \mathcal{B}. \tag{4.50}$$

However, (4.49–4.50) imply, for any $\varepsilon < \varepsilon_0$,

$$\frac{2\mathbf{b}}{3} \leq \left|\{(x,t) \in \Omega^T \setminus \mathcal{B} : S = 0\}\right| \leq \left|\{(x,t) \in \Omega^T \setminus \mathcal{B} : S^\varepsilon \leq \mu\}\right| \leq \frac{\mathbf{b}}{3}, \tag{4.51}$$

that is in contradiction to $\mathbf{b} \neq 0$. So $0 < S$. By a similar argument, one can prove $S < 1$. Moreover, a similar argument will show the lemma if one of the conditions *A7.b)*, *A7.c)*, and *A7.d)* holds. So proof of this lemma is completed. ■

Lemma 4.8 *$\{\mathcal{D}(\mathcal{G}^\varepsilon)\}$ is a Cauchy sequence in $L^2(\Omega^T)$. See (2.1) for $\mathcal{D}$.*

Proof: *CASE 1:* Suppose *A7.a)* or *A7.c)* holds. If $\mathcal{D}$ is a bounded function on $(s_l, s_r]$, the lemma is obvious by **Lemmas 4.6, 4.7**. If not, for any $\delta > 0$, one can find $\varpi_0, \varpi_1 \in \mathbf{N}$ and a positive number $\mathbf{b}$ such that, by *A7.a)*$_3$ or *A7.c)*$_5$,

$$\begin{cases} 2 < \varpi_0 < \varpi_1 - 2, \\ \frac{2\mathbf{k}_1}{2^{\varpi_0}} \leq \min\left\{\mathcal{J}(\frac{\mathbf{k}_1}{2}) - s_l, s_r - \mathcal{J}(1 - \frac{\mathbf{k}_1}{2})\right\}, \\ \mathcal{D}(\mathcal{J}(\frac{\mathbf{k}_1}{2^{\varpi_1}})) < 0, \\ \sum\limits_{\varpi=\varpi_1}^{\infty} \left|\mathcal{D}(\mathcal{J}(\frac{\mathbf{k}_1}{2^{\varpi+1}}))\right|^2 \frac{|c_0 T|^{\varpi - \varpi_0 + 1}}{(\varpi - \varpi_0)^{(\varpi - \varpi_0)\mathbf{f}_\varpi}} < \delta, \\ \left(|\mathcal{D}(\mathcal{J}(\frac{\mathbf{k}_1}{2^{\varpi_1}}))|^2 + |\mathcal{D}(\mathcal{J}(1))|^2\right) \max\{\mathbf{b}, \frac{c_0 |c_0 T|^{\varpi_1 - \varpi_0}}{(\varpi_1 - \varpi_0)^{(\varpi_1 - \varpi_0)\mathbf{f}_{\varpi_1}}}\} < \delta, \end{cases} \tag{4.52}$$

where $\mathbf{k}_1$ is the one in *A5* and $c_0, \mathbf{f}_\varpi$ are those in **Lemma 4.2**. Suppose $\frac{\mathbf{k}_1}{2^{\varpi_*+1}} \leq \varepsilon < \frac{\mathbf{k}_1}{2^{\varpi_*}} \leq \frac{\mathbf{k}_1}{2^{\varpi_1}}$. Because of (3.14), we define

$$\begin{cases} B^{\varepsilon,\varpi_1} \stackrel{\text{def}}{=} \{(x,t) \in \Omega^T : S^\varepsilon < \frac{\mathbf{k}_1}{2^{\varpi_1}}\}, \\ B_\varpi \stackrel{\text{def}}{=} \{(x,t) \in \Omega^T : \frac{\mathbf{k}_1}{2^{\varpi+1}} \leq S^\varepsilon < \frac{\mathbf{k}_1}{2^\varpi}\}, \text{ for } \varpi_1 \leq \varpi \leq \varpi_* - 1, \\ B_{\varpi_*} \stackrel{\text{def}}{=} \{(x,t) \in \Omega^T : \frac{\mathbf{k}_1}{2^{\varpi_*+1}} \leq \varepsilon \leq S^\varepsilon < \frac{\mathbf{k}_1}{2^{\varpi_*}}\}. \end{cases}$$

Then $B^{\varepsilon,\varpi_1} = \bigcup\limits_{\varpi=\varpi_1}^{\varpi_*} B_\varpi$. **Lemma 4.2** and (4.52)$_4$ imply

$$\begin{aligned} \int_{\Omega^T} |\mathcal{D}(\mathcal{J}(S^\varepsilon))|^2 \mathcal{X}_{B^{\varepsilon,\varpi_1}} &= \int_{\Omega^T} |\mathcal{D}(\mathcal{J}(S^\varepsilon))|^2 \sum_{\varpi=\varpi_1}^{\varpi_*} \mathcal{X}_{B_\varpi} \\ &\leq \sum_{\varpi=\varpi_1}^{\varpi_*} \left|\mathcal{D}(\mathcal{J}(\frac{\mathbf{k}_1}{2^{\varpi+1}}))\right|^2 \frac{|c_0 T|^{\varpi - \varpi_0 + 1}}{(\varpi - \varpi_0)^{(\varpi - \varpi_0)\mathbf{f}_\varpi}} < \delta. \end{aligned} \tag{4.53}$$

See (4.3) for characteristic function $\mathcal{X}_{\mathcal{B}}$.

Let both $\varepsilon_i, \varepsilon_j < \frac{\mathbf{k}_1}{2^{\varpi_1}}$. Define

$$\begin{cases} \mathcal{K}^{i,j} \stackrel{\text{def}}{=} \{(x,t) \in \Omega^T : \ \frac{\mathbf{k}_1}{2^{\varpi_1}} \le \min\{S^{\varepsilon_i}, S^{\varepsilon_j}\}\}, \\ \widetilde{\mathcal{K}}_{i,j} \stackrel{\text{def}}{=} \{(x,t) \in \Omega^T : \ S^{\varepsilon_i} \le \frac{\mathbf{k}_1}{2^{\varpi_1}} \le S^{\varepsilon_j}\}. \end{cases}$$

Consider the following

$$\begin{aligned} \int_{\Omega^T} |\mathcal{D}(\mathcal{J}(S^{\varepsilon_i})) - \mathcal{D}(\mathcal{J}(S^{\varepsilon_j}))|^2 &\le \int_{\Omega^T} |\mathcal{D}(\mathcal{J}(S^{\varepsilon_i})) - \mathcal{D}(\mathcal{J}(S^{\varepsilon_j}))|^2 \mathcal{X}_{\mathcal{K}^{i,j}} \\ &+ \int_{\Omega^T} |\mathcal{D}(\mathcal{J}(S^{\varepsilon_i}))|^2 \mathcal{X}_{\widetilde{\mathcal{K}}_{j,i}} + \int_{\Omega^T} |\mathcal{D}(\mathcal{J}(S^{\varepsilon_j}))|^2 \mathcal{X}_{\widetilde{\mathcal{K}}_{i,j}} \\ &+ \int_{\Omega^T} |\mathcal{D}(\mathcal{J}(S^{\varepsilon_i}))|^2 \mathcal{X}_{\mathcal{B}^{\varepsilon_i,\varpi_1}} + \int_{\Omega^T} |\mathcal{D}(\mathcal{J}(S^{\varepsilon_j}))|^2 \mathcal{X}_{\mathcal{B}^{\varepsilon_j,\varpi_1}}. \end{aligned} \tag{4.54}$$

By (4.53),

$$\int_{\Omega^T} |\mathcal{D}(\mathcal{J}(S^{\varepsilon_i}))|^2 \mathcal{X}_{\mathcal{B}^{\varepsilon_i,\varpi_1}} + \int_{\Omega^T} |\mathcal{D}(\mathcal{J}(S^{\varepsilon_j}))|^2 \mathcal{X}_{\mathcal{B}^{\varepsilon_j,\varpi_1}} \le 2\delta. \tag{4.55}$$

By **Lemma 4.2** and $(4.52)_5$,

$$\int_{\Omega^T} |\mathcal{D}(\mathcal{J}(S^{\varepsilon_i}))|^2 \mathcal{X}_{\widetilde{\mathcal{K}}_{j,i}} + \int_{\Omega^T} |\mathcal{D}(\mathcal{J}(S^{\varepsilon_j}))|^2 \mathcal{X}_{\widetilde{\mathcal{K}}_{i,j}} \le c_1\delta. \tag{4.56}$$

Lemmas 4.6, 4.7 imply that $\mathcal{D}(\mathcal{J}(S^\varepsilon))$ converges to $\mathcal{D}(\mathcal{J}(S))$ pointwise almost everywhere. By Egoroff's theorem [15], one can select a set $\mathcal{B}$ such that (i) $|\mathcal{B}| < \mathbf{b}$ ($\mathbf{b}$ is the one in $(4.52)_5$) and (ii) $\mathcal{D}(\mathcal{J}(S^\varepsilon))$ converges to $\mathcal{D}(\mathcal{J}(S))$ uniformly in $\Omega^T \setminus \mathcal{B}$. By $(4.52)_5$,

$$\int_{\Omega^T} |\mathcal{D}(\mathcal{J}(S^{\varepsilon_i})) - \mathcal{D}(\mathcal{J}(S^{\varepsilon_j}))|^2 \mathcal{X}_{\mathcal{K}^{i,j}\cap\mathcal{B}} < c_2\delta, \tag{4.57}$$

and there is a $\varepsilon_0 < \frac{\mathbf{k}_1}{2^{\varpi_1}}$ so that, for both $\varepsilon_i, \varepsilon_j < \varepsilon_0$,

$$\int_{\Omega^T} |\mathcal{D}(\mathcal{J}(S^{\varepsilon_i})) - \mathcal{D}(\mathcal{J}(S^{\varepsilon_j}))|^2 \mathcal{X}_{\mathcal{K}^{i,j}\setminus\mathcal{B}} \le \delta. \tag{4.58}$$

Therefore, by (4.54–4.58), for any $\delta > 0$, there is a ε_0 so that, as $\varepsilon_i, \varepsilon_j \le \varepsilon_0$,

$$\int_{\Omega^T} |\mathcal{D}(\mathcal{J}(S^{\varepsilon_i})) - \mathcal{D}(\mathcal{J}(S^{\varepsilon_j}))|^2 \le c_3\delta. \tag{4.59}$$

So convergence of $\{\mathcal{D}(\mathcal{J}(S^\varepsilon))\}$ (that is, $\{\mathcal{D}(\mathcal{G}^\varepsilon)\}$) is proved.

CASE 2: If *A7.b)* or *A7.d)* holds, the convergence of $\{\mathcal{D}(\mathcal{G}^\varepsilon)\}$ can be shown by a similar argument as *CASE 1.* ∎

Lemma 4.9 *There is a convergent subsequence of* $\{s^\varepsilon\}$ *in* $L^2(\mathcal{Q}^T)$.

Proof: *STEP 1:* By (3.11), **Theorem 3.1**, and **Lemmas 4.1, 4.6, 4.8**, there is a subsequence (not be relabelled) of $\{S^\varepsilon, s^\varepsilon\}$ such that, as $\varepsilon \to 0$,

$$\begin{cases} S^\varepsilon, \mathcal{G}^\varepsilon \to S, \mathcal{G}, & \text{in } L^2(\Omega^T) \text{ strongly}, \\ \mathcal{D}(s^\varepsilon) \to \widehat{\mathcal{D}}, & \text{in } L^2(0,T;\mathcal{U}) \text{ weakly}, \\ s^\varepsilon \to s, & \text{in } L^2(\mathcal{Q}^T) \text{ weakly}, \\ \partial_t s^\varepsilon \to \partial_t s, & \text{in } \textit{dual } L^2(0,T;\mathcal{U}_0) \text{ weakly}, \\ s^\varepsilon(T) \to \widehat{s}, & \text{in } L^2(\mathcal{Q}) \text{ weakly}, \\ s^\varepsilon(0) \to s_0, & \text{in } L^2(\mathcal{Q}) \text{ strongly}. \end{cases} \tag{4.60}$$

Suppose $\mathbf{v}_i$ ($i \in \mathbf{N}$) is a smooth function in $\mathcal{Q}$ and $\{\mathbf{v}_i\}_{i=1}^\infty$ forms a basis of $\mathcal{U}_0$. For each i and $f \in C^1[0,T]$, one obtains, by (2.1) and (3.18),

$$-\int_{\mathcal{Q}^T} s^\varepsilon\, \partial_t f(t)\mathbf{v}_i + \int_{\mathcal{Q}^T} \nabla_y \mathcal{D}(s^\varepsilon)\, f(t)\nabla_y \mathbf{v}_i = \int_{\mathcal{Q}} (s^\varepsilon(0)f(0) - s^\varepsilon(T)f(T))\mathbf{v}_i. \tag{4.61}$$

As $\varepsilon \to 0$, (4.60) implies

$$-\int_{\mathcal{Q}^T} s\, \partial_t f(t)\mathbf{v}_i + \int_{\mathcal{Q}^T} \nabla_y \widehat{\mathcal{D}}\, f(t)\nabla_y \mathbf{v}_i = -\int_{\mathcal{Q}} \widehat{s}\, f(T)\mathbf{v}_i + \int_{\mathcal{Q}} s_0\, f(0)\mathbf{v}_i. \tag{4.62}$$

Applying Green's theorem for (4.62) in the t variable yields

$$\begin{aligned} &\int_{\mathcal{Q}^T} \partial_t s\, f(t)\mathbf{v}_i + \int_{\mathcal{Q}^T} \nabla_y \widehat{\mathcal{D}}\, f(t)\nabla_y \mathbf{v}_i \\ &\quad = -\int_{\mathcal{Q}} (\widehat{s} - s(T))f(T)\mathbf{v}_i + \int_{\mathcal{Q}} (s_0 - s(0))f(0)\mathbf{v}_i. \end{aligned} \tag{4.63}$$

Since $\{\mathbf{v}_i\}_{i=1}^\infty$ is a basis of $\mathcal{U}_0$, (4.63) implies

$$\widehat{s} = s(T), \quad s(0) = s_0, \tag{4.64}$$

and

$$\int_{\mathcal{Q}^T} \partial_t s\, \eta + \int_{\mathcal{Q}^T} \nabla_y \widehat{\mathcal{D}}\, \nabla_y \eta = 0, \qquad \text{for } \eta \in L^2(0,T;\mathcal{U}_0). \tag{4.65}$$

STEP 2: We claim $\mathcal{D}^{-1}(\widehat{\mathcal{D}}) = s$. See **Remark 2.1** for $\mathcal{D}^{-1}$. Let us find $\varphi^\varepsilon, \varphi \in L^2(0,T;\mathcal{U}_0)$ by solving, for all $(x,t) \in \Omega^T$,

$$\begin{cases} -\Delta_y \varphi^\varepsilon = s^\varepsilon, & y \in \mathcal{M}, \\ \varphi^\varepsilon|_{\partial\mathcal{M}} = 0, \end{cases} \qquad \begin{cases} -\Delta_y \varphi = s, & y \in \mathcal{M}, \\ \varphi|_{\partial\mathcal{M}} = 0. \end{cases} \tag{4.66}$$

(3.18), (4.66), and Green's theorem imply

$$\begin{aligned} \int_{\mathcal{Q}^T} \mathcal{D}(s^\varepsilon)s^\varepsilon &= \int_{\mathcal{Q}^T} \mathcal{D}(\mathcal{G}^\varepsilon)s^\varepsilon - \int_{\mathcal{Q}^T} (\mathcal{D}(s^\varepsilon) - \mathcal{D}(\mathcal{G}^\varepsilon))\Delta_y \varphi^\varepsilon \\ &= \int_{\mathcal{Q}^T} \mathcal{D}(\mathcal{G}^\varepsilon)s^\varepsilon - \int_{\mathcal{Q}^T} \partial_t s^\varepsilon \varphi^\varepsilon. \end{aligned} \tag{4.67}$$

Note

$$-\int_{Q^T} \partial_t s^\varepsilon \varphi^\varepsilon = -\int_Q \frac{|\nabla_y \varphi^\varepsilon|^2}{2}(T) + \int_Q \frac{|\nabla_y \varphi^\varepsilon|^2}{2}(0). \tag{4.68}$$

By $(4.60)_5$ and (4.64), $s^\varepsilon(T)$ converges weakly to $s(T)$ in $L^2(Q)$. (4.66), Hölder inequality, and Green's theorem imply

$$\int_Q |\nabla_y \varphi|^2(T) \leq \varliminf_{\varepsilon\to 0} \int_Q |\nabla_y \varphi^\varepsilon|^2(T). \tag{4.69}$$

Take limit supremum both sides of (4.67) to obtain, by (4.60) and **Lemma 4.8**,

$$\varlimsup_{\varepsilon\to 0} \int_{Q^T} \mathcal{D}(s^\varepsilon)s^\varepsilon \leq \int_{Q^T} \mathcal{D}(\mathcal{G})s - \int_Q \frac{|\nabla_y \varphi|^2}{2}(T) + \int_Q \frac{|\nabla_y \varphi|^2}{2}(0). \tag{4.70}$$

Set $\eta = \varphi$ in (4.65) to obtain

$$0 = \int_Q \frac{|\nabla_y \varphi|^2}{2}(T) - \int_Q \frac{|\nabla_y \varphi|^2}{2}(0) + \int_{Q^T} (\widehat{\mathcal{D}} - \mathcal{D}(\mathcal{G}))s. \tag{4.71}$$

By (4.70–4.71),

$$\varlimsup_{\varepsilon\to 0} \int_{Q^T} \mathcal{D}(s^\varepsilon)s^\varepsilon \leq \int_{Q^T} \widehat{\mathcal{D}}s. \tag{4.72}$$

Since $\mathcal{D}^{-1}$ is strictly increasing on $\Re$, for any $f \in L^2(Q^T)$,

$$0 \leq \int_{Q^T} (\mathcal{D}^{-1}(\mathcal{D}(s^\varepsilon)) - \mathcal{D}^{-1}(f))(\mathcal{D}(s^\varepsilon) - f). \tag{4.73}$$

By (4.60), (4.72–4.73), and monotonicity argument [16], one can easily obtain

$$\mathcal{D}^{-1}(\widehat{\mathcal{D}}) = s. \tag{4.74}$$

STEP 3: We claim that $\{s^\varepsilon\}$ is a convergent sequence in $L^2(Q^T)$. By (4.60), (4.72), and (4.74),

$$\lim_{\varepsilon\to 0} \int_{Q^T} (\mathcal{D}(s^\varepsilon) - \widehat{\mathcal{D}})(s^\varepsilon - s) = 0. \tag{4.75}$$

Define $\mathcal{F}_{1,\varepsilon} \stackrel{\text{def}}{=} (\mathcal{D}(s^\varepsilon) - \widehat{\mathcal{D}})(s^\varepsilon - s)$. By (4.74–4.75), $\mathcal{F}_{1,\varepsilon}$ converges to 0 in $L^1(Q^T)$. So there is a subsequence (not be relabelled) of $\{\mathcal{F}_{1,\varepsilon}\}$ converging to 0 pointwise almost everywhere.

Let us consider a point $(x_0, y_0, t_0) \in Q^T$ which satisfies $\lim_{\varepsilon\to 0} \mathcal{F}_{1,\varepsilon}(x_0, y_0, t_0) = 0$. It is not difficult to see $\{\mathcal{D}(s^\varepsilon(x_0, y_0, t_0))\}$ is a bounded set. For any accumulation point $\mathcal{D}_{x_0,y_0,t_0}$ of $\{\mathcal{D}(s^\varepsilon(x_0, y_0, t_0))\}$, one may find a subsequence (not be relabelled) of $\{\mathcal{D}(s^\varepsilon(x_0, y_0, t_0))\}$ such that $\lim_{\varepsilon\to 0} \mathcal{D}(s^\varepsilon(x_0, y_0, t_0)) = \mathcal{D}_{x_0,y_0,t_0}$. Since $\mathcal{D}^{-1}$ is continuous,

$$\begin{aligned} 0 &= \lim_{\varepsilon\to 0}(\mathcal{D}(s^\varepsilon(x_0, y_0, t_0)) - \widehat{\mathcal{D}}(x_0, y_0, t_0))(s^\varepsilon(x_0, y_0, t_0) - s(x_0, y_0, t_0)) \\ &= (\mathcal{D}_{x_0,y_0,t_0} - \widehat{\mathcal{D}}(x_0, y_0, t_0))(\mathcal{D}^{-1}(\mathcal{D}_{x_0,y_0,t_0}) - s(x_0, y_0, t_0)). \end{aligned} \tag{4.76}$$

(4.74) and (4.76) imply $\mathcal{D}_{x_0,y_0,t_0} = \widehat{\mathcal{D}}(x_0, y_0, t_0)$. So $\{\mathcal{D}(s^\varepsilon(x_0, y_0, t_0))\}$ has only one accumulation point $\widehat{\mathcal{D}}(x_0, y_0, t_0)$. Since $\mathcal{F}_{1,\varepsilon}$ converges to 0 pointwise almost everywhere, $\mathcal{D}(s^\varepsilon)$ converges to $\widehat{\mathcal{D}}$ pointwise almost everywhere. By continuity of $\mathcal{D}^{-1}$ and boundedness of $\{s^\varepsilon\}$ in Q^T, s^ε converges to s in $L^2(Q^T)$. ■

By **Lemma 4.9** and a similar argument as **Lemma 4.7**, one can obtain the following result:

Lemma 4.10 $s_l < s < s_r$.

Proof of Theorem 2.1: By **Theorem 3.1** and integration by parts,

$$\int_{\Omega^T} \partial_t S^\varepsilon \zeta + \int_{Q^T} \partial_t s^\varepsilon (\zeta + \eta) = \int_{\Omega^T} (S_0^\varepsilon - S^\varepsilon)\partial_t \zeta + \int_{Q^T} (s_0^\varepsilon - s^\varepsilon)\partial_t(\zeta + \eta),$$

for $\zeta \in L^2(0, T; \mathcal{V}) \cap H^1(\Omega^T), \eta \in L^2(0, T; \mathcal{U}_0) \cap H^1(0, T; L^2(\mathcal{Q}))$, and $\zeta(T) = \eta(T) = 0$. By (3.11) and **Lemmas 4.6, 4.9**, we obtain (2.6). Indeed, **Theorem 2.1** is a direct consequence of **Theorem 3.1, Lemmas 4.1, 4.3, 4.6, 4.7, 4.8, 4.9, 4.10.**

5. Existence of The Auxiliary Problem

Now we prove **Theorem 3.1**, which is done by Galerkin's method. Let $I = (0, T]$, $\ell \in \mathbb{N}$, $h = \frac{T}{\ell}$, $t_m = mh$, and $I_m^* = (t_{m-1}, t_m]$. For a Banach space X,

$$I_h(X) \stackrel{\text{def}}{=} \{f \in L^\infty(0, T; X) : f \text{ is constant in time on each } I_m^* \subset I\}. \tag{5.1}$$

If $f \in I_h(X)$, $f|_{I_m^*} = f(t_m)$ for $m \le \ell = \frac{T}{h}$. For $f \in L^\infty(Q^T)$,

$$\wp(f)(x, y, t) \stackrel{\text{def}}{=} \frac{1}{h} \int_{I_m^*} f(x, y, \tau) d\tau, \quad \text{for } t \in I_m^*. \tag{5.2}$$

One approximates $\mathcal{G}_b^\varepsilon, P_b^\varepsilon, E_\alpha$ for $\alpha \in \{w, o\}$ by

$$\mathcal{G}_b^{\varepsilon,h} \stackrel{\text{def}}{=} \wp(\mathcal{G}_b^\varepsilon), \quad P_b^{\varepsilon,h} \stackrel{\text{def}}{=} \wp(P_b^\varepsilon), \quad E_\alpha^h \stackrel{\text{def}}{=} \wp(E_\alpha). \tag{5.3}$$

By *A4* and (3.11–3.12), it is not difficult to see, for $\alpha \in \{w, o\}$,

$$\begin{cases} \mathcal{G}_b^{\varepsilon,h} \to \mathcal{G}_b^\varepsilon, \\ P_b^{\varepsilon,h} \to P_b^\varepsilon, \\ E_\alpha^h \to E_\alpha, \end{cases} \qquad \text{in } L^2(0, T; H^1(\Omega)) \text{ as } h \to 0. \tag{5.4}$$

Suppose that $\{\mathrm{e}_i\}_{i=1}^\infty$ (resp. $\{\mathrm{v}_i\}_{i=1}^\infty$) is a basis of $\mathcal{V}$ (resp. $\mathcal{U}_0$), and v_i satisfies

$$\begin{cases} -\Delta_y \mathrm{v}_i = c_i \mathrm{v}_i, & \text{in } \mathcal{Q}, \\ \mathrm{v}_i|_{\Omega \times \partial\mathcal{M}} = 0, \end{cases} \tag{5.5}$$

for some constant c_i. Let $\mathcal{V}^h$ (resp. $\mathcal{U}_0^h$) denote the linear span of $\{\mathrm{e}_i\}_{i=1}^\ell$ (resp. $\{\mathrm{v}_i\}_{i=1}^\ell$) where $\ell = \frac{T}{h}$. $\mathcal{W}_1^h \stackrel{\text{def}}{=} \mathcal{V}^h \times \mathcal{V}^h \times \mathcal{U}_0^h$. By $(3.10)_4$, one may find $\mathcal{G}_0^{\varepsilon,h}$ such that $\mathcal{G}_0^{\varepsilon,h} - \mathcal{G}_b^{\varepsilon,h}(0)$ is the L^2 projection of $\mathcal{G}_0^\varepsilon - \mathcal{G}_b^\varepsilon(0)$ on $\mathcal{V}^h$. Let $s_0^{\varepsilon,h} \stackrel{\text{def}}{=} \mathcal{G}_0^{\varepsilon,h}$.

A discretized scheme for (3.13–3.19) is to find $\{S^{\varepsilon,h}, \mathcal{G}^{\varepsilon,h}, P^{\varepsilon,h}, s^{\varepsilon,h}\}$ such that

$$(\mathcal{G}^{\varepsilon,h} - \mathcal{G}_b^{\varepsilon,h}, P^{\varepsilon,h} - P_b^{\varepsilon,h}, s^{\varepsilon,h} - \mathcal{G}^{\varepsilon,h}) \in I_h(\mathcal{W}_1^h), \tag{5.6}$$

$$S^{\varepsilon,h}(0) = \mathcal{J}^{\varepsilon,-1}(\mathcal{G}^{\varepsilon,h}(0)), \quad \mathcal{G}^{\varepsilon,h}(0) = \mathcal{G}_0^{\varepsilon,h}, \quad s^{\varepsilon,h}(0) = s_0^{\varepsilon,h}, \tag{5.7}$$

and if $\{S^{\varepsilon,h}, \mathcal{G}^{\varepsilon,h}, s^{\varepsilon,h}\}(t_{m-1})$ is given, then $(\mathcal{G}^{\varepsilon,h} - \mathcal{G}_b^{\varepsilon,h}, P^{\varepsilon,h} - P_b^{\varepsilon,h}, s^{\varepsilon,h} - \mathcal{G}^{\varepsilon,h})(t_m)$ is a zero of the mapping $\mathcal{H}^{\varepsilon,h} : \Re^{3\ell} \to \Re^{3\ell}, \ell = \frac{T}{h}$ defined by

$$\mathcal{H}^{\varepsilon,h}(\xi_{1,1}, \cdot, \xi_{1,\ell}, \xi_{2,1}, \cdot, \xi_{2,\ell}, \xi_{3,1}, \cdot, \xi_{3,\ell}) = (\overline{\xi}_{1,1}, \cdot, \overline{\xi}_{1,\ell}, \overline{\xi}_{2,1}, \cdot, \overline{\xi}_{2,\ell}, \overline{\xi}_{3,1}, \cdot, \overline{\xi}_{3,\ell}) \tag{5.8}$$

where

$$(\mathcal{G}^{\varepsilon,h} - \mathcal{G}_b^{\varepsilon,h}, P^{\varepsilon,h} - P_b^{\varepsilon,h}, s^{\varepsilon,h} - \mathcal{G}^{\varepsilon,h})(t_m) = \sum_{i=1}^{\ell} (\xi_{1,i}\mathbf{e}_i,\ \xi_{2,i}\mathbf{e}_i,\ \xi_{3,i}\mathbf{v}_i) \in \mathcal{W}_1^h, \tag{5.9}$$

$$S^{\varepsilon,h}(t_m) = \mathcal{J}^{\varepsilon,-1}(\mathcal{G}^{\varepsilon,h}(t_m)), \tag{5.10}$$

$$\begin{aligned}\overline{\xi}_{1,i} = &\int_\Omega \partial^{-h} S^{\varepsilon,h}(t_m)\mathbf{e}_i + \int_\Omega \widetilde{\Lambda}_w^\varepsilon(S^{\varepsilon,h})\nabla_x(P^{\varepsilon,h} - E_w^h)(t_m)\nabla_x\mathbf{e}_i \\ &- \int_\Omega \frac{\Lambda_w^\varepsilon \Lambda_o^\varepsilon(S^{\varepsilon,h})\frac{dv^\varepsilon}{ds}(\mathcal{G}^{\varepsilon,h})}{\Lambda^\varepsilon(S^{\varepsilon,h})}\nabla_x\mathcal{G}^{\varepsilon,h}(t_m)\nabla_x\mathbf{e}_i + \int_Q \partial^{-h}s^{\varepsilon,h}(t_m)\mathbf{e}_i,\end{aligned} \tag{5.11}$$

$$\overline{\xi}_{2,i} = \beta_\varepsilon \int_\Omega \left(\widetilde{\Lambda}^\varepsilon(S^{\varepsilon,h})\nabla_x P^{\varepsilon,h}(t_m) - \sum_{\alpha\in\{w,o\}} \widetilde{\Lambda}_\alpha^\varepsilon(S^{\varepsilon,h}(t_m))\nabla_x E_\alpha^h\right)\nabla_x\mathbf{e}_i, \tag{5.12}$$

$$\overline{\xi}_{3,i} = \int_Q \partial^{-h}s^{\varepsilon,h}(t_m)\mathbf{v}_i - \int_Q \frac{\lambda_w^\varepsilon\lambda_o^\varepsilon\frac{dv^\varepsilon}{ds}(s^{\varepsilon,h})}{\lambda^\varepsilon(s^{\varepsilon,h})}\nabla_y s^{\varepsilon,h}(t_m)\nabla_y\mathbf{v}_i. \tag{5.13}$$

See (3.9) for $\mathcal{J}^{\varepsilon,-1}$ and $(2.1)_1$ for time differentiation. β_ε in (5.12) is a constant satisfying $\beta_\varepsilon > \sup_{z\in\Re} \frac{2\Lambda_w^\varepsilon(z)}{\Lambda_o^\varepsilon(z)|\frac{dv^\varepsilon}{ds}(\mathcal{J}^\varepsilon(z))|}$.

Theorem 3.1 is proved by the following steps: First we show that zeros of (5.8–5.13) exist and are bounded independently of h (see **Lemma 5.1**), next prove a subset of these zeros forms a convergent sequence (see **Lemmas 5.2, 5.3**), and finally conclude the existence of a weak solution of (3.13–3.25) (see **Lemmas 5.4, 5.5**). Let us define a nonnegative function $\Gamma : \Re \to \Re_0^+$ by

$$\Gamma(z) \stackrel{\text{def}}{=} \int_0^z (\mathcal{J}^{\varepsilon,-1}(z) - \mathcal{J}^{\varepsilon,-1}(\xi))d\xi.$$

By (3.9), $\mathcal{J}^{\varepsilon,-1}$ is a strictly increasing function. As **Remark 1.2** of [2],

$$\begin{cases} \Gamma(z_1) - \Gamma(z_2) \le (\mathcal{J}^{\varepsilon,-1}(z_1) - \mathcal{J}^{\varepsilon,-1}(z_2))z_1, & \text{for } z_1, z_2 \in \Re, \\ |\mathcal{J}^{\varepsilon,-1}(z)| \le \varpi\Gamma(z) + \sup_{|\xi|\le 1/\varpi} |\mathcal{J}^{\varepsilon,-1}(\xi)|, & \text{for } z \in \Re,\ \varpi > 0. \end{cases} \tag{5.14}$$

Lemma 5.1 *Under* (3.7–3.11), (5.6–5.13) *are solvable for all* $h(= T/\ell)$, *and solutions satisfy, for* $(\zeta_1, \zeta_2, \eta) \in \mathcal{W}_1^h$, *in* I_m^*,

$$S^{\varepsilon,h}(t_m) = \mathcal{J}^{\varepsilon,-1}(\mathcal{G}^{\varepsilon,h}(t_m)), \tag{5.15}$$

$$0 = \int_\Omega \partial^{-h} S^{\varepsilon,h}(t_m)\zeta_1 + \int_\Omega \widetilde{\Lambda}^\varepsilon_w(S^{\varepsilon,h})\nabla_x(P^{\varepsilon,h} - E^h_w)(t_m)\nabla_x\zeta_1$$

$$- \int_\Omega \frac{\Lambda^\varepsilon_w \Lambda^\varepsilon_o}{\Lambda^\varepsilon}(S^{\varepsilon,h})\nabla_x \Upsilon^\varepsilon(S^{\varepsilon,h})(t_m)\nabla_x\zeta_1 + \int_Q \partial^{-h} s^{\varepsilon,h}(t_m)\zeta_1, \quad (5.16)$$

$$0 = \int_\Omega \widetilde{\Lambda}^\varepsilon(S^{\varepsilon,h})\nabla_x P^{\varepsilon,h}(t_m)\nabla_x\zeta_2 - \sum_{\alpha\in\{w,o\}} \int_\Omega \widetilde{\Lambda}^\varepsilon_\alpha(S^{\varepsilon,h})\nabla_x E^h_\alpha(t_m)\nabla_x\zeta_2, \quad (5.17)$$

$$0 = \int_Q \partial^{-h} s^{\varepsilon,h}(t_m)\eta - \int_Q \frac{\lambda^\varepsilon_w \lambda^\varepsilon_o}{\lambda^\varepsilon}(s^{\varepsilon,h})\nabla_y v^\varepsilon(s^{\varepsilon,h})(t_m)\nabla_y\eta. \quad (5.18)$$

Moreover,

$$\sup_{t\le T}\|s^{\varepsilon,h}(t)\|_{L^2(Q)} + \|\mathcal{G}^{\varepsilon,h}\|_{L^2(0,T;H^1(\Omega))} + \|P^{\varepsilon,h}\|_{L^2(0,T;H^1(\Omega))}$$
$$+\|s^{\varepsilon,h}\|_{L^2(0,T;\mathcal{U})} \le c_0, \quad (5.19)$$

where c_0 is a constant independent of h.

Proof: The solvability of (5.6–5.13) is done by induction. $\{S^{\varepsilon,h}, \mathcal{G}^{\varepsilon,h}, s^{\varepsilon,h}\}(0)$ is given in (5.7). Suppose $\{S^{\varepsilon,h}, \mathcal{G}^{\varepsilon,h}, s^{\varepsilon,h}\}(t_{m-1})$ is solved. Since $\mathcal{H}^{\varepsilon,h}$ of (5.8) is continuous, (5.4) and (5.9–5.13) imply

$$\mathcal{H}^{\varepsilon,h}(\xi_{1,1},\cdots,\xi_{3,\ell})(\xi_{1,1},\cdots,\xi_{3,\ell}) \ge \int_\Omega (\mathcal{G}^{\varepsilon,h} - \mathcal{G}^{\varepsilon,h}_b)\partial^{-h}S^{\varepsilon,h}(t_m)$$
$$+c_1\left(\int_Q \frac{|s^{\varepsilon,h}|^2}{h} + \int_\Omega |\nabla_x \mathcal{G}^{\varepsilon,h}|^2 + \int_\Omega |\nabla_x P^{\varepsilon,h}|^2 + \int_Q |\nabla_y s^{\varepsilon,h}|^2\right)(t_m) - c_2, \quad (5.20)$$

where c_1, c_2 are positive constants. By $(5.14)_1$,

$$\partial^{-h}\Gamma(\mathcal{G}^{\varepsilon,h})(t_m) \le (\mathcal{G}^{\varepsilon,h} - \mathcal{G}^{\varepsilon,h}_b)\partial^{-h}S^{\varepsilon,h}(t_m) + \mathcal{G}^{\varepsilon,h}_b\partial^{-h}S^{\varepsilon,h}(t_m). \quad (5.21)$$

(5.20–5.21) and $(5.14)_2$ imply

$$\mathcal{H}^{\varepsilon,h}(\xi_{1,1},\cdots,\xi_{3,\ell})(\xi_{1,1},\cdots,\xi_{3,\ell})$$
$$\ge c_3\left(\int_\Omega \frac{\Gamma(\mathcal{G}^{\varepsilon,h})}{h} + |\nabla_x \mathcal{G}^{\varepsilon,h}|^2 + |\nabla_x P^{\varepsilon,h}|^2 + \int_Q \frac{|s^{\varepsilon,h}|^2}{h} + |\nabla_y s^{\varepsilon,h}|^2\right) - c_4. \quad (5.22)$$

If norm of $(\xi_{1,1},\cdots,\xi_{3,\ell})$ is large enough, right hand side of (5.22) is strictly positive. So $\mathcal{H}^{\varepsilon,h}$ has a zero for $t = t_m$. By induction, is is easy to see (5.6–5.13) are solvable. Clearly the zero of (5.6–5.13) satisfies (5.16–5.18).

If $(\mathcal{G}^{\varepsilon,h} - \mathcal{G}^{\varepsilon,h}_b, P^{\varepsilon,h} - P^h_b, s^{\varepsilon,h} - \mathcal{G}^{\varepsilon,h}) = \sum_{i=1}^{\ell}(\xi_{1,i}\mathbf{e}_i,\ \xi_{2,i}\mathbf{e}_i,\ \xi_{3,i}\mathbf{v}_i)$ is a zero of (5.8), then

$$\mathcal{H}^{\varepsilon,h}(\xi_{1,1},\cdots,\xi_{3,\ell})(\xi_{1,1},\cdots,\xi_{3,\ell}) = 0. \quad (5.23)$$

Integrating (5.23) over $[0, t_m]$, one obtains, by (5.4),

$$\int_0^{t_m}\int_\Omega (\mathcal{G}^{\varepsilon,h} - \mathcal{G}^{\varepsilon,h}_b)\partial^{-h}S^{\varepsilon,h} + \int_0^{t_m}\int_Q (s^{\varepsilon,h} - \mathcal{G}^{\varepsilon,h}_b)\partial^{-h}s^{\varepsilon,h}$$
$$+c_5\left(\int_0^{t_m}\int_\Omega |\nabla_x \mathcal{G}^{\varepsilon,h}|^2 + \int_0^{t_m}\int_\Omega |\nabla_x P^{\varepsilon,h}|^2 + \int_0^{t_m}\int_Q |\nabla_y s^{\varepsilon,h}|^2\right) \le c_6, \quad (5.24)$$

where c_5, c_6 are constants independent of h. By $(5.14)_1$,

$$\partial^{-h}\Gamma(\mathcal{G}^{\varepsilon,h})(t) \le (\mathcal{G}^{\varepsilon,h} - \mathcal{G}_b^{\varepsilon,h})\partial^{-h}S^{\varepsilon,h}(t) + \mathcal{G}_b^{\varepsilon,h}\partial^{-h}S^{\varepsilon,h}(t). \tag{5.25}$$

Integrate (5.25) over $\Omega \times [0, t_m]$ to get

$$\begin{aligned}\frac{1}{h}\int_{t_m-h}^{t_m}\int_\Omega \Gamma(\mathcal{G}^{\varepsilon,h}) &\le \int_0^{t_m}\int_\Omega (\mathcal{G}^{\varepsilon,h} - \mathcal{G}_b^{\varepsilon,h})\partial^{-h}S^{\varepsilon,h} + \int_\Omega \Gamma(\mathcal{G}^{\varepsilon,h}(0)) \\ &- \int_0^{t_m-h}\int_\Omega (S^{\varepsilon,h} - S^{\varepsilon,h}(0))\partial^h \mathcal{G}_b^{\varepsilon,h} + \frac{1}{h}\int_{t_m-h}^{t_m}\int_\Omega (S^{\varepsilon,h} - S^{\varepsilon,h}(0))\mathcal{G}_b^{\varepsilon,h},\end{aligned} \tag{5.26}$$

where $S^{\varepsilon,h}(t) = S^{\varepsilon,h}(0)$ for $-h < t < 0$. Similar to (5.26), we have

$$\begin{aligned}\frac{1}{h}\int_{t_m-h}^{t_m}\int_Q \frac{|s^{\varepsilon,h}|^2}{2} &\le \int_0^{t_m}\int_Q (s^{\varepsilon,h} - \mathcal{G}_b^{\varepsilon,h})\partial^{-h}s^{\varepsilon,h} + \int_Q \frac{|s^{\varepsilon,h}(0)|^2}{2} \\ &- \int_0^{t_m-h}\int_Q (s^{\varepsilon,h} - s^{\varepsilon,h}(0))\partial^h \mathcal{G}_b^{\varepsilon,h} + \frac{1}{h}\int_{t_m-h}^{t_m}\int_Q (s^{\varepsilon,h} - s^{\varepsilon,h}(0))\mathcal{G}_b^{\varepsilon,h},\end{aligned} \tag{5.27}$$

where $s^{\varepsilon,h}(t) = s^{\varepsilon,h}(0)$ for $-h < t < 0$. Note $\|\partial^h \mathcal{G}_b^{\varepsilon,h}\|_{L^1(0,T;L^\infty(\Omega))\cap L^2(\Omega^T)}$ and $\|\mathcal{G}_b^{\varepsilon,h}\|_{L^\infty(\Omega^T)}$ are bounded by a constant independent of h. (5.24), (5.26–5.27), $(5.14)_2$, and discrete Gronwall's inequality imply (5.19). ■

Lemma 5.2 *For any small $\varpi(>0)$, solutions of* (5.15–5.18) *satisfy*

$$\int_\varpi^T\int_\Omega |\mathcal{G}^{\varepsilon,h}(t) - \mathcal{G}^{\varepsilon,h}(t-\varpi)|^2 \le c_0\varpi,$$

where c_0 is a constant independent of $\varpi, h(= T/\ell)$.

Proof: For fixed μ, we add (5.16) (resp. (5.18)) for $m = j+1, \cdots, j+\mu$, and test the resulting equation by $\zeta_j = h^2\mu\partial^{-h\mu}(\mathcal{G}^{\varepsilon,h} - \mathcal{G}_b^{\varepsilon,h})(t_{j+\mu})$ (resp. $\eta_j = h^2\mu\partial^{-h\mu}(s^{\varepsilon,h} - \mathcal{G}^{\varepsilon,h})(t_{j+\mu})$), where $t_{j+\mu} = (j+\mu)h$. Then we sum above two equations for $j = 1, \cdots, \ell-\mu$ to obtain, by **Lemma 5.1**,

$$\begin{aligned}&\sum_{j=1}^{\ell-\mu}\left\{\int_\Omega |h\mu|^2\partial^{-h\mu}S^{\varepsilon,h}(t_{j+\mu})\partial^{-h\mu}\mathcal{G}^{\varepsilon,h}(t_{j+\mu}) + \int_Q |h\mu\partial^{-h\mu}s^{\varepsilon,h}(t_{j+\mu})|^2\right\} \\ &= \sum_{j=1}^{\ell-\mu}\left\{\int_\Omega |h\mu|^2\partial^{-h\mu}S^{\varepsilon,h}\ \partial^{-h\mu}\mathcal{G}_b^{\varepsilon,h}(t_{j+\mu}) + \int_Q |h\mu|^2\partial^{-h\mu}s^{\varepsilon,h}\ \partial^{-h\mu}\mathcal{G}_b^{\varepsilon,h}(t_{j+\mu})\right\} \\ &- \sum_{j=1}^{\ell-\mu}\sum_{m=j+1}^{j+\mu}\left\{\int_\Omega \left(\widetilde{\Lambda}_w^\varepsilon(S^{\varepsilon,h})\nabla_x(P^{\varepsilon,h} - E_w^h) - \frac{\Lambda_w^\varepsilon\Lambda_o^\varepsilon}{\Lambda^\varepsilon}(S^{\varepsilon,h})\nabla_x v^\varepsilon(\mathcal{G}^{\varepsilon,h})\right)(t_m)\nabla_x\zeta_j\right. \\ &\qquad\left. - \int_Q \frac{\lambda_w^\varepsilon\lambda_o^\varepsilon}{\lambda^\varepsilon}\nabla_y v^\varepsilon(s^{\varepsilon,h})(t_m)\nabla_y\eta_j\right\}.\end{aligned} \tag{5.28}$$

By **Lemma 5.1** and rearranging the indices j and m, right hand side of (5.28) is bounded by $c\mu$. So

$$\int_{h\mu}^{T}\int_{\Omega}|h\mu|^2\partial^{-h\mu}S^{\epsilon,h}(t)\partial^{-h\mu}\mathcal{G}^{\epsilon,h}(t)\leq ch\mu. \tag{5.29}$$

Since $\mathcal{G}^{\epsilon,h}$ is a step function in time, inequality (5.29) is also satisfied if one replaces $h\mu$ by any positive constant ϖ. So the lemma is complete. ■

Arguing as **Lemmas 4.5, 4.6**, one obtains:

Lemma 5.3 *There is a subsequence of* $\{\mathcal{G}^{\epsilon,h},S^{\epsilon,h}\}$ *converging to* $\{\mathcal{G}^{\epsilon},S^{\epsilon}\}$ *pointwise almost everywhere and in* $L^2(\Omega^T)$ *strongly.*

Remark 5.1 *Let us define* $\mathcal{D}^\epsilon:\Re\to\Re$ *by*

$$\mathcal{D}^\epsilon(z)\stackrel{\text{def}}{=}\int_{\mathcal{J}(0.5)}^{z}\frac{\lambda_w^\epsilon\lambda_o^\epsilon}{\lambda^\epsilon}\left|\frac{dv^\epsilon}{ds}\right|(\xi)d\xi.$$

By $(3.9)_1$ *and* **Lemma 5.3**, $\mathcal{D}^\epsilon(\mathcal{G}^{\epsilon,h})$ *converges to* $\mathcal{D}^\epsilon(\mathcal{G}^\epsilon)$ *in* $L^2(\Omega^T)$, *and* $\mathcal{D}^\epsilon(s^{\epsilon,h})$ *is bounded in* $L^2(0,T;\mathcal{U})$.

Lemma 5.4 *There is* $\{S^\epsilon,\mathcal{G}^\epsilon,P^\epsilon,s^\epsilon\}$ *such that, for* $(\zeta_1,\zeta_2,\eta)\in L^2(0,T;\mathcal{W}_1)$,

$$\partial_t S^\epsilon+\int_{\mathcal{M}}\partial_t s^\epsilon\in dual\ L^2(0,T;\mathcal{V}),\quad \partial_t s^\epsilon\in dual\ L^2(0,T;\mathcal{U}_0), \tag{5.30}$$

$$\mathcal{G}^\epsilon=\mathcal{J}^\epsilon(S^\epsilon),\quad (\mathcal{G}^\epsilon-\mathcal{G}_b^\epsilon,P^\epsilon-P_b^\epsilon,s^\epsilon-\mathcal{G}^\epsilon)\in L^2(0,T;\mathcal{W}_1), \tag{5.31}$$

$$\int_{\Omega^T}\partial_t S^\epsilon\zeta_1+\int_{\Omega^T}\left(\widetilde{\Lambda}_w^\epsilon(S^\epsilon)\nabla_x(P^\epsilon-E_w)-\frac{\Lambda_w^\epsilon\Lambda_o^\epsilon}{\Lambda^\epsilon}(S^\epsilon)\nabla_x\Upsilon^\epsilon(S^\epsilon)\right)\nabla_x\zeta_1 + \int_{Q^T}\partial_t s^\epsilon\zeta_1=0, \tag{5.32}$$

$$\int_{\Omega^T}\widetilde{\Lambda}^\epsilon(S^\epsilon)\nabla_x P^\epsilon\nabla_x\zeta_2-\sum_{\alpha\in\{w,o\}}\int_{\Omega^T}\widetilde{\Lambda}_\alpha^\epsilon(S^\epsilon)\nabla_x E_\alpha\nabla_x\zeta_2=0, \tag{5.33}$$

$$\int_{Q^T}\partial_t s^\epsilon\eta-\int_{Q^T}\frac{\lambda_w^\epsilon\lambda_o^\epsilon}{\lambda^\epsilon}\nabla_y v^\epsilon(s^\epsilon)\nabla_y\eta=0, \tag{5.34}$$

$$\mathcal{G}^\epsilon(x,0)=\mathcal{G}_0^\epsilon,\quad s^\epsilon(x,y,0)=s_0^\epsilon. \tag{5.35}$$

Proof: By (5.7) and **Lemmas 5.1, 5.3**, there is $\{S^\epsilon,\mathcal{G}^\epsilon,P^\epsilon,s^\epsilon,\widehat{\mathcal{D}}^\epsilon,\widehat{s}^\epsilon\}$ such that, as $h\to 0$,

$$\left\{\begin{array}{ll}
S^{\epsilon,h},\mathcal{G}^{\epsilon,h}\to S^\epsilon,\mathcal{G}^\epsilon, & \text{in } L^2(\Omega^T)\text{ strongly},\\
S^{\epsilon,h},\mathcal{G}^{\epsilon,h},P^{\epsilon,h}\to S^\epsilon,\mathcal{G}^\epsilon,P^\epsilon, & \text{in } L^2(0,T;H^1(\Omega))\text{ weakly},\\
s^{\epsilon,h},\mathcal{D}^\epsilon(s^{\epsilon,h})\to s^\epsilon,\widehat{\mathcal{D}}^\epsilon, & \text{in } L^2(0,T;\mathcal{U})\text{ weakly},\\
s^{\epsilon,h}(T)\to\widehat{s}^\epsilon, & \text{in } L^2(Q)\text{ weakly},\\
s^{\epsilon,h}(0)\to s_0^\epsilon, & \text{in } L^2(Q)\text{ strongly},\\
\partial^{-h}S^{\epsilon,h}+\int_{\mathcal{M}}\partial^{-h}s^{\epsilon,h}\to\partial_t S^\epsilon+\int_{\mathcal{M}}\partial_t s^\epsilon, & \text{in } dual\ L^2(0,T;\mathcal{V})\text{ weakly},\\
\partial^{-h}s^{\epsilon,h}\to\partial_t s^\epsilon, & \text{in } dual\ L^2(0,T;\mathcal{U}_0)\text{ weakly}.
\end{array}\right. \tag{5.36}$$

If one can show

$$\begin{cases} \widehat{s}^\varepsilon = s^\varepsilon(T), \\ \widehat{\mathcal{D}}^\varepsilon = \mathcal{D}^\varepsilon(s^\varepsilon), \end{cases} \tag{5.37}$$

then (5.15–5.18) imply the lemma as $h \to 0$.

For each $i \geq 1$ and $f \in C^1[0,T]$, (5.18) implies

$$\begin{aligned} -\int_0^{T-h}\int_Q s^{\varepsilon,h}\partial^h\wp(f)(t)\mathbf{v}_i + \int_{Q^T}\nabla_y\mathcal{D}^\varepsilon(s^{\varepsilon,h})\wp(f)(t)\nabla_y\mathbf{v}_i \\ = -\frac{1}{h}\int_{T-h}^T\int_Q s^{\varepsilon,h}(T)\wp(f)(t)\mathbf{v}_i + \int_Q s^{\varepsilon,h}(0)f(0)\mathbf{v}_i. \end{aligned} \tag{5.38}$$

See (5.2) for $\wp(f)$. Letting $h \to 0$ and following the argument in *STEP 1* of **Lemma 4.9**, one obtains (i) $\widehat{s}^\varepsilon = s^\varepsilon(T)$ (that is, $(5.37)_1$), and (ii) for $\eta \in L^2(0,T;\mathcal{U}_0)$,

$$\int_{Q^T}\partial_t s^\varepsilon\eta + \int_{Q^T}\nabla_y\widehat{\mathcal{D}}^\varepsilon\nabla_y\eta = 0. \tag{5.39}$$

To show $\widehat{\mathcal{D}}^\varepsilon = \mathcal{D}^\varepsilon(s^\varepsilon)$, one follows the argument in *STEP 2* of **Lemma 4.9** and employs (5.5). ■

Lemma 5.5 *$\varepsilon \leq S^\varepsilon \leq 1-\varepsilon$ and $\mathcal{J}(\varepsilon) \leq s^\varepsilon \leq \mathcal{J}(1-\varepsilon)$.*

Proof: By (3.6), (3.10) and (5.31), $\zeta \stackrel{\text{def}}{=} \max\{\mathcal{G}^\varepsilon - \mathcal{J}^\varepsilon(1-\varepsilon), 0\} \in L^2(0,T;\mathcal{V})$. Let $\zeta_1 = \zeta_2 = \zeta$ in (5.32–5.33) and $\eta = \max\{s^\varepsilon - \mathcal{J}^\varepsilon(1-\varepsilon), 0\} - \zeta_1$ in (5.34). By $(3.9)_4$ and (5.35), we see $S^\varepsilon \leq 1-\varepsilon, s^\varepsilon \leq \mathcal{J}(1-\varepsilon)$. Similarly, letting $\zeta_1 = \max\{-\mathcal{G}^\varepsilon + \mathcal{J}^\varepsilon(\varepsilon), 0\}$ in (5.32) and $\eta = \max\{-s^\varepsilon + \mathcal{J}^\varepsilon(\varepsilon), 0\} - \zeta_1$ in (5.34), one gets $S^\varepsilon \geq \varepsilon, s^\varepsilon \geq \mathcal{J}(\varepsilon)$. ■

Based on **Lemmas 5.4, 5.5, Theorem 3.1** is proved below.

Proof of Theorem 3.1: (3.13–3.19) is a direct result of **Lemmas 5.4, 5.5**. Define

$$\begin{cases} P_w^\varepsilon \stackrel{\text{def}}{=} P^\varepsilon - \frac{1}{2}\Big(\Upsilon^\varepsilon(S^\varepsilon) + \int_0^{\Upsilon^\varepsilon(S^\varepsilon)}\big(\frac{\Lambda_o^\varepsilon}{\Lambda^\varepsilon}(\Upsilon^{\varepsilon,-1}) - \frac{\Lambda_w^\varepsilon}{\Lambda^\varepsilon}(\Upsilon^{\varepsilon,-1})\big)d\xi\Big), \\ P_o^\varepsilon \stackrel{\text{def}}{=} \Upsilon^\varepsilon(S^\varepsilon) + P_w^\varepsilon, \\ p^\varepsilon \stackrel{\text{def}}{=} P^\varepsilon - \frac{1}{2}\Big(\int_0^{\Upsilon^\varepsilon(S^\varepsilon)}\big(\frac{\Lambda_o^\varepsilon}{\Lambda^\varepsilon}(\Upsilon^{\varepsilon,-1}) - \frac{\Lambda_w^\varepsilon}{\Lambda^\varepsilon}(\Upsilon^{\varepsilon,-1}) - \frac{\lambda_o^\varepsilon}{\lambda^\varepsilon}(v^{\varepsilon,-1}) + \frac{\lambda_w^\varepsilon}{\lambda^\varepsilon}(v^{\varepsilon,-1})\big)d\xi\Big), \\ p_w^\varepsilon \stackrel{\text{def}}{=} p^\varepsilon - \frac{1}{2}\Big(v^\varepsilon(s^\varepsilon) + \int_0^{v^\varepsilon(s^\varepsilon)}\big(\frac{\lambda_o^\varepsilon}{\lambda^\varepsilon}(v^{\varepsilon,-1}) - \frac{\lambda_w^\varepsilon}{\lambda^\varepsilon}(v^{\varepsilon,-1})\big)d\xi\Big), \\ p_o^\varepsilon \stackrel{\text{def}}{=} v^\varepsilon(s^\varepsilon) + p_w^\varepsilon. \end{cases}$$

Clearly $\{S^\varepsilon, \mathcal{G}^\varepsilon, P^\varepsilon, s^\varepsilon, P_\alpha^\varepsilon, p_\alpha^\varepsilon(\alpha = w, o)\}$ satisfies (3.20–3.25).

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